Topic today: homotopy groups $\pi_n(M,m_0)$ for manifold M with base point m_0.

Want to compute these so we can tell when a Lie group G has a covering group: happens when $\pi_1(G,e)$ nontrivial.

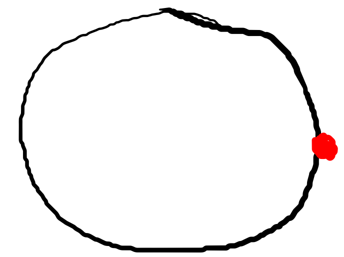
Main technique: if H is a closed subgroup of G, get long exact sequence

$$\begin{aligned}
&\to \pi_2(H) \to \pi_2(G) \to \pi_2(G/H)\\
&\to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H)\\
&\to \pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to 1
\end{aligned}$$

Conclusion: if we can control π_1 on subgroup H and homogeneous space G/H, control $\pi_1(G)$.

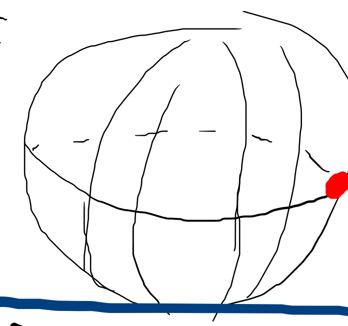
$S^0 = \{x \in \mathbb{R}^1 \mid x^2 = 1\} = \{\pm 1\}$. $\cdot\,(1)$

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\} =$

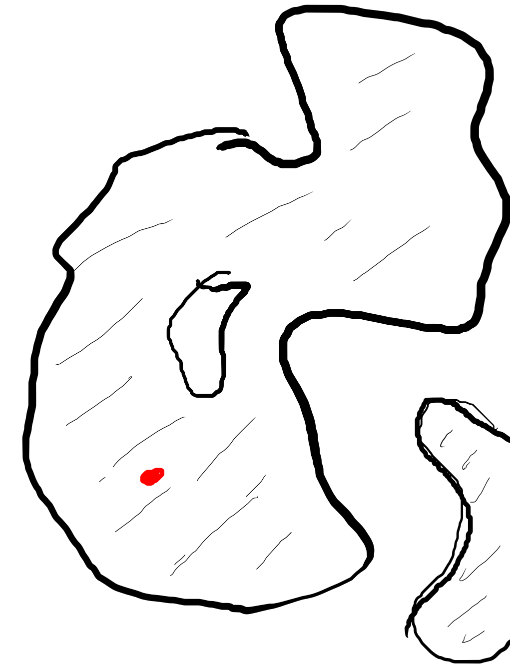


$\cdot\,(1,0)$

$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\} =$

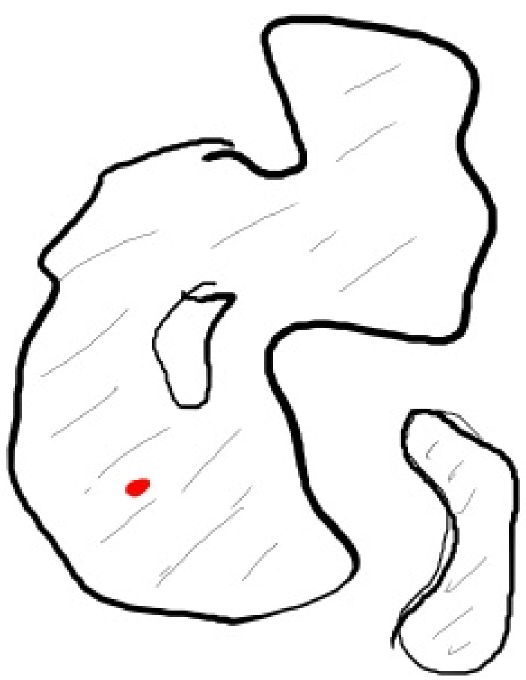


$\cdot\,(1,0,0)$



M m_0

$$\pi_n(M, m_0) = \text{homotopy classes of continuous maps } [S^n, (1,0,0)] \xrightarrow{\gamma} (M, m_0)$$

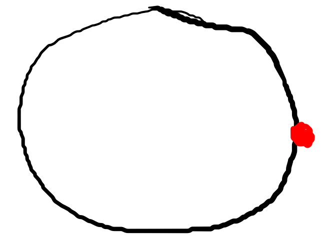


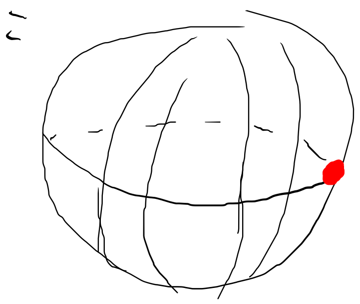
M m_0

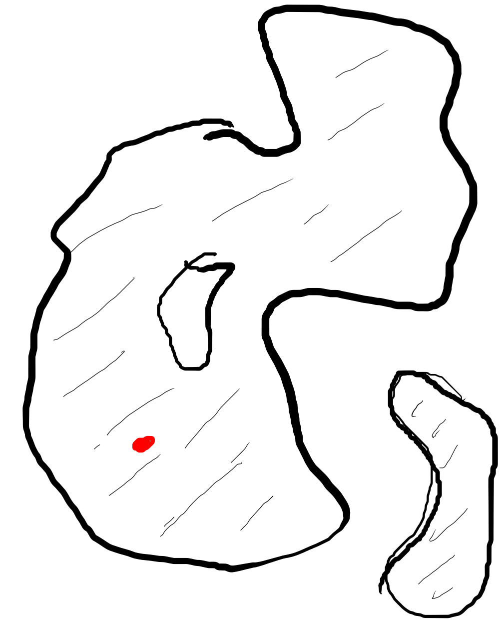
This is a group for $n \geq 1$, abelian for $n \geq 2$.

If $(M, m_0) = (G, e)$ is a Lie group, then $\pi_0(G,e) \cong G/G_0$ is also a group

$S^0 = \{x \in \mathbb{R}^1 \mid x^2 = 1\} = \{\pm 1\}$ ·   · (1)

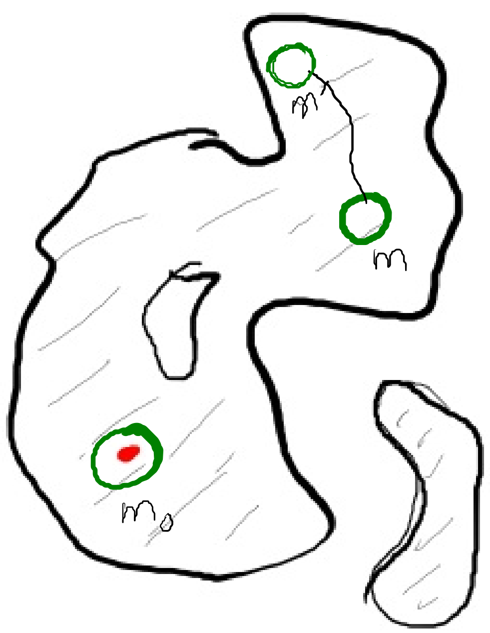
$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\} =$  · $(1,0)$

$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\} =$  · $(1,0,0)$



M m_0

$\pi_0(M, m_0)$ = homotopy classes of continuous maps $(S^0, x_0) \xrightarrow{\gamma} (M, m_0)$



M m_0

$(1) \longmapsto m_0$ $(-1) \longmapsto m$ <u>anything</u>

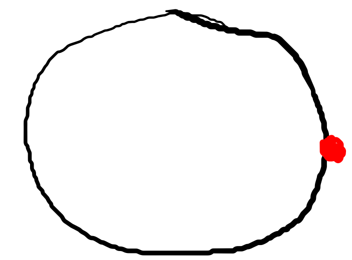
homotopy: wiggle m continuously to m'

CAN WIGGLE m to any other point of M in same connected component as m.

$\boxed{\pi_0(M, m_0) = \text{set of connected components of } M}$

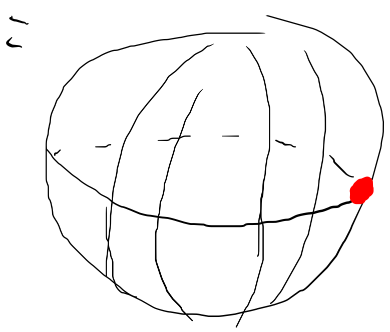
$S^0 = \{x \in \mathbb{R}^1 \mid x^2 = 1\} = \{\pm 1\}$ · $\cdot\ (1)$

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\} =$

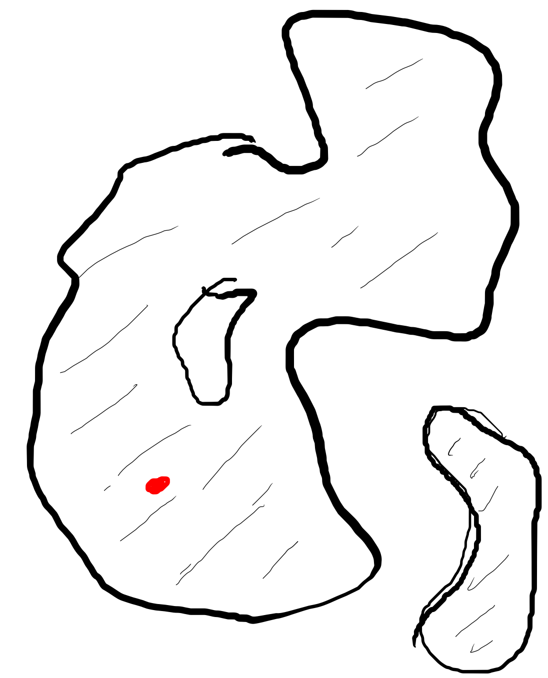


$\cdot\ (1,0)$

$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\} =$



$\cdot\ (1,0,0)$



M m_0

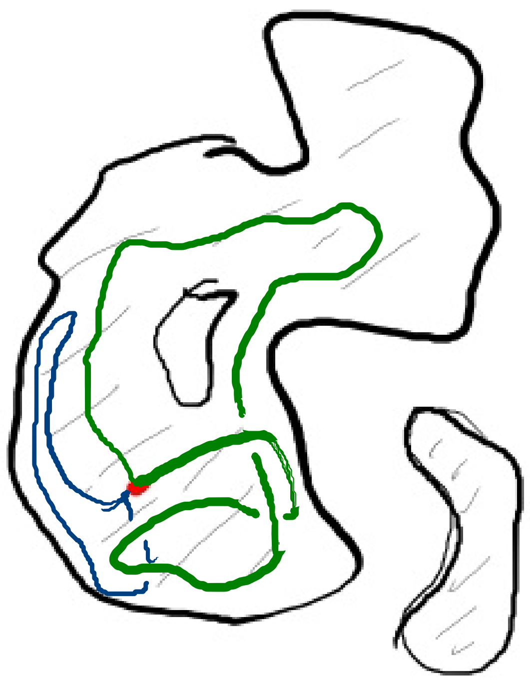
$\pi_1(M, m_0) =$ homotopy classes of loops

$\gamma : [0,1] \to M \qquad \gamma(0) = \gamma(1) = m_0$

Identity element is $\gamma_e(t) = m_0$ (constant)

Blue path is homotopic to γ_e, because it extends continuously ~~from~~ S^1 to ball

$D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$



M m_0

$$\cdots \longrightarrow \pi_1(H,e) \longrightarrow \overset{\mathbb{Z}}{\pi_1(G,e)} \xrightarrow{2\cdot} \overset{\mathbb{Z}}{\pi_1(G/H,eH)} \longrightarrow \overset{\{\pm 1\}}{\pi_0(H,e)} \longrightarrow \overset{\{1\}}{\pi_0(G,e)} \longrightarrow \overbrace{\pi_0(G/H,eH)}^{\text{point}} \to 1$$

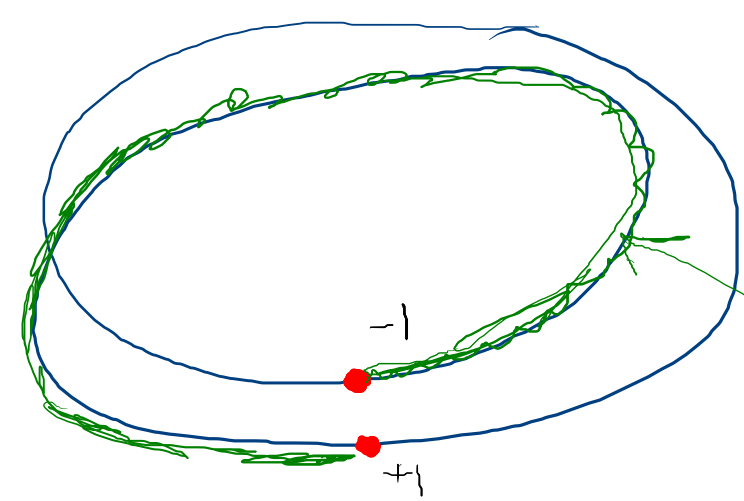
Example: G = circle, $H=\{\pm 1\}$, G/H = circle

Nontrivial element of $\pi_0(H,e)$ given by $\gamma: S^0 \to H$

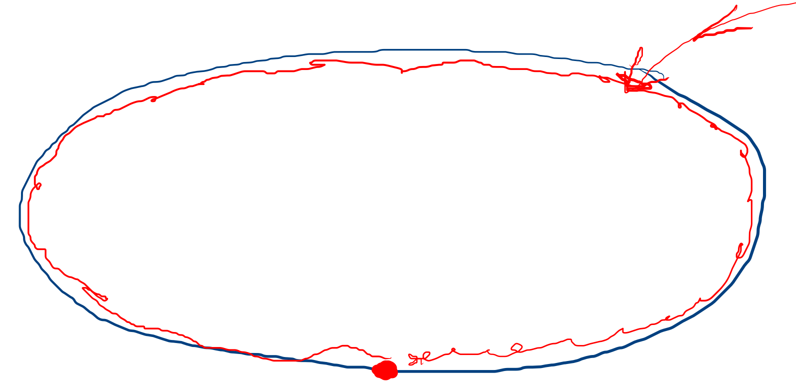
$\gamma(1)=1$, $\gamma(-1)=-1$

Maps to trivial elt of $\pi_0(G,e)$ because of homotopy. This homotopy in G maps to loop in G/H, defining class in $\pi_1(G/H)$ mapping to γ.

G



G/H



Fundamental groups of classical groups

SO(n)/SO(n−1)= S^{n−1}

SU(n)/SU(n−1)= S^{2n−1}

Sp(n)/Sp(n−1)= S^{4n−1}

π_1(SO(1)) = {1}   π_1(SO(2) = Z

π_1(SO(n)) = Z/2Z n >= 3

π_1(SU(n)) = {1} n>= 1

π_1(Sp(n)) ={1} n >= 0

Need from topology: $\pi_1(S^{n-1}) = 0, \; n \geq 3$

$\pi_2(S^{n-1}) = 0, \; n \geq 4$

To start induction:

$SU(2) \cong Sp(1) =$ unit quaternions $\cong S^3$

$SO(3) \cong SU(2)/\{\pm 1\}$

Byproduct: $\dim G = \dim H + \dim G/H$ $\dim S^{n-1} = n-1$

$\dim SO(n) = n(n-1)/2 = 1+2+ \ldots n-1$

$\dim SU(n) = n^2 - 1 = 3+5+ \ldots 2n-1$

$\dim Sp(n) = 2n^2 + n = 3 + 7 + 11 + \ldots 4n-1$

Fundamental groups of classical groups

SO(n)/SO(n−1)= S^{n−1}

SU(n)/SU(n−1)= S^{2n−1}

Sp(n)/Sp(n−1)= S^{4n−1}

$$\boxed{\begin{gathered}\pi_1(SO(1)) = \{1\} \quad \pi_1(SO(2)) = \mathbb{Z} \\ \pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z} \quad n \geqslant 3 \\ \pi_1(SU(n)) = \{1\}, \quad n \geqslant 1 \\ \pi_1(Sp(n)) = \{1\} \quad n \geqslant 0\end{gathered}}$$

Need from topology: $\pi_1(S^{n-1}) = 0, \; n \geqslant 3$

$\pi_2(S^{n-1}) = 0, \; n \geqslant 4$

To start induction:

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$\dim SU(n) = n^2 - 1 = 3+5+\ldots 2n-1$

$\dim Sp(n) = 2n^2 + n = 3+7+11+\ldots 4n-1$

Frobenius theorem

N.B. USED theorem to construct 1-param subgroups of a Lie group!

Statement in a moment. First,

what question is it addressing?

Theorem. X vector field on manifold M;

for each m0 in M there is a unique γ_{X,m0}:

(a,b) --> M with γ_{X,m0}(0) = m0,

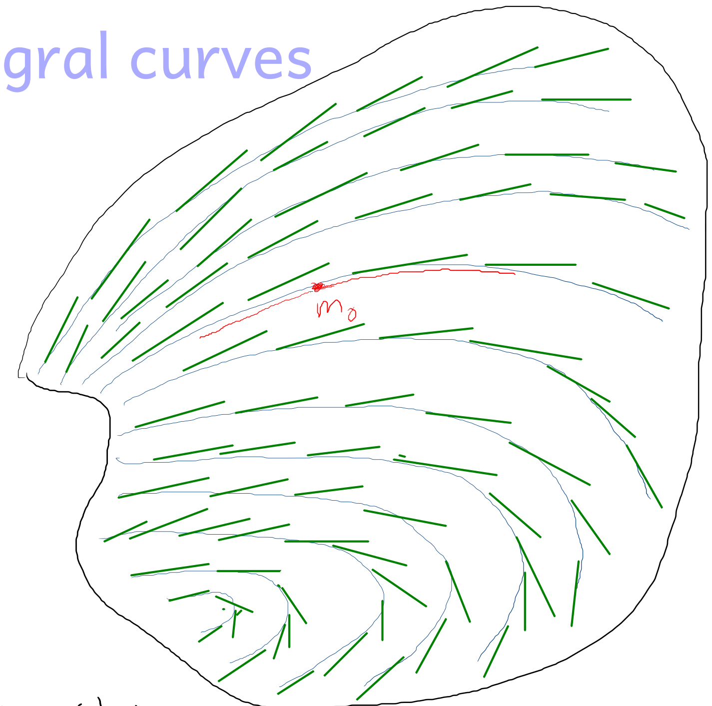
γ_{X,m0}'(t) = X(γ_{X,m0}(t)).

Manifold is nicely covered (foliated) by 1-dimensional submanifolds (as long as X(m) is never zero).

is a theorem about manifolds and submanifolds.

vector field on a manifold,

integral curves



$X(m_1) = 0$:

$\boxed{\gamma_{\{X, m_1\}}(t) = m_1}$ ← POINT, not curve

What about TWO vector fields X and Y?

X(m) and Y(m) define a 2-dimensional subspace of the tangent space T_m(M) at each m in M. (unless they don't!)

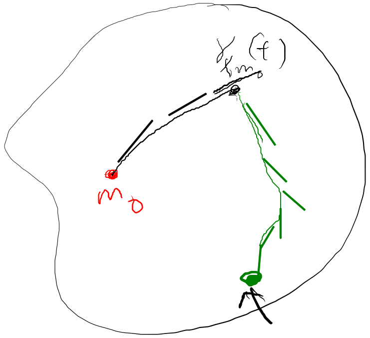
Hope: through each m0 in M passes unique 2-diml submanifold γ_{X,Y,m0}

γ_{X,Y,m0}(s,t)=γ_{Y,(γ_{X,m0}(t))} (s)

"Works": this is 2-diml submanifold (as long as $X(m_0)$, $Y(m_0)$ lin ind and s,t small)

That is, start at m0, follow curve defined by X for a while, then follow curve defined by Y for a while.

X: black
Y: green



$\gamma_{X,Y,m_0}(s,t)$
s, t small real

FAILURE: tangent space at $\gamma_c(s,t)$ is NOT (necessarily) spanned by $\underbrace{X(\gamma(s,t))}_{\text{LOST}}$, $\underbrace{Y(\gamma(s,t))}_{\text{IS in tangent space to subman.}}$

$\boxed{\text{Frobenius theorem}}$

<u>Definition</u> M: a $\boxed{d\text{-dimensional smooth distrib.}}$ $\mathcal{D}$ on M is an assignment $\underset{\substack{\cap \\ M}}{m} \longrightarrow \underline{\mathcal{D}_m}$

d-dimensional subspace of $T_m M$

SUCH THAT [<u>smooth</u>]: near every $m_1 \in M$, there are d smooth vector fields $D_1 \ldots D_d$ defined near m_1,

$$\mathcal{D}_m = \operatorname{span}\left(D_1(m), \ldots D_d(m)\right), \text{ all } m \text{ near } m_1$$

<u>Smoothly varying family of subspaces</u> $\{ S_\alpha \subset V \mid \alpha \in \underset{\text{manifold}}{\underline{A}} \}$ should mean: near each $\alpha_0 \in A$, have $\underline{d}$ ~~vectors~~ smooth functions $V_1(\alpha), \ldots V_d(\alpha)$, $S_\alpha = \langle V_1(\alpha), \ldots V_d(\alpha) \rangle$, all α near α_0

When is $\mathcal{D}$ family of tangent spaces to nice d-diml submanifolds of M?

1-diml $\mathcal{D}$: ALWAYS

2-diml $\mathcal{D}$: NOT ALWAYS

Vector field X on M "belongs to $\mathcal{D}$" if $X(m) \in \mathcal{D}_m$, all m

tangent vector at m

A-diml subspace of $T_m(M)$

Defn $\mathcal{D}$ is INVOLUTIVE if whenever X, Y belong to $\mathcal{D}$, $[X, Y]$ also belongs to $\mathcal{D}$

Has to have dimension at least 2

(NON) EXAMPLE

If M has dimension 2, ANY distribution is involutive [WHY?]

need 2-diml $\mathcal{D}$ in $\mathbb{R}^3$

$$\mathcal{D} = \text{span of } \left\langle \frac{\partial}{\partial x}, \underbrace{\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}}_{\text{lin comb of } \frac{\partial}{\partial y},\, \frac{\partial}{\partial z};\ x \in \mathbb{R}} \right\rangle \subseteq \underbrace{T_{x,y,z}\mathbb{R}^3}_{\text{span}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)}$$

EASY: $\mathcal{D}$ is smooth 2-diml

Spanned by

$$X = \frac{\partial}{\partial x}$$

$$Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$$

not this

hard

solve $\frac{\partial f}{\partial x} = 0$

$\frac{\partial f}{\partial y} = 0$

EASY

$$[X, Y] = \boxed{\frac{\partial}{\partial z} \notin \text{smooth} \cdot X + \text{smooth} \cdot Y}$$

NOT INVOLUTIVE

follow X, then follow Y $\rightsquigarrow$ DIFFERENT surfaces in $\mathbb{R}^3$ from follow Y, then follow X

FROBENIUS

Thm $\mathcal{D}$ involutive d-diml distn on M; through any $m_0 \in M$ $\exists$ unique connect d-diml embedded submanifold $\Gamma_{\mathcal{D},m_0}$ such that

$$\boxed{\underbrace{T_m(\Gamma_{\mathcal{D},m_0})}_{= \mathcal{D}(m)} \subset T_m M}$$

$d=1$: $\mathcal{D}$ = span of 1 vector field X (locally). Involutive automatic

Thm follows from "1-param group of diffeos" thm (1st page)

Higher: thm about differential equations There are $\frac{\partial}{\partial}$ in it; PDE? ← harder

INVOLUTIVE lets you reduce to ~~d~~ ODE's.

SK IP proof: Try to add a reference to proof accessible online

1-dim'l version $\longrightarrow$

<u>Thm</u> G Lie group, $X \in \mathfrak{g}$ $\rightsquigarrow$ get one-parameter subgroup ~~ex~~ $\boxed{t \mapsto \exp(tX) \in G}$

$$\boxed{\mathbb{R} \xrightarrow{\gamma_X} G}$$ <u>Lie group hom.</u>

---

d-dim'l:

<u>Thm</u> G Lie gp, $\mathfrak{h} \subset \mathfrak{g}$ d-dimensional Lie subalgebra, get embedded Lie subgroup $\boxed{\gamma_{\mathfrak{h}} : H \to G}$ of dim d

<u>FRIDAY</u>

Lie subgroups from Lie subalgebras, Lie group maps from Lie algebra maps

Friday, April 3, 2020 8:34 AM

What came pretty easily from Lie group definitions:

H, G Lie groups with Lie algebras $\mathfrak{h}$, $\mathfrak{g}$, ϕ: H --> G Lie group homomorphism $\Rightarrow$ dϕ: $\mathfrak{h}$ ---> $\mathfrak{g}$ is a Lie algebra homomorphism.

H $\subset$ G is an immersed subgroup $\Rightarrow$ $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra.

Topic today: CONVERSES

$\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra $\Rightarrow$ $\exists$ H $\subset$ G an immersed subgroup

dϕ: $\mathfrak{h}$ ---> $\mathfrak{g}$ a Lie algebra homomorphism $\Rightarrow$ sometimes $\exists$ ϕ: H --> G Lie group homomorphism.

sometimes: ALWAYS if H is connected and simply connected.

How to make a Lie subgroup

Friday, April 3, 2020 9:39 AM

G Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{h}$ a Lie subalgebra of G.

Recall that $\mathfrak{g}$ consists of left-invariant vector fields on G. To any d-diml vector subspace $\mathfrak{s} \subset \mathfrak{g}$ can attach a d-dimensional distribution

$$D(\mathfrak{s}) = \{ \Sigma f_j X_j \mid f_j \in C^\infty(G),\ X_j \in \mathfrak{s} \}.$$

Easy to check that $D(\mathfrak{s})$ is involutive if and only if $\mathfrak{s}$ is a Lie subalgebra. So back to our Lie subalgebra $\mathfrak{h}$.

Frobenius: through each point g of G there is a unique maximal d-dimensional connected immersed submanifold H(g), characterized by

$g \in H(g)$, $T_x(H(g)) = \mathrm{span}(X(x) \mid X \text{ in } \mathfrak{h})$ for all $x \in H(g)$, H(g) maximal.

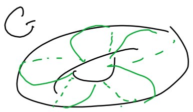
H(g) and H(g') either coincide or are disjoint; so G is disjoint union of all, and H(g) = H(g'), all $g' \in H(g)$.

Since the vector fields in X are preserved by the left translation operations $x \mapsto g_0.x$, left translation must permute the submanifolds H(g). So $g_0.H(g) = H(g_0 g)$, which implies that $H(e) =_{def} H$ is a subgroup!

Also shows that H(g) = g.H: the foliation of G is by cosets of H.

Easy properties of $\mathfrak{h} \mapsto H$

Friday, April 3, 2020 1:41 PM



Bad example to understand: $G = \mathbb{R}^2/\mathbb{Z}^2$
$\mathfrak{h} = \mathbb{R}(1,s) \rightsquigarrow H = \mathbb{R}(1,s) / \mathbb{Z}^2 \cap \mathbb{R}(1,s)$
$\mathfrak{h}$ ← line of slope s
$\cong$ 2-dim'l torus G
s rational: H is circle in G
s irrational: H is line, winds around ∞ often

H is generated by the elements exp(X), X in $\mathfrak{h}$

$$\mathfrak{h} = \{ Y \in \mathfrak{g} \mid \exp(tY) \in H, \text{ all real } t \}$$

G is locally a product of H and any complement: if S is a submanifold of G passing through e, and $T_e(G) = T_e(H)$ direct sum $T_e(S)$, then there are neighborhoods U_H of e in H and U_S of e in S so that $U_H \times U_S$ maps diffeomorphically by multiplication to an open neighborhood of e in G:

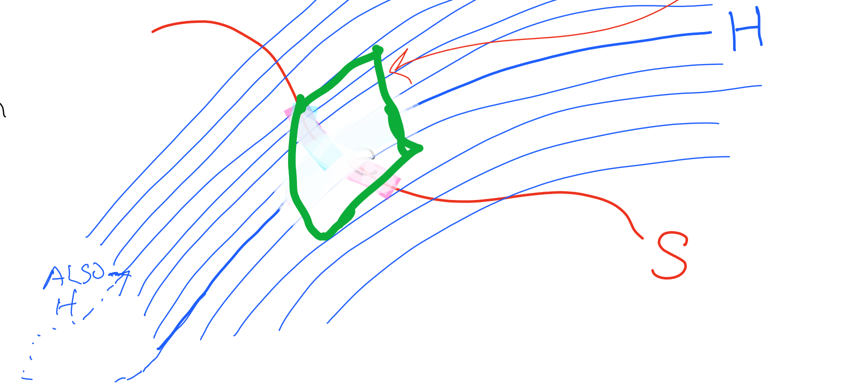
countable ∞ of horiz stripes are in S

$U_S \times U_H \to V \subset G$, $(s,h) \mapsto s.h$ $H(s) \cap V \supset s.U_H$

WISH: $G \simeq S \times H$ by map $(s,h) \mapsto sh$
usually FALSE: locally true

Global difficulty: ← bit of coset of H
could wind back to different coset

V =



Homomorphisms

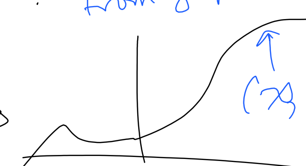
Friday, April 3, 2020 3:33 PM

GIVEN Lie groups H, G, Lie alg hom $\varphi: \mathfrak{h} \to \mathfrak{g}$. WANT Lie group hom $\Phi: H \to G$, $d\Phi = \varphi$.

USE subgroup result: strategy

maps $\longleftrightarrow$ graphs

think of $f: \mathbb{R} \to \mathbb{R}$ as (SPECIAL) curve in $\mathbb{R}^2$



CONSTRUCT GRAPH of Φ $\subset H \times G$

$\Gamma = \{(h, \overset{\text{"hope"}}{\Phi(h)}) \mid h \in H\}$ ← since Φ is group hom., Γ has to be SUBGP of $H \times G$!

Construct $\Phi \longrightarrow$ construct subgroup of $H \times G$.

Use prev thm: $\mathrm{Lie}(H \times G) = \mathfrak{h} \times \mathfrak{g}$ (easy)

Make subalgebra $\mathfrak{f} = \{(X, \varphi(X)) \mid X \in \mathfrak{h}\} \subset \mathfrak{h} \times \mathfrak{g}$ GRAPH of φ!

EASY: φ Lie alg hom $\Longrightarrow$ $\mathfrak{f}$ is Lie subalg of $H \times G$.

like φ

$\Longrightarrow$ have embedded Lie subgroup $F \subset H \times G$ ← candidate for Γ

Thm from Frob

Lie alg $= \mathfrak{f}$

From F to Γ

Friday, April 3, 2020 3:43 PM

Piatetski-Shapiro: collaborator of Gelfand
→ prof at Yale
AUTOM FORMS

COLLOQUIUM ~ late 1960s? ⤳ TITLE: Automorphic forms on GL_2

2nd COLLOQUIUM ~ 1980? ⤳ Title: Automorphic forms on G_2

Ten years: managed to erase L.

Hope: come back in ten years, erase the 2.

$\mathfrak{f} = \{(X, \varphi(X)) \mid X \in \mathfrak{h}\} \subset \mathfrak{h} \times \mathfrak{g}$

$\Downarrow$ restrict

$\mathfrak{h}$

$\boxed{\text{Restriction to } \mathfrak{f} \text{ is ISOM}}$

$\mathfrak{g} \downarrow \mathfrak{h}$ $\boxed{\text{ONTO Lie alg hom}}$

$F \subset H \times G$ (connected)

$\downarrow$ proj on 1st factor

H $\qquad$ H

restrict proj to subgroup F

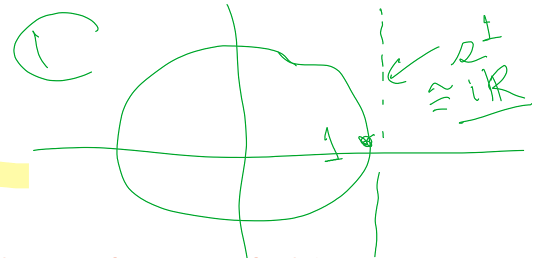
SINCE Lie alg map $\mathfrak{f} \to \mathfrak{h}$ is ISOM, Lie group map $F \to H$ is local diffeo, image is open, COVERING of H_0 = id comp of H. SO if H connected, F is a covering of H.

Get smooth map $F \to G$ proj on 2nd factor; diff is $\varphi : \mathfrak{h} \to \mathfrak{g}$

$\boxed{\text{DEFINES THE } \Phi \text{ on COVERING } F. \text{ IF } H \text{ SIMPLY CONN, } F \cong H}$

Tori

Sunday, April 5, 2020 8:30 AM



Big picture: aiming to describe all compact Lie groups K.

Medium picture: last pset asked you to describe Lie algebras of vector fields on R using [d/dx,∗]; d/dx spans a maximal commutative subalgebra. We'll describe compact Lie groups in a parallel way.

Small picture: today want to understand tori T, which will appear as maximal commutative subgroups of K.

TORUS used in algebraic groups in related but DIFFERENT ways

DEFINITION: a COMPACT TORUS is a compact connected abelian Lie group T. Write

$\mathfrak{t}$ = Lie(T), $\mathfrak{t}_{\mathbf{C}}$ = complexification of $\mathfrak{t} = \mathfrak{t} \otimes_{\mathbf{R}} \mathbf{C}$, $\mathfrak{t}^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{t}, \mathbf{R})$

LIE GROUP: Roman upper case G
Lie algebra: fraktur lower $\mathfrak{g}$
complexify: $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$
$\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$ dual vec space

FUNDAMENTAL EXAMPLE: $S^1 = \{ e^{it} \mid t \in \mathbf{R} \}$, $\mathfrak{s}^1 = i\mathbf{R}$

all exs: PRODUCTS of S^1

$e^x = 1 \Leftrightarrow x \in 2\pi i\mathbb{Z}$

LATTICE OF COCHARACTERS OF T $X_*(T)$ = kernel of exp $\subset \mathfrak{t}$; $X_*(S^1) = 2\pi i\,\mathbf{Z}$

A abelian Lie

LATTICE OF CHARACTERS OF T $X^*(T) = \mathrm{Hom}(T,S^1) \subset i\mathfrak{t}\mathbf{R}^*$; $X^*(S^1) = \mathbf{Z}$

$\exp: \mathfrak{a} \to A$ HOMOM. of groups

Lie gr hom ⤳ $d: \mathfrak{t} \to i\mathbb{R}$

Lie alg hom = linear map

maps = $i\mathfrak{t}^*$

Main theorem about tori

Sunday, April 5, 2020 2:26 PM

T compact comm. abelian Lie

THEOREM $X_*(T)$ and $X^*(T)$ are lattices (finitely generated free abelian groups) dual to each other ($\mathrm{Hom}_{\mathbf{Z}}(X_*(T),\mathbf{Z}) = X^*(T)$). X_* is a covariant equivalence of categories

another version of $X_*(T)$

inverse functors

[compact tori] $\to$ [lattices] $\quad T \mapsto \mathrm{Hom}_{Lie}(S^1,T) \quad X \otimes_{\mathbf{Z}} S^1 \leftarrow X$

Similarly, X^* is a contravariant equivalence of categories

REVERSES MAPS

[compact tori] $\to$ [lattices] $\quad H \mapsto X^*(H) = \mathrm{Hom}_{Lie}(H,S^1) \quad \mathrm{Hom}_{\mathbf{Z}}(Y,S^1) \leftarrow Y$

Proof is an exercise once you understand X_* and X^*. Look at three examples of tori:

$U(1)^n$ = diag unitary $n\times n$ matrices $\simeq \mathbb{R}^n/(2\pi\mathbb{Z})^n$ "n-diml torus"

$U = \{ (e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \}$ $\quad S = \{ (e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \mid \Sigma\, \theta_j = 0 \}$ $\quad P = U/\{ e^{i\phi}(1, \ldots, 1) \}$

det 1 diag unitary $\quad$ diag unitary / scalar matrices

$X_*(U) = \mathrm{Hom}_{Lie}(S^1,U) = \mathbf{Z}^n = \{ m = (m_1, \ldots, m_n) \}$, $m(e^{i\theta}) = (e^{im_1\theta}, \ldots, e^{im_n\theta})$.

$X^*(U) = \mathrm{Hom}_{Lie}(U,S^1) = \mathbf{Z}^n = \{ \mu = (\mu_1, \ldots, \mu_n) \}$, $\mu(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) = e^{i(\mu_1\theta_1 + \ldots + \mu_n\theta_n)}$

$X_*(S) = \mathrm{Hom}_{Lie}(S^1,S) = \{ m = (m_1, \ldots, m_n) \mid \Sigma\, m_j = 0\}$, $m(e^{i\theta}) = (e^{im_1\theta}, \ldots, e^{im_n\theta})$.

$X^*(S) = \mathrm{Hom}_{Lie}(S,S^1) = \{ \mu = (\mu_1,\ldots,\mu_n) \}/\mathbf{Z}(1,\ldots,1)$, $\mu(e^{i\theta_1},e^{i\theta_2},\ldots,e^{i\theta_n}) = e^{i(\mu_1\theta_1+\ldots+\mu_n\theta_n)}$

$X_*(P) = \mathrm{Hom}_{Lie}(S^1,P) = \{ m = (m_1, \ldots, m_n) \mid \Sigma\, m_j = 0\}/\mathbf{Z}(1,\ldots,1)$, $m(e^{i\theta}) = (e^{im_1\theta},\ldots,e^{im_n\theta})$.

$X^*(P) = \mathrm{Hom}_{Lie}(P,S^1) = \{ \mu = (\mu_1,\ldots,\mu_n) \mid \Sigma\, \mu_j = 0\}$, $\mu(e^{i\theta_1},e^{i\theta_2},\ldots,e^{i\theta_n}) = e^{i(\mu_1\theta_1 + \ldots + \mu_n\theta_n)}$

Passage from U to S is passage to subtorus; from U to P is passage to a quotient.

More ways to change T

Sunday, April 5, 2020 4:13 PM

$U = \{ (e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \} = U(1)^n$ $\quad X_*(U) = \mathbf{Z}^n$ $\quad X^*(U) = \mathbf{Z}^n$

only def up to mults of π

$\supset \{\pm 1\}$ two-element subgroup ⇢ quotient group $\bar{U} = U/\{\pm 1\}$ ← image in $\bar{U}$

extra cocharacter $(\tfrac12 \ldots \tfrac12)(e^{i\theta}) = (e^{i\theta/2}, \ldots, e^{i\theta/2})$ Not well-def in U; IS well-def in $\bar{U}$

$X_*(\bar{U}) = \langle \mathbf{Z}^n, (1/2,\ldots,1/2)\rangle = \mathbf{Z}^n \cup (\mathbf{Z} + 1/2)^n$ (twice as many) $\quad X^*(\bar{U}) = \{ \mu \in \mathbf{Z}^n \mid \Sigma \mu_j$ ~~even~~ $\}$

Not all chars on U are not triv on $\pm I$; don't factor to $\bar{U}$

MORE cocharacters, ~~FEWER~~ characters. half as many

1st coord

$\tilde{U} =_{def} \{ ((e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}), z) \mid z^2 = e^{i\Sigma\theta_j} \}$ ~~two-to-one cover~~ of U

STARE AT THIS & convince yourself it's double cover!

U = basic ex
S = subtorus
P = quotient torus

$X_*(\tilde{U}) = \{ m \in \mathbf{Z}^n \mid \Sigma m_j$ even $\}$ $\quad X^*(\tilde{U}) = \langle \mathbf{Z}^n, (1/2,\ldots,1/2)\rangle = \mathbf{Z}^n \cup (\mathbf{Z} + 1/2)^n$

$m(e^{i\theta}) = ((e^{im_1\theta}, \ldots, e^{im_n\theta}), e^{i[(m_1+\ldots+m_n)/2]\theta})$ $\quad \mu((e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}), z) = e^{i(\mu_1\theta_1 + \ldots +}$

$\mu_n\theta_n)$

CONVENTION: elts of X_* ⟷ Roman
X^* ⟷ Greek

$[\mu + (1/2,\ldots,1/2)]((e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}), z) = z \cdot e^{i(\mu_1\theta_1 + \ldots +}$

values in S codim 1 sublattice

cochar from U has to land in S

half as many

$\mu_n\theta_n)$

char of U, divide by chars restr. to be 1.

quotient by 1 dim sublattice

FEWER cocharacters, MORE characters.

char of U, triv. on scalars

Representations of tori

fin. diml REPRESENTATION of Lie group G is fin. diml complex vector space V, Lie group hom $\pi: G \to GL(V)$ ← really useful, complicated to understand

Sunday, April 5, 2020 5:24 PM

THEOREM Suppose T is a compact torus, V is an finite-dimensional complex vector space, and $\pi: T \to GL(V)$ is a representation of T ($=_{\text{def}}$ Lie group homomorphism). For each character $\xi: T \to$ circle, define the ξ-weight space of V to be (subspace of V)

UNDERSTAND reps of basic Lie groups COMPACT TORI

$\xi \in X^*(T)$

$$V_\xi = \{v \in V \mid \pi(t)v = \xi(t)\cdot v \ \ (\text{all } t \in T)\}.$$

"Eigenspace decomp for T repns"

Then V is the direct sum of its weight spaces. In particular, there are a basis $(v_1,\ldots,v_N)$ of V and weights $(\xi_1,\ldots,\xi_N)$ in $X^*(T)$ so that $\pi(t)$ is diagonal in this basis, with diagonal entries $\xi_j(t)$. We have

$$\dim V_\xi = \#\{ j \mid \xi_j = \xi \} =_{\text{def}} \text{multiplicity of } \xi \text{ in } V = m(\xi,V), \qquad \Sigma_\xi\, m(\xi,V) = \dim V$$

analog: A diag on V, dim V = $\sum_\lambda$ mult of eigval λ

Proof. Linear algebra is always easier in the presence of an inner product, which allows us to make subspaces W into direct sum decompositions $V = W \oplus W^\perp$. In the presence of a group, we need an inner product preserved by the group, and this is not quite so easy to get. We will start with any inner product and then to "average over the group" to get a better one. Here's how.

WED.

Like any torus, T is isomorphic to the quotient of its Lie algebra by its cocharacter lattice, and therefore to $\mathbf{R}^n/\mathbf{Z}^n = [S^1]^n = \{ (e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \mid 0 \le \theta_j < 2\pi \}$. This choice of coordinates allows us to integrate continuous functions on T:

$$\int_T f(t)\,dt =\text{def} \int_0^{2\pi} \cdots \int_0^{2\pi} f(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})\, d\theta_1 \cdots d\theta_n .$$

Since integration on $\mathbf{R}^n$ is translation-invariant, $\int_T f(t\cdot t_0)\,dt = \int_T f(t)\,dt$. This integral is what we need.

Start with any positive definite inner product $\langle,\rangle_0$ on V. Define a new one by averaging over T:

$$\langle v_1, v_2\rangle =_{\text{def}} \int_T \langle \pi(t)v_1, \pi(t)v_2\rangle_0\, dt \qquad \langle \pi(t_0)v_1, \pi(t_0)v_2\rangle = \langle v_1, v_2\rangle;$$

The red formula follows by applying the blue one to the definition. Conclusion is that $\pi(t)$ is unitary.

Topic for today 4/8: representations of tori and how to use them

Wednesday, April 8, 2020 3:05 PM

1. ~~Finish proof of theorem from Monday.~~
2. Definition of the adjoint representation of any Lie group.
3. Center of a connected Lie group (topic of pset 9, due 4/15).

Remember that GOAL of course is to describe and classify compact Lie groups.

Method: study weights of the adjoint representation.

"easier" to manipulate than lots of sin / cos

How does unitary help?

Sunday, April 5, 2020 8:17 PM

We're proving

Simultaneous diag of ops $\pi(t)$

$$\underbrace{\pi(t) = \begin{pmatrix} \xi_1(t) & & 0 \\ & \ddots & \\ 0 & & \xi_n(t) \end{pmatrix}, \text{ all } t \in T}_{\pi \text{ group hom} \iff \text{each } \xi_i \text{ is group hom. to scalars}}$$

THEOREM Suppose T is a compact torus, V is an finite-dimensional complex vector space, and $\pi: T \to GL(V)$ is a representation of T ($=_{\text{def}}$ Lie group homomorphism). For each character $\xi: T \to$ circle, define the ξ-weight space of V to be

$$V_\xi = \{v \in V \mid \pi(t)v = \xi(t)\cdot v \ \ (\text{all } t \in T)\}.$$

Then V is the direct sum of its weight spaces. In particular, there are a basis $(v_1,\ldots,v_N)$ of V and weights $(\xi_1,\ldots,\xi_N)$ in $X^*(T)$ so that $\pi(t)$ is diagonal in this basis, with diagonal entries $\xi_j(t)$. We have

$$\dim V_\xi = \#\{ j \mid \xi_j = \xi \} =_{\text{def}} \text{multiplicity of } \xi \text{ in } V = m(\xi,V), \qquad \Sigma_\xi\, m(\xi,V) = \dim V$$

So far we proved that V has an inner product $\langle,\rangle$ making π unitary: $\pi(t)^{-1} =$ complex conj of $\pi(t)^t$. Since T is abelian, all the operators $\pi(t)$ commute. By linear algebra, commuting unitary operators can be simultaneously diagonalized; that is, there are an orthonormal basis $(v_1,\ldots,v_N)$ of V and functions $(\xi_1,\ldots,\xi_N)$ on T so that $\pi(t)$ is diagonal in this basis, with diagonal entries $\xi_j(t)$. A diagonal matrix is unitary if and only if the diagonal entries have absolute value 1; so each ξ_j takes values in S^1. That π is a smooth group homomorphism means that all the $\xi_j: T \to S^1$ are smooth group homomorphisms; that is, $\xi_j \in X^*(T)$. Everything else in the theorem is easy linear algebra. **QED**.

Example of weights of a representation

Wednesday, April 8, 2020 3:40 PM

$T = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{r(\theta)}$ $V_{\mathbb{R}} = \mathbb{R}^3$ $\pi_{\mathbb{R}}(r(\theta)) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

Check: $\pi(r(\theta_1)\, r(\theta_2)) = \pi(r(\theta_1+\theta_2))$

$\underline{\pi \text{ REAL REP}}$

Matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ <u>NOT</u> diagonalizable/$\mathbb{R}$: eigenvals $\underline{1, i, -i}$ ← This is why we work over $\mathbb{C}$

COMPLEXIFY: $V =_{def} V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^3$ get <u>complex rep</u> $\pi_{\mathbb{C}}$ by same formula

Over $\mathbb{C}$, <u>can</u> diagonalize

~~Simulta~~ Eigenvectors for $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ $\boxed{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}}$ orthogonal basis of simultaneous eigvecs for $\pi(r(\theta))$

$\pi(r(\theta)) \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = \begin{pmatrix} \cos\theta + i\sin\theta \\ 0 \\ -\sin\theta + i\cos\theta \end{pmatrix} = \boxed{\begin{matrix} e^{i\theta} \\ 0 \\ ie^{i\theta} \end{matrix}} = e^{i\theta} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$

Weights of V: 3 chars $\boxed{\begin{matrix} r(\theta) \to e^{i\theta} \\ r(\theta) \to 1 \\ r(\theta) \to e^{-i\theta} \end{matrix}}$ all in $X^*(T)$

$X^*(T) \cong \underset{m}{\underset{\psi}{\mathbb{Z}}}$ $\boxed{\xi_m(r(\theta)) = e^{im\theta}}$

$V = V_{\xi_1} \oplus V_{\xi_0} \oplus V_{\xi_{-1}}$

Question: why are two definitions of cocharacters the same?

Wednesday, April 8, 2020 3:25 PM

T compact torus

1st def $X_*(T) =_{def} \ker(\exp)$

2nd def $X'_*(T) = \mathrm{Hom}(S^1, T)$

Why same?

If $l \in X'_*(T)$

$\boxed{\begin{array}{l} l : S^1 \to T \\ dl : \mathbb{R} \to \mathfrak{t} \end{array}}$ "functoriality of Lie algebra"

[Lin map $\mathbb{R} \to V$] $= v \in V$. So $\mathrm{Hom}_{Liealg}(\mathbb{R}, \mathfrak{t}) \cong \mathfrak{t}$

$\varphi \longmapsto \varphi(1)$

$$\begin{tikzcd} S^1 \arrow[rr, dashed, "\Phi"] & & T \\ \mathbb{R} \arrow[u, "\exp"] \arrow[rr, "\varphi"] & & \mathfrak{t} \arrow[u, "\exp"'] \end{tikzcd}$$

$e^{2\pi i t}$ $\uparrow$ t

Which of these exponentiate?

$\boxed{\begin{array}{l}\underline{\text{GIVEN}}\ \varphi \text{ Lie alg map; when} \\ \underline{\text{does it come from Lie group map?}} \\ \underline{\text{ANSWER}}: \text{when } \varphi(\ker \exp \text{ in } \mathbb{R}) \\ \qquad\qquad\qquad\qquad\quad \parallel \\ \text{contained in } \ker(\exp \text{ in } \mathfrak{t}) \quad \mathbb{Z} \end{array}}$

That is, $\boxed{\varphi(1) \in X_*(T)}$

How to tell when a Lie algebra homomorphism exponentiates to Lie group homomorphism

Wednesday, April 8, 2020 3:31 PM

Suppose H is a connected Lie group with universal cover $H^\sim$:

$$1 \to \pi_1(H) \to H^\sim \to H \to 1.$$

Here $\pi_1(H)$ is a discrete subgroup of the center $Z(H^\sim)$.

Theorem. Suppose $\phi: \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism. Write $H^\sim$ for the universal covering group of H. By the big theorem from last week, we automatically get a Lie group homomorphism $\Phi^\sim: H^\sim \to G$. This homomorphism

descends to $\Phi^\sim: H^\sim \to G \iff \Phi^\sim(\pi_1(H)) = \{e\} \subseteq G$

Adjoint representation

Wednesday, April 8, 2020 3:50 PM

$\mathfrak{g}$ Lie algebra $\rightsquigarrow$ Lie group $\mathrm{Aut}(\mathfrak{g})$

$= \{ T : \mathfrak{g} \to \mathfrak{g}$ linear, invertible, $T([X,Y]) = [TX, TY] \}$

Clear: closed subgroup of $GL(\mathfrak{g})$ $\leftarrow$ $n \times n$ matrices, inv $n = \dim \mathfrak{g}$

$\boxed{\text{General thm (never proved) Closed subgp of Lie group is a Lie group}}$

BIG idea: 20th century big idea - always understand not just things, but MAPS

Not just groups, group HOMOMORPHISMS

Not just classification of things, but AUTOMORPHISM

Return on FRIDAY

Question: is exp:$\mathfrak{g} \to G$ onto?

Wednesday, April 8, 2020 3:58 PM

$\underline{SL(2,\mathbb{R})}$ 2x2 real, det = 1

exp: $\underline{sl(2,\mathbb{R})} \to SL(2,\mathbb{R})$ 2x2 trace 0

$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \notin$ image(exp)

Question asked for a simply connected G with exp not onto. Same matrix works in $SL(2,\mathbb{C})$: exp(diagonalizable) must be diagonalizable, so if $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = \exp(X)$, then X $\underline{\text{cannot}}$ be diagonalizable. Also X must have eigenvalues = odd mults of $i\pi$. Since X in $sl(2,\mathbb{C})$, $tr(X)=0$; so eigenvalues must be $mi\pi$, and must be $-mi\pi$ for some odd m. So X has distinct eigvalues $\Longrightarrow$ diagonalizable $\Longleftarrow$

Roots for a compact Lie group K

Friday, April 10, 2020 11:09 AM

1. Automorphisms of a Lie group
2. Adjoint representation of a Lie group
3. Definition of maximal torus, examples
4. Definition of roots, examples

We're aiming to describe compact connected Lie groups in completely combinatorial way. So far succeeded in the abelian case:

$$\text{compact connected abelian } T \quad \leftrightarrow \quad \text{lattice } X^*(T) \simeq \mathbf{Z}^n$$

$$\text{Lie group maps } T \to T' \quad \leftrightarrow \quad \text{lattice maps } X^*(T') \to X^*(T)$$

$$\leftrightarrow \quad n \times n' \text{ integer matrices}$$

So compact connected abelian Lie groups are classified by nonnegative integers, and maps are integer matrices.

Just like vector spaces and linear algebra.

Today: start the push toward nonabelian compact groups.

Lie group automorphisms

Friday, April 10, 2020 11:27 AM

Aut of group G: $\underset{\text{smooth}}{\text{isom}}$ $\boxed{\gamma: G \to G}$

G Lie group with Lie algebra $\mathfrak{g}$.

Aut(G) = group of smooth automorphisms γ: G → G.

Differentials of automorphisms defines natural homomorphism

$$d: \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g}), \quad \gamma \mapsto d\gamma; \qquad \mathrm{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g}).$$

The kernel of d consists of automorphisms trivial on the identity component G_e, and may therefore be regarded as a subgroup of the discrete group $\underset{\pi_0(G)}{\mathrm{Aut}(G/G_e)}$. ← "group of conn. comps of G"

Proposition. Suppose B is a finite-dimensional algebra over **R** (real vector space equipped with bilinear map $*$ from B x B to B). Then Aut(B) is a closed Lie subgroup of GL(B), with Lie algebra the vector space of derivations of B:

$$\mathrm{Der}(B) = \{ D \in \mathrm{End}(B) \mid \underbrace{D(b * b') = (Db) * b' + b * (Db')}_{} \} \quad \text{Leibnitz rule}$$

The map d can fail to be surjective for two reasons. First (if G_e is not simply connected) some Lie algebra automorphisms of $\mathfrak{g}$ may fail to exponentiate to automorphisms of G_e. Second (if G is not connected) some automorphisms of G_e may fail to extend to G. The conclusion is that there is a short exact sequence

$$\cancel{1 \to (\text{subgroup of } \mathrm{Aut}(G/G_e)) \to \mathrm{Aut}(G) \to (\text{subgroup of } \mathrm{Aut}(\mathfrak{g}))} \to 1 \qquad \text{"INNER"}$$

Easy smooth automorphisms are inner automorphisms Ad(g)(x) = gxg^{-1}. Image is Int(G):

Ad

$$1 \to Z(G) \to G \to \mathrm{Int}(G) \to 1, \qquad 1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1.$$

Adjoint representation: reprise

Friday, April 10, 2020 12:31 PM

We defined for any Lie group G

$\mathrm{Ad}: G \to \mathrm{Int}(G) \subset \mathrm{Aut}(G)$ $\qquad d: \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$;

also write for the composition

Lie group hom. Lie alg hom.

$\mathrm{Ad}: G \to \mathrm{Aut}(\mathfrak{g})$, $\qquad d\mathrm{Ad} =_{\mathrm{def}} \mathrm{ad}: \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ $\qquad \mathrm{ad}(X)(Y) = [X,Y]$.

Lie alg aut def $\qquad$ Lie alg deriv def

$\mathrm{Ad}(g)([X,Y]) = [\mathrm{Ad}(g)(X), \mathrm{Ad}(g)(Y)]$ $\qquad \mathrm{ad}(Z)([X,Y]) = [Z,[X,Y]] = [[Z,X],Y] + [X,[Z,Y]]$

FACT

$\mathrm{Ad}(Z) = [Z, \cdot]$

$\mathrm{Ad}(Z)(X) = [Z,X]$

If $H \subset G$ is a Lie subgroup, can restrict to H:

$\mathrm{Ad}_G: H \to \mathrm{Aut}(\mathfrak{g})$, $\qquad d\mathrm{Ad}_G =_{\mathrm{def}} \mathrm{ad}_G: \mathfrak{h} \to \mathrm{Der}(\mathfrak{g})$.

Ad_G is a real representation of H on the real vector space $\mathfrak{g}$.

COMPLEXIFIES to complex representation

$\mathrm{Ad}_{G,\mathbb{C}}: H \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$.

General groups

Maximal tori

Friday, April 10, 2020 11:26 AM

weak notion of maximal

Suppose K is a compact Lie group. A maximal torus in K is a compact torus T so that whenever $T \subset T'$, with T' a compact torus, then T=T'. Always EXIST: $\{e\} \leftarrow$ 0-dim'l torus

One of maximal dim, it's maximal.

A compact torus $T \subset K$ is maximal if and only if $Z\_K(T)_e = T$. Equivalently,

$$\{ X \in \mathfrak{k} \mid Ad(t)(X) = X, \text{ all } t \in T \} = \mathfrak{t}.$$

$\leftarrow$ Lie alg of $Z_K(T)$. Always $\exists\ \mathfrak{t}$

Ad_K is a representation of T on $\mathfrak{k}$, so the complexification has a weight space decomp

complexified Lie alg $\rightarrow$ $\mathfrak{k}_{\mathbf{C}} = \bigoplus_{\xi \text{ in } X^*(T)} \mathfrak{k}_{\mathbf{C},\xi}$ T wt spaces $\qquad \mathfrak{k}_{\mathbf{C},\xi} = \{ X \in \mathfrak{k}_{\mathbf{C}} \mid Ad(t)(X) = \xi(t)\cdot X \text{ all } t \in T \}$ condition for T maximal

If T is maximal, zero weight space is the complexified Lie algebra of T: $\mathfrak{k}_{\mathbf{C},0} = \mathfrak{t}_{\mathbf{C}}$. Also

$$[\mathfrak{k}_{\mathbf{C},\alpha}, \mathfrak{k}_{\mathbf{C},\beta}] \subset \mathfrak{k}_{\mathbf{C},\alpha+\beta}$$

Example: K = U(2), 2 x 2 unitary matrices, $\quad \mathfrak{k} = \mathfrak{u}(2)$, 2 x 2 skew-hermitian matrices. One maximal torus is T=U(1) x U(1) diagonal unitary matrices. Because every 2 x 2 complex matrix Z can be written uniquely as Z=A+iB with A and B skew-hermitian,

$$\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}.$$

$\mathfrak{k}_{\mathbf{C}} = \mathfrak{gl}(2,\mathbf{C})$ 2 x 2 complex matrices, T

=

$$Ad(t)(Z) = tZt^{-1}, \quad Ad(t)(e_{pq}) = e^{i(\theta p - \theta q)}$$

Writing down U(2)

Friday, April 10, 2020 1:46 PM

$K = U(2) = 2\times 2$ cplx k, $\boxed{\bar{k}^t = k^{-1}}$

$T = \left\{ \begin{pmatrix} e^{i\theta_1} & \\ & e^{i\theta_2} \end{pmatrix} \right\}$

$\mathfrak{u}(2) = 2\times 2$ cplx A, $\boxed{\bar{A}^t = -A}$

$A = \begin{pmatrix} ix & z \\ -\bar{z} & iy \end{pmatrix}$ $x, y \in \mathbb{R}$, $z \in \mathbb{C}$

$\mathfrak{k}_{\mathbb{C}} \cong 2\times 2$ complex

$\boxed{\mathrm{Ad}(t)(Z) = tZt^{-1}}$ ← true for any Lie group of matrices / $\mathbb{R}, \mathbb{C}, \mathbb{H}$

Compute WEIGHTS of $\mathrm{Ad}(T)$:

$\mathrm{Ad}(t)(e_{pq}) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} e_{pq} \begin{pmatrix} e^{-i\theta_1} & \\ & e^{-i\theta_2} \end{pmatrix}$

mult row p by $e^{i\theta_p}$

mults col q by $e^{-i\theta_q}$

LEFT matrix mult ⟷ ROW operations

↑ "matrix unit"

1 in pth row, qth column

zeros elsewhere

$$\boxed{\mathrm{Ad}(t)(e_{pq}) = e^{i(\theta_p - \theta_q)} e_{pq}}$$

Roots for U(2)

Friday, April 10, 2020 3:44 PM

General fact: $\overset{n\times n}{t}$ has diag entries $z_1 \dots z_n$

$$\boxed{Ad(t)(e_{pq}) = z_p \cdot z_q^{-1} \cdot e_{pq}}$$

U(2) case: $X^*(U(1)\times U(1)) = \mathbb{Z}^2 = \{(\cancel{m},m)\}$

$= \{(\mu_1, \mu_2)\}$

$$\boxed{\mu\left(t = \begin{pmatrix} e^{i\theta_1} & \\ & e^{i\theta_2}\end{pmatrix}\right) = e^{i(\mu_1\theta_1 + \mu_2\theta_2)}}$$

$\uparrow$ in $X^*(T)$

e_{11} = weight vector (0,0)
e_{12} = weight vector (1,−1)
e_{21} = weight vector (−1,1)
e_{22} = weight vector (0,0)

basis of 2×2 matrices of weight vecs for T

0 weight space = Span$(e_{11}, e_{22}) = \left\{\begin{pmatrix} z_1 & 0 \\ 0 & z_2\end{pmatrix} \middle| z_i \in \mathbb{C}\right\}$

$= t_{\mathbb{C}}$ complexified Lie alg (T)

Definition of roots

Friday, April 10, 2020 3:49 PM

General compact Lie K: pick T max torus

$$\mathfrak{k}_{\mathbb{C}} = \bigoplus_{\xi \in X^*(T)} \mathfrak{k}_{\mathbb{C},\xi} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\xi \in X^*(T)\setminus\{0\}} \mathfrak{k}_{\mathbb{C},\xi}$$

zero weight space

Def $\Delta(K,T)$ = roots of T in K $\subset X^*(T)\setminus\{0\}$ are nonzero weights of T on Ad_K on $\mathfrak{k}_{\mathbb{C}}$

finite set of nonzero elements of lattice $X^*(T)$

EX $K = U(2)$, T = diagonal $= U(1)\times U(1)$ $\boxed{X^*(T) = \mathbb{Z}^2}$

$$\Delta(K,T) = \{(1,-1), (-1,1)\}$$

← "combinatorial description of $U(2)$"

See how to use $\Delta(K,T)$ to recover K

Root SU(2)

Monday, April 13, 2020 8:14 AM

Setting: K compact Lie group, T maximal torus in K, $X^*(T)$ lattice of characters

$\mathfrak{k}$ = Lie(K), $\mathfrak{k}_{\mathbf{C}}$ = complexification. Defined roots of T in K $\Delta(K,T)$ ← NONZERO chars

VECTOR SPACE

$\mathfrak{k}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} + \bigoplus_{\alpha \text{ in } \Delta(K,T)} \mathfrak{k}_{\mathbf{C},\alpha}$ ← zero weight space (on $\mathfrak{t}_{\mathbf{C}}$) root space decomposition

$\mathfrak{k}_{\mathbf{C},\alpha} = \{ Z \in \mathfrak{k}_{\mathbf{C}} \mid Ad(t)Z = \alpha(t) \cdot Z \text{ all } t \in T \}$

Roots control how nonabelian K is.

Topic today: structure of subgroup generated by each root space.

Theorem. If X_α belongs to the root space $\mathfrak{k}_{\mathbf{C},\alpha}$, there is a unique homomorphism

ϕ_α: SU(2) $\to$ K, ϕ_α(diagonal) $\subseteq$ T, $[d\phi_\alpha]_{\mathbf{C}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in (\mathbb{R}^{>0}) \cdot X_\alpha$.

same as $\varphi_\alpha : \mathfrak{su}(2) \to \mathfrak{k}$ since SU(2) SIMPLY CONN

in $\mathfrak{su}(2)_{\mathbb{C}}$

POSITIVE mult of X_α

We call ϕ_α a root SU(2) homomorphism.

Conclusion is that K is built of little SU(2)s.

Analogous: vector space built from lines

better: $n \times n$ matrix "assembled" from 2×2 blocks

GAUSSIAN ELIM

General Stuff

Monday, April 13, 2020 10:04 AM

σ : use $\bar{\ }$ on $\mathbb{C}$

If $\mathfrak{g}$ is any REAL Lie algebra, complexification $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}\otimes_{\mathbf{R}}\mathbf{C} = \{ X + i\,Y \mid X, Y \text{ in } \mathfrak{g} \}$ has real structure $\sigma : \mathfrak{g}_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}$, $\sigma(X + i\,Y) = X - iY$, $\mathfrak{g}_{\mathbf{C}}{}^{\sigma} = \mathfrak{g}$.

σ is conjugate-linear [$\sigma(zX_1+X_2) = \bar{z}\sigma(X_1) + \sigma(Y_2)$] Lie algebra aut, $\sigma^2=1$.

Conversely, if σ is any conjugate-linear order 2 automorphism of complex $\mathfrak{G}$, then $\mathfrak{G}$ is isomorphic to the complexification of the real Lie algebra $\mathfrak{g} = \mathfrak{G}^{\sigma}$. Important, but won't use today.

Main idea today: real Lie algebra homomorphism $\phi : \mathfrak{h} \to \mathfrak{g}$ are the same as complex Lie algebra homomorphisms $\phi_{\mathbf{C}} : \mathfrak{h}_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}$, $\phi_{\mathbf{C}} \circ \sigma_{\mathfrak{h}} = \sigma_{\mathfrak{g}} \circ \phi_{\mathbf{C}}$.

Use this to construct Lie group homs $\phi : SU(2) \to G$ from $\phi_{\mathbf{C}} : \mathfrak{sl}(2,\mathbf{C}) \to \mathfrak{g}_{\mathbf{C}}$.

K compact, X in $\mathfrak{k} \Rightarrow$ Ad(X) is diagonalizable with purely imaginary eigenvalues.

$[\, \mathfrak{k}_{\mathbf{C},\alpha} , \mathfrak{k}_{\mathbf{C},\beta} \,] \subset \mathfrak{k}_{\mathbf{C},\alpha+\beta}$. In particular, $[\, \mathfrak{k}_{\mathbf{C},\alpha} , \mathfrak{k}_{\mathbf{C},-\alpha} \,] \subset \mathfrak{t}_{\mathbf{C}} =$ Lie(T)$_{\mathbf{C}}$.

$\mathfrak{k}_{\mathbf{C},0}$

$\sigma(\mathfrak{k}_{\mathbf{C},\alpha}) = \mathfrak{k}_{\mathbf{C},-\alpha}$

Reason: ad X takes purely imag values on $\mathfrak{t} = \mathrm{Lie}(T)$

Know this for any torus, any Lie alg of any complex rep of Torus

$\exp(tX) =$ conn. commutative subg of K

so closure is a torus S

$X \in \mathrm{Lie}(S)$

$\mathrm{Ad}_{\mathbb{C}}|_S =$ complex repn of S

SU(2)

Monday, April 13, 2020 10:54 AM

Know: Lie algebra of SU(2) is 2 x 2 skew hermitian matrices of trace 0; basis

more famous in physics REMEMBER

$T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $[T, U] = 2V$, $[V, T] = 2U$, $[U, V] = 2T$.

preserved by cyclic perm of (T, U, V)

Complexified Lie algebra of SU(2) is 2 x 2 complex matrices of trace 0; basis

eigenvalues of Ad(H) obvious!

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$.

$[H, H] = 0$

$\mathrm{Ad}(H)(X) = 2X$

The real structure σ is minus conjugate transpose, so (in this ex)

more famous for math

$H = iT$, $X = (U-iV)/2$, $Y = -(U + iV)/2$ $\sigma H = -H$, $\sigma X = -Y$, $\sigma Y = -X$.

$\boxed{SU(2) \cong S^3}$

"Generalization" of spher coords on S^2

Idea: look in $\mathfrak{k}_{\mathbf{C}}$ for elements H' and X' satisfying $[H',X'] = 2X'$, $[X',\sigma(X')] = H'$.

These will automatically give $\phi' : SU(2) \to K$, $\phi'(H) = H'$, $\phi'(X) = X'$.

You should figure out exactly why that is true; ingredients are on last two pages.

Quantum Computing: Anything in SU(2) is $\exp(\theta T)\exp(\varphi U)\exp(\psi V)$

more or less: φ, ψ, θ "coords" on SU(2)

cons in SU(2)

$\exp(\theta T) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ eigvals $e^{\pm i\theta}$

$\exp(\varphi U) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ ← eigvals $e^{\pm i\varphi}$

$\exp(\psi V) = \begin{pmatrix} \cos\psi & i\sin\psi \\ i\sin\psi & \cos\psi \end{pmatrix}$ ← eigvals $e^{\pm i\psi}$

Example of SU(n) $= n\times n$ complex g, $\bar{g}^t = g^{-1}$

Monday, April 13, 2020 12:29 PM

Inside K = SU(n) we can choose a maximal torus T = $S(U(1))^n$, diagonal matrices of det 1, diagonal entries $e^{i\theta_p}$. (This torus was called S on 4/6 lecture.) Shows

$$X^*(T) = \{ (\lambda_1, \ldots, \lambda_n) \text{ in } \mathbf{Z}^n\} / \mathbf{Z}(1, \ldots, 1).$$

in $X^*(T)$ $e_p = (0 \ldots 1 \ldots 0)$ pth coord

Complexified Lie algebra is n x n complex trace 0 matrices. Roots are $e_p - e_q$ (p not q between 1 and n). Root spaces are

$$\mathfrak{k}_{\mathbf{C}, e_p - e_q} = \mathbf{C}\cdot e_{pq} \quad = \text{matrices with at most one nonzero entry, in row p/col q.}$$

$\sigma =$ negative conj transpose $(e_{pq})^t = e_{qp}$

Choose a root vector $X_{pq} = z\cdot e_{pq}$; then

$$Y_{pq} =_{\text{def}} -\sigma(X_{pq}) = \bar{z}\cdot e_{qp},\ H_{pq} =_{\text{def}} [X_{pq}, Y_{pq}] = |z|^2\cdot(e_{pp} - e_{qq})$$

$$\begin{array}{c} p \\ \end{array}\begin{bmatrix} & z \\ & \end{bmatrix}_{q},\quad \begin{bmatrix} & \\ \bar{z} & \end{bmatrix}_{p}^{\,q}$$

$$[H_{pq}, X_{pq}] = 2|z|^2\cdot X_{pq} \qquad [H_{pq}, Y_{pq}] = -2|z|^2\cdot Y_{pq}$$

So X_{pq}, Y_{pq}, H_{pq} ALMOST satisfy bracket relations for SU(2); to correct, need to scale X_{pq} by $1/|z|$. Resulting SU(2) subgp of SU(n) acts on coordinates p,q.

$\frac{1}{|z|}X_{pq} = X'$, $\frac{1}{|z|}Y_{pq} = Y'$, $\frac{1}{|z|^2}H_{pq} = H'$

$[X', Y'] = H'$, $[H', X'] = 2X'$, $[H', Y'] = -2Y'$

$$= \frac{1}{|z|^2}\begin{bmatrix} 0 & & & \\ & |z|^2 & & \\ & & -|z|^2 & \\ & & & 0 \end{bmatrix} = H_{p,q}$$

GET $\mathfrak{su}(2) \to \mathfrak{su}(n) \leftarrow \phi_{pq}$

SU(2)s from root vectors

Monday, April 13, 2020 12:02 PM

Thm from 1st page

Theorem. If X_α belongs to the root space $\mathfrak{k}_{\mathbb{C},\alpha}$, there is a unique homomorphism

$\phi_\alpha : SU(2) \to K$, $\phi_\alpha(\text{diagonal}) \subset T$, $[d\phi_\alpha]_{\mathbb{C}}$*[Math Processing Error]*∈ $(\mathbb{R}^{>0})X_\alpha$.

Seek: X, H
$[H, X] = 2X$
$[X, -\sigma X] = H$

to get SU(2)

Proof Define $Y_\alpha = -\sigma(X_\alpha) \in \mathfrak{k}_{\mathbb{C},-\alpha}$

$X^*(T) = \text{homs } T \to S^1$

$\text{Lie}(S^1) = i\mathbb{R}$

diff of any $\alpha \in X^*(T)$ takes purely imag values on $\mathfrak{t}$

$$H_\alpha = [X_\alpha, Y_\alpha] \in \mathfrak{k}_{\mathbb{C},0} = \mathfrak{t}_{\mathbb{C}}$$
$$= [X_\alpha, -\sigma(X_\alpha)]$$
$$= \sigma[\sigma X_\alpha, -X_\alpha] = -\sigma[X_\alpha, \sigma X_\alpha] \overset{?}{=} -H_\alpha$$

purely imag in $\mathfrak{t}_{\mathbb{C}}$

CONCLUDE

$$[H_\alpha, X_\alpha] = \alpha(H_\alpha) \cdot X_\alpha$$

REAL r_α

def of "$X_\alpha \in \mathfrak{k}_{\mathbb{C},\alpha}$"

wish: $r_\alpha = 2$

CAN ARRANGE (by replace X_α by $s \cdot X_\alpha$, $s > 0$

$H_\alpha \rightsquigarrow s^2 H_\alpha \rightsquigarrow$ replace r by $s^2 r$

EITHER

$r_\alpha = 2$ or $r_\alpha = -2$ or $r_\alpha = 0$

WANT POOR AWFUL

Wednesday: POOR + AWFUL CONTRADICT K COMPACT

Root SU(2) continued

Wednesday, April 15, 2020 8:51 AM

Setting from Monday: K compact Lie, T maximal torus, $X^*(T)$ lattice of characters
$\mathfrak{k}$ = Lie(K), $\mathfrak{k}_{\mathbf{C}}$ = complexification. Defined roots of T in K $\Delta(K,T)$

decompose representation Ad of T on $\mathfrak{k}_{\mathbb{C}}$

$$\mathfrak{k}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} + \bigoplus_{\alpha \text{ in } \Delta(K,T)} \mathfrak{k}_{\mathbf{C},\alpha}$$ root space decomposition

Roots control how nonabelian K is.

Continuing topic today: structure of subgroup generated by each root space.

Theorem. If X_α belongs to the root space $\mathfrak{k}_{\mathbf{C},\alpha}$, there is a unique homomorphism

$$\phi_\alpha : SU(2) \to K, \qquad \phi_\alpha(\text{diagonal}) \subset T, \qquad [d\phi_\alpha]_{\mathbf{C}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in (\mathbb{R}^{>0}) \cdot X_\alpha.$$

$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$

called X

positive mult of X_α

We call ϕ_α a root SU(2) homomorphism.

Conclusion is that K is built of little SU(2)s.

Two goals today: finish proof, give some examples.

$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$\sigma(U + iV) = U - iV$
$U, V \in \mathfrak{su}(2)$

Strategy for finding a map from SU(2)

Wednesday, April 15, 2020 10:52 AM

Recall complexified Lie algebra of SU(2) had basis H, X, Y satisfying

$*$ $[H,X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = H$ complex conj $\sigma X = -Y$, $\sigma H = -H$

Consequence: in any complexified Lie algebra $\mathfrak{g}_C$, elements X', H' satisfying

$H' = [X',-\sigma X']$ $[H',X'] = 2X'$ ➡ Lie alg hom $\mathfrak{su}(2) \to \mathfrak{g}$ ➡ Lie gp hom $SU(2) \to G$

deduce rest of $*$ SINCE SU(2) is simply con

Idea: look in $\mathfrak{k}_C$ for elements H' and X' satisfying $[H',X'] = 2X'$, $[X',-\sigma(X')] = H'$.

What we did in the last slide ~~Wednesday~~ Mon day: given root decomposition of compact Lie group $K \supset T$, nonzero root vector X_α, found positive multiple X'_α so that

SCALE r_α

$[X'_\alpha, -\sigma(X'_\alpha)] = H'_\alpha \in i\mathfrak{t}$, $[H'_\alpha, X'_\alpha] = r_\alpha X'_\alpha$ ($r_\alpha = 2$ or 0 or -2)

If $r_\alpha = 2$, we get $\phi_\alpha :\ SU(2) \to K$ that we want. Need to rule out other two cases.

Rule out $r_\alpha = 0$

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Element X in $\mathfrak{k}_{\mathbb{C}}$ $\quad [X, -\sigma X] = H \quad [H, X] = 0$

TRY to get contradiction

$X + \sigma X \in \mathfrak{k}$ real Lie alg of $\mathfrak{k}$ (like $z + \bar{z}$ is real, any cplx z)

$[X + \sigma X, H] = 0$ (apply σ to relations above: 1st gives $\sigma H = -H$, then 2nd)

$[X + \sigma X, X - \sigma X] = 2H$

SO $\underbrace{[X + \sigma X, \bullet]}_{\text{linear map } \mathfrak{k}_{\mathbb{C}} \to \mathfrak{k}_{\mathbb{C}}}$ preserves span$(\overbrace{X - \sigma X, H}^{\text{basis of 2-diml vec space}})$ acts by $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ NILPOTENT

BUT proved: for any $A \in \mathfrak{k} = \mathrm{Lie}(\text{compact})$, $[A, \bullet]$ is DIAGONALIZABLE, purely imag eigenvalues $\Rightarrow \Leftarrow$ CONTRADICTION

in prob set, then about torus reps

Why is Ad(X) diagonalizable on $\mathfrak{k}_{\mathbf{C}}$ with imaginary eigenvalues?

Wednesday, April 15, 2020 3:27 PM

For any X in any Lie(G), the subgroup { exp(tX) | t in **R** } is connected and abelian; so its closure S is also connected and abelian. If G=K is compact, this makes S a torus. According to the theorem from class 4/8/20, this implies that in every complex representation of S (like Ad on $\mathfrak{k}_{\mathbf{C}}$) every element of Lie(S) (like X) acts diagonalizably with purely imaginary eigenvalues.

In a noncompact G, S is still connected abelian, but need not be compact. All you can say about its representations is what you learned in linear algebra for a commuting family of complex matrices: there must be a common eigenvalue, but it need not be purely imaginary, and the matrices need not be diagonalizable (even one at a time, and certainly not simultaneously.

How do we rule out r_α = -2?

Wednesday, April 15, 2020 3:32 PM

In this case we start with a nonzero root vector X in $\mathfrak{k}_{\mathbf{C},\alpha}$; we defined Y=-σX, H = [X,Y], and the assumption meant [H , X] = -2X, [H , Y] = -2Y. It's convenient (for my memory; no mathematical need) to write X' = X, Y' = -Y = σX', H' = -H; then we have

[H' , X'] = 2X', [H' , Y'] = -2Y', [X',Y'] = H', σX' = Y', σY' = X', σH' = -H'.

Proposition The Lie algebra $\mathfrak{sl}(2,\mathbf{R})$ has a basis of its complexification H, X, Y satisfying [H,X] = 2X, [H,Y] = 2Y, [X,Y] = H, σH = -H, σX = Y, σY = x.

Proof. The complexified Lie algebra is 2 x 2 complex matrices of trace 0; the complex conjugation map σ is complex conjugation of matrices. Here are the basis elements: H = $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ X = $\frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$ Y = $\frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$. QED

Now it's clear that in the case r_α = -2 we get a subalgebra of $\mathfrak{k}$ isomorphic to $\mathfrak{sl}(2,\mathbf{R})$. This subalgebra has elements (like $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$) not diagonalizable in the adjoint representation; so this is a contradiction.

Comments from class about the end of the proof.

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Reason K can't have a subalgebra isom to $\mathfrak{sl}(2,\mathbb{R})$:

$\mathfrak{sl}(2,\mathbb{R})$ has elts

$T=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $N=\begin{pmatrix}0&1\\0&0\end{pmatrix}$

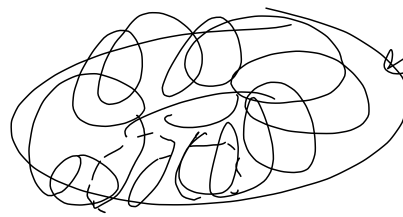
$[T,N]=2N$

so $\mathrm{ad}(T)$ has eigenvalue 2 ← real

CONTRADICTS: any elt of $X\in\mathfrak{k} = \mathrm{Lie}(K)$ has to have $\mathrm{ad}(X)$ diag, purely imaginary eig vals

[Obvious idea: $SL(2,\mathbb{R})$ is noncompact; shouldn't map to compact?

Problem: image might not be closed



dense line $\cong \mathbb{R}$ in 2-dim torus

$\underbrace{\mathbb{R}}_{\text{not compact}} \hookrightarrow \underbrace{S^1\times S^1}_{\text{compact}}$

This page is not needed after the cleaning
up of the previous page, but I'll leave it.

FRIDAY

Wednesday, April 15, 2020 3:53 PM

Look at / calculate roots for

COMPLEX $SU(n) \leftarrow$ DONE

REAL $SO(3) \leftarrow$ painful

$SO(n) \leftarrow$ built from $SO(3)$ stuff

QUATERNIONIC $Sp(n)$

Teaser

$\boxed{SO(4) \text{ is nearly } SU(2) \times SU(2)}$

4x4 real orthogonal

Computing roots

Friday, April 17, 2020 9:23 AM

Topic today is computing the roots in various compact Lie groups. Of course this begins with finding maximal tori. Aiming at three examples:

most of these are nearly simple. All but ≈ 5 simple compact Lie are on this list

O(n) = n x n real matrices preserving inner product on $\mathbf{R}^n$

U(n) = n x n complex matrices preserving inner product on $\mathbf{C}^n$

Sp(n) = n x n quaternionic matrices preserving inner product on $\mathbf{H}^n$

roots of T in K belong to $X^*(T)$

Maximal tori are built from

$U(1) = \{ (e^{i\theta}) \mid \theta \text{ real}\}$, 1 x 1 complex or quaternionic matrix $X^*(U(1)) = \mathbf{Z}$

$C(2) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ 2 x 2 real matrix $X^*(C(2)) = \mathbf{Z}$

$e_2 + e_4$ in $\mathbb{Z}^4$ means $(0, 1, 0, 1)$

$3e_2 - 2e_3$ means $(0, 3, -2, 0)$

Here are the maximal tori:

$C(2)^{[n/2]} \subset O(n)$ $U(1)^n \subset U(n)$ $U(1)^n \subset Sp(n)$

roots $\binom{[n/2]}{2} \cdot 4$ $[n/2] \cdot 2$

Here are the roots:

$\Delta(O(n), C(2)^{[n/2]}) = \{ \pm e_p \pm e_q \mid 1 \le p < q \le [n/2] \} \cup \{ \pm e_p \}$ (if n is odd).

#roots $n(n-1)$

$\Delta(U(n), U(1)^n) = \{ e_p - e_q \mid 1 \le p \ne q \le n \}$

roots $4 \cdot \binom{n}{2} + 2n$

$\Delta(Sp(n), U(1)^n) = \{ \pm e_p \pm e_q \mid 1 \le p < q \le n \} \cup \{ \pm 2e_p \}$

explanations to follow

In these formulas, $\{ e_1, e_2, \ldots, e_m \}$ is the standard basis of $\mathbf{Z}^m$

K = U(n)

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ROOTS $\leftrightarrow$ $k_{\mathbb{C}}$

T = U(1)^n = { diagonal($e^{i\theta_1}$, $e^{i\theta_2}$, ... , $e^{i\theta_n}$) }

Characters of T = X*(T) = { χ_m | m in $\mathbf{Z}^n$ } χ_m (e^{iq_1}, e^{iq_2}, ... , e^{iq_n}) = $e^{i(m_1 q_1 + \dots + m_n q_n)}$

NOT complex vec space

~~T = n x n complex A,~~ $\boxed{{}^{t}\bar{A} = -A}$ 2x2 case $\begin{pmatrix} \text{imag} & z \\ -\bar{z} & \text{imag} \end{pmatrix}$

n×n skew-Herm matrices

general: (imag on diagonal, anything above diag, $-$conj below diag)

$k_{\mathbb{C}} \cong$ all n×n complex matrices

EASY → Reason: any n×n complex Z is uniquely $\boxed{X + iY = Z}$ X, Y skew-Herm.

ROOTS for U(n): decompose conjugation action of T = U(1)^n on all n×n complex matrices

$t = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} e_{pq} \begin{pmatrix} e^{-i\theta_1} & & \\ & \ddots & \\ & & e^{-i\theta_n} \end{pmatrix} = e^{i(\theta_p - \theta_q)} e_{pq}$

$\| \ \mathrm{Ad}(t)(e_{pq})$

1 in pth row, qth column

e_{pq} = simultaneous eigenvector for all Ad(t), $t \in T$

character of $\chi_{(0 \dots \overset{p}{1} \ \overset{q}{-1} \dots 0)}$

CONCLUSION: ROOTS of T in U(n) are

$\{ e_p - e_q \mid 1 \le p \ne q \le n \} = \Delta(K, T) \subseteq X^*(T)$

introduced on 4/10

T acts TRIVIALLY on $\underbrace{\{ e_{pp} \mid 1 \le p \le n \}}_{\text{"zero root space"}} = \mathrm{Lie}(T)_{\mathbb{C}}$

equal proves T is maximal

$O(n)$ – see it's too hard

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$$T = \begin{pmatrix} \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} & & & 0 \\ & \ddots & & \\ & & \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} & \\ 0 & & & 1 \end{pmatrix}$$

$n = 2m + \varepsilon$

$\uparrow$

0 or 1

if ~~m~~ n odd

This is a torus in $O(n)$

$\dim = m = \left[\frac{n}{2}\right]$

$\mathfrak{o}(n) = n \times n$ real skew-symmetric $\boxed{{}^tA = -A}$

$\mathfrak{o}(n)_{\mathbb{C}} = n \times n$ complex skew-symmetric

basis $\boxed{e_{kl} - e_{lk} \quad 1 \le k < l \le n}$

$U(n)$ example: suggests computing

~~Let~~ $t(e_{kl} - e_{lk})t^{-1} \quad t \in T$

PAINFUL, since t isn't diagonal.

FOUR CASES

$k = 2p$	$l = 2q$	two more
$k = 2p-1$	$l = 2q$	

Get sum of four terms, sin, cos coeffs. PAIN

Go back to $U(n)$, give "natural" explanation for success of e_{pq}; apply that idea to $O(n)$

U(n) revisited

Friday, April 17, 2020 3:39 PM

Before hard problem of studying T action on matrices, do easier problem of T acting on $\mathbb{C}^n$

$$\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} e^{i\theta_1} z_1 \\ \vdots \\ e^{i\theta_n} z_n \end{pmatrix}$$

SIMULTANEOUS EIGENVECTORS for T on $\mathbb{C}^n$ are standard basis vectors $e_p = \begin{array}{c}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\end{array} \leftarrow$ pth row

$$t \cdot e_p = \chi_{(0 \ldots 1 \ldots 0)}(t) e_p$$

weights of $\mathbb{C}^n$ rep of T are $e_1, \ldots e_n$

$(1 0 \ldots 0)$
$(0 1 \ldots 0)$
$\vdots$
$(0 \ldots 0 1)$
} std basis of $\mathbb{Z}^n$

Friday, April 17, 2020 3:44 PM

T acts on $\underline{\text{dual}}$ vector space $(\mathbb{C}^n)^* \underset{\text{def}}{=} \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$

$\cong$ ROW VECTORS

$\lambda = (\lambda_1 \dots \lambda_n) \in (\mathbb{C}^n)^*$

$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$ Apply linear functional λ to z:

$\lambda_1 z_1 + \dots + \lambda_n z_n = \underset{1\times n}{\lambda} \cdot \underset{n\times 1}{z}$ $\underline{\text{matrix mult}}$

Action of $U(n)$ on $(\mathbb{C}^n)^*$ is

$$\boxed{\underset{\text{action}}{u \cdot \lambda} = \underbrace{\lambda}_{1\times n} \underbrace{u^{-1}}_{n\times n} = 1\times n}$$

Need inverse to make $u_1 \cdot (u_2 \cdot \lambda) = (u_1 u_2) \cdot \lambda$

Natural basis of $(\mathbb{C}^n)^* = (f_1 \dots f_n)$

$f_q = (0 \dots \underset{q^{th}\text{ place}}{1} \dots 0)$ $\boxed{t \cdot f_q = e^{-i\theta_q} f_q}$

f_q is a weight vector, weight $-e_q = (0 \dots \underset{q^{th}\text{ place.}}{-1} \dots 0)$

Tensor products:

Friday, April 17, 2020 3:49 PM

Fact if V, W fin. dim'l vector spaces

$$\mathrm{Hom}(V, W) \cong W \otimes V^*$$

$$\underset{\substack{w\otimes\lambda \\ \shortparallel \\ \boxed{w \cdot \lambda(v)}}}{T}(v) \longleftarrow w \otimes \lambda \qquad w \in W, \quad \lambda : V \to \text{field}$$

← think of W as ~~right~~ vector space (Makes quaternion case easier

---

$\mathbb{C}^n, \mathbb{C}^n$ V, W basis $e_1 .. e_n$ for $\mathbb{C}^n$, $f_1 .. f_n$ for $(\mathbb{C}^n)^*$

$$(e_p \otimes f_q)(v \in \mathbb{C}^n) = (q^{th} \text{ coord of } v) \cdot e_p$$

$$\boxed{e_p \otimes f_q = \text{matrix } e_{pq}}$$

$$\begin{array}{cc} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{row } p \\ \quad\quad\text{col } q & \end{array}$$

$$t \cdot e_{pq} = t \cdot (e_p \otimes f_q) = \chi_{e_p}(t) e_p \otimes \chi_{-e_q}(t) \cdot f_q$$
$$= t e_p \otimes t \cdot f_q$$

$$\boxed{t \cdot e_{pq} = \chi_{e_p - e_q}(t)(e_{pq})}$$

pull scalars past $\otimes$

No class MONDAY
WEDNESDAY: diagonal $T \subset O(n)$ on $\mathbb{C}^n, (\mathbb{C}^n)^*$
ROOTS

Root data and compact groups

Wednesday, April 22, 2020 2:58 PM

1. First goal today is to finish calculating the roots of O(n), talk a bit about Sp(n).
2. Define root data.
3. State relation between root data and compact groups.

O(n) roots

Wednesday, April 22, 2020 3:00 PM

Recall n = 2m + ε with ε = 0 or 1

$T = SO(2)^m$ ← $[n/2]$ block diagonal

A Describe weights of T on $\mathbb{C}^n = (\mathbb{R}^n)_{\mathbb{C}}$

CASE $m=1$

($O(n)$ = group of $n \times n$ real matrices, acts on $\mathbb{R}^n$
Theory of weights is for complex reps)

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos\theta + i\sin\theta \\ -\sin\theta + i\cos\theta \end{pmatrix} = e^{i\theta}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

Says $\mathbb{C}\cdot\begin{pmatrix} 1 \\ i \end{pmatrix} \subseteq$ weight space of $\mathbb{C}^2$ repn, character

$$\chi_m\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = e^{im\theta} \qquad m=1$$

SAME: $\mathbb{C}\begin{pmatrix} 1 \\ -i \end{pmatrix} \subseteq$ weight space for χ_{-1}

weight space decomp under $T = SO(2)$

$$\mathbb{C}^2 = \mathbb{C}^2_{(1)} \oplus \mathbb{C}^2_{(-1)}$$

$$\mathbb{C}^2_{(1)} = \mathbb{C}\cdot\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \mathbb{C}^2_{(-1)} = \mathbb{C}\cdot\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

General n

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$\mathbb{R}^{2m+\varepsilon}$; $\mathbb{C}^{2m+\varepsilon}$ basis $e_{2p-1} + i e_{2p} =_{def} a_p$ $1 \le p \le m$

$e_{2p-1} - i e_{2p} =_{def} b_p$ ← 2m basis vecs

THIS IS A BASIS OF WEIGHT VECTORS

$\varepsilon = 1$ e_{2m+1} ← ε basis vector

$t \cdot a_p = \chi_{e_p}(t)\, a_p$

$\chi_{e_p}\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & \\ -\sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_m & \sin\theta_m \\ & & & -\sin\theta_m & \cos\theta_m \end{pmatrix} = e^{i\theta_p}$

pth block

Characters of $T = SO(2)^m \cong \mathbb{Z}^m$ basis $e_1, \dots e_m$

$\chi_l(t) = e^{i l_1\theta_1 + \dots + i l_m \theta_m}$

$l \in \mathbb{Z}^m$

$e_p = (0 \dots 1 \dots 0)$ pth place

Weights of T on $\mathbb{C}^m$ are

$\underbrace{e_1, -e_1, e_2, -e_2 \dots e_m, -e_m}_{2m}, 0 \in \mathbb{Z}^m$

weight of e_{2m+1}: only n odd

$\mathrm{Lie}(O(n)) \subset n \times n$ matrices

WANT: compute weights of T on $n \times n$ matrices

$\mathrm{Hom}(V, W) \cong W \otimes V^*$

COMPUTE wts of T on $\mathbb{C}^n$

O(n) continued

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T weights on $\mathbb{C}^n$ $\{\pm e_p\}_{1\le p\le m}$, maybe 0

ALWAYS: weights of T on V^* are MINUS weights on V

dual vector space

SO weights of T on $(\mathbb{C}^n)^*$ are $\pm e_1, \dots \pm e_p$, maybe 0

Get immediately weights of T on $n \times n$ matrices $= \mathbb{C}^n \otimes (\mathbb{C}^n)^*$

ALWAYS: weights of T on $V_1 \otimes V_2$ are (weights on V_1) + (weight on V_2)

Weights on matrices are $\pm e_p \pm e_q$ $p \neq q$, maybe $\pm e_p$, 0 (n odd)

$\pm 2e_p$

Weight e_p+e_q: use $a_p \otimes b_q^*$ or $b_q \otimes a_p^*$

wt e_p ; dual basis; wt = minus wt of b_q, $-(-e_q)=e_q$

Matrix units:

$a_p \otimes b_q^*$: $(e_{2p-1}+ie_{2p}) \otimes \cancel{(e^*_{2q-1} - ie^*_{2q})}$

Corrected on next page: should have been $(e^*_{2q-1}+ie^*_{2q})/2$

$\frac{1}{2}\left[e_{2p-1,2q-1} \mp e_{2p,2q} + i(e_{2p,2q-1} \pm e_{2p-1,2q})\right]$

$n \times n$ matrix, 4 nonzero entries; weight vector for conj by T — elementary

rows $2p-1, 2p$
columns $2q-1, 2q$

$$\frac{1}{2}\begin{pmatrix} 1 & i \\ \pm i & -1 \end{pmatrix}$$

Details about O(n) on $\mathbf{C}^n$ calculation

Wednesday, April 22, 2020 5:52 PM

This material added after class to clarify/correct preceding page

Writing $n=2m+\varepsilon$, found basis of weight vectors for $\mathbf{C}^n$

$a_p = e_{2p-1} + ie_{2p}$ $\quad$ $b_p = e_{2p-1} - ie_{2p}$ $\quad$ maybe e_{2m+1}, $\quad$ weights e_p, $-e_p$, 0 in $\mathbf{Z}^m$

Need also dual basis of $[\mathbf{C}^n]^*$ a_p^* b_p^* maybe e_{2m+1}^*, $\quad$ weights $-e_p$, e_p, 0 in $\mathbf{Z}^m$

Therefore we get a basis for Hom($\mathbf{C}^n$, $\mathbf{C}^n$) with the indicated weights:

$a_p \otimes a_q^*$, $b_p \otimes a_q^*$, $a_p \otimes b_q^*$, $b_p \otimes b_q^*$, maybe $a_p \otimes e_{2m+1}^*$, $b_p \otimes e_{2m+1}^*$, $e_{2m+1} \otimes a_p^*$, $e_{2m+1} \otimes b_p^*$, $e_{2m+1} \otimes e_{2m+1}^*$

$e_p - e_q$, $-e_p - e_q$ $\quad$ $e_p + e_q$, $-e_p + e_q$ $\quad$, $\quad$ e_p $\quad$, $\quad$ $-e_p$ $\quad$, $\quad$ $-e_p$ $\quad$, $\quad$ e_p $\quad$, $\quad$ 0

In order to see which of these weights appear in $\mathfrak{so}(n)$ = Lie(O(n)) = skew-symmetric matrices,
we need to write the new basis in terms of the matrix basis $e_{ij} = e_i \otimes f_j$ discussed 4/17.

a_p^* defined by $a_p^*(a_q) = \delta_{pq}$, $a_p^*(b_q) = 0$, etc., so $a_p^* = (e_{2p-1}^* - i\, e_p^*)/2$, $b_p^* = (e_{2p-1}^* + i\, e_p^*)/2$.

Since $[e_p \otimes e_q^*]^t = e_q \otimes e_p^*$, we calculate

$[\, a_p \otimes a_q^*]^t = [\, b_q \otimes b_p^*]$, etc. SO a weight basis of skew-symmetric matrices is

$a_p \otimes a_q^* - b_q \otimes a_p^*$, $a_p \otimes b_q^* - a_q \otimes b_p^*$, $b_p \otimes a_q^* - b_q \otimes a_p^*$, maybe $a_p \otimes e_{2m+1}^* - e_{2m+1} \otimes b_p^*$, $b_p \otimes e_{2m+1}^* - e_{2m+1} \otimes a_p^*$

$e_p - e_q$ $\quad$, $\quad$ $e_p + e_q$ $\quad$, $\quad$ $-e_p - e_q$ $\quad$, $\quad$ e_p $\quad$, $\quad$ $-e_p$

O(n) end

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Need basis for SKEW-SYMM $n \times n$ complex matrices of weight vectors for T

CORRECTION from previous page:

$$\frac{1}{2}\left[e_{2p-1,2q-1} \pm e_{2p,2q} + i\left(e_{2p,2q-1} \mp e_{2p-1,2q}\right)\right];$$ ADD neg-transpose

NOT skew-symm

$n = 2m+1$: $2m^2$
$n = 2m$: $2(m^2 - m)$

Theorem ROOTS of T in $O(n)$ are

$\pm e_p \pm e_q$ $\quad 1 \le p \ne q \le m \quad (n = 2m+\varepsilon)$

$\pm e_p \quad 1 \le p \le m \quad$ (if $n = 2m+1$)

Gave most details of proof

Theorem Roots of T in $Sp(n)$ are $\pm e_p \pm e_q$ $\quad 1 \le p \ne q \le n$

$\pm 2e_p \quad 1 \le p \le n$

$(S^1)^n \cong$ diag matrices, entries in unit circle in $\mathbb{C}$

NUMBER OF ROOTS $\binom{n}{2} \cdot 4 + 2n = 2n^2$

All rootspaces ONE-DIMENSIONAL, both cases

Recall $U(n)$ Theorem Roots of T in $U(n)$ are $e_p - e_q$, $1 \le p \ne q \le n$

$T \cong (S^1)^n$

Number of roots $n(n-1)$

Friday preview

Wednesday, April 22, 2020 3:52 PM

Definition of root datum

1) lattice X^*, dual lattice $X_* = \mathrm{Hom}(X^*, \mathbb{Z})$
2) finite subsets $R \subset X^*$, $R^\vee \subset X_*$

roots ← BIJECTION → coroots

3) $\langle \alpha, \alpha^\vee \rangle = 2$, all $\alpha \in R$ → define $s_\alpha : X^* \to X^*$

$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$

${}^t s_\alpha : X_* \to X_*$

4) s_α permutes R, ${}^t s_\alpha$ permutes $R^\vee$

Question: why are $\pm 2e_p$ not roots for O(n)?

Wednesday, April 22, 2020 3:42 PM

Weight vector for $2e_p$

corrections!

Weight on $n \times n$ matrices, NOT on Lie(O(n))

$$a_p \otimes a_p^* \longleftrightarrow \frac{1}{2}(e_{2p-1} + ie_{2p}) \otimes (e_{2p-1}^* \cancel{+} - ie_{2p}^*)$$

$$\frac{1}{2}\left[e_{2p-1,2p-1} + e_{2p,2p} + i\left[e_{2p,2p-1} \cancel{+} - ie_{2p-1,2p}\right]\right]$$

slightly wrong?? YES;

~~OR~~ weight space has dim 1

MATRIX IS NOT SKEW-SYMMETRIC: weight doesn't appear in Lie(O(n))

correction

$$\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Root data for compact groups

Friday, April 24, 2020 10:44 AM

What we know so far: given compact group K, choose maximal torus T.

Get dual lattices X_* = Hom(S^1,T) , X^* = Hom(T,S^1).

cocharacter lattice character lattice

lattice = finitely generated free abelian $\cong \mathbb{Z}^n$

Roots of T in K = nonzero weights of Ad(T) on $\mathfrak{k}_{\mathbb{C}}$ is

Δ(K,T) $\subset X^{**}$ finite subset.

Each root α defines ϕ_α: SU(2) $\to$ K "root SU(2)," unique up to conj by T.

diag $\longrightarrow$ T

root is a character
coroot is a cocharacter

Coroot for α is $\alpha^\vee = \phi_\alpha|_{\text{diagonal}}$, $\alpha^\vee(\exp(i\theta)) = \phi_\alpha \begin{bmatrix} exp(i\theta) & 0 \\ 0 & exp(-i\theta) \end{bmatrix}$

$\Delta^\vee$(K,T) = { $\alpha^\vee$ } $\subset X_*$ finite subset in bijection with Δ(K,T)

ROOT DATUM FOR T IN K is $\mathcal{R}$(K,T) = (X^* , Δ(K,T) , X_* , $\Delta^\vee$(K,T)) ,

COMBINATORIAL
$X^* \cong \mathbb{Z}^n$, $X_* \cong \mathbb{Z}^n$
root datum integer n ≥ 0
finite set of integer vectors of size n
(roots, coroots)

a pair of dual lattices equipped with finite subsets in bijection.

diff geom, pointset topology COMBINATORIAL

~~THEOREM. K compact connected Lie group: $\mathcal{R}$(K,T) determines K up to isomorphism.~~
~~Any quadruple subject to axioms for a reduced root datum is allowed~~; so
isomorphism classes of compact connected Lie groups are the same as isomorphism ~~classes~~ of reduced root data.

Say what root data are allowed combinatorially!

Question: is "root data" FUNCTOR?

Friday, April 24, 2020 3:26 PM

Can you relate maps of root data to maps of compact groups?

First answer: NO.

$K_1 = SO(4)$

$T_1 = SO(2) \times SO(2)$

roots $(\pm 1, \pm 1)$ (4)

coroots $(\pm 1, \pm 1)$

$K_2 = Sp(2)$ (acts on $\mathbb{H}^2$)

$T_2 = S^1 \times S^1 \cong SO(2) \times SO(2)$

roots $(\pm 1, \pm 1)$, $(\pm 2, 0)$, (0 ± 2)

$(\pm 1, \pm 1)$ $(\pm 1, 0)$, (0 ± 1)

LOOKS LIKE: $R(K_1) \subset R(K_2)$

SUGGESTS: $K_1 \xrightarrow{?} K_2$

$K_1 = SO(3) \hookrightarrow SU(3) = K_2$

real orth, det 1

cplx unitary det 1

$T_2 = S(U(1)^3) = \{(t_1, t_2, t_3) \mid t_1 t_2 t_3 = 1\}$ $\cong$ product two circles

$T_1 = SO(2)$

$T_1 \to T_2$

$t \longmapsto (t, 1, t^{-1})$

gives $X_*(T_1) \hookrightarrow X_*(T_2)$

$X^*(T_2) \twoheadrightarrow X^*(T_1)$

$m \longmapsto (m, 0, -m)$

$(\lambda_1, \lambda_2, \lambda_3) \to \lambda_1 - \lambda_3$

(sends $(1,1,1)$ to 0)

$X_*(T_1) = X^*(T_1) = \mathbb{Z}$

$X_*(T_2) = \{m \in \mathbb{Z}^3 \mid \sum m_i = 0\}$

$X^*(T_2) = \mathbb{Z}^3 / \mathbb{Z} \cdot (1,1,1)$

Root datum doesn't know whether to be covariant or contravariant

REALLY IMPORTANT TO FIX THIS!

Research problem.

Root SU(2)s for O(n)

Friday, April 24, 2020 11:35 AM

We saw that it was easy to get root $e_p - e_q$ for U(n) using matrix units e_{pq} ;
corresponding SU(2) acts just on p and q coordinates.

$n \times n$ complex g, ${}^t\bar{g} = g^{-1}$

$$\varphi_\alpha \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\text{in } SU(2)} = \begin{array}{c|c} \text{rows} & \\ \begin{array}{c} \\ p \\ \\ q \\ \\ \end{array} & \begin{pmatrix} \ddots & & & & \\ & a & & b & 0 \\ 0 & & \ddots & & \\ & c & & d & \\ & & 0 & & \ddots \end{pmatrix} \\ \text{cols} & \begin{array}{ccccc} & p & & q & \end{array} \end{array} \in U(n)$$

Not so easy to see the same thing in O(n). Here's one way. $\mathbf{C}^m \simeq \mathbf{R}^{2m}$, giving

respects length of vectors

To try at home: $\det_{\mathbb{R}}(\Phi(A)) \overset{?}{\longleftrightarrow} \det_{\mathbb{C}} A$

Φ: GL(m,C) → GL(2m,R)

$$\Phi(A)_{2p-1,2q-1} = \mathrm{Re}(A_{pq}) = \Phi(A)_{2p,2q} \quad \Phi(A)_{2p,2q-1} = \mathrm{Im}(A_{pq}) = -\,\Phi(A)_{2p-1,2q}\,.$$

The embedding of nonzero complex numbers in invertible 2 x 2 real matrices is

$$x + iy \longmapsto \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

RESPECTS MULTIPLICATION

It isn't hard to check that $\Phi(U(m)) \subset SO(2m)$, so we get in particular an inclusion

$$SU(2) \to SO(4) \leftarrow T = SO(2) \times SO(2)$$

and this is ϕ_{e1-e2}. The other ϕ_{ep-eq} arise by using other sets of four coordinates, and the $e_p + e_q$ by a judicious sprinkling of complex conjugates. Finally, when we talked about quaternions and SU(2), I described a two-to-one covering map

$$\underbrace{SU(2)}_{\text{unit quaternions}} \to \underbrace{SO(3)}_{\mathbb{R}^3 = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k \text{ "imaginary quaternions"}} ; \leftarrow \text{torus } SO(2)$$

roots $\pm e_1$

this is ϕ_{e1} .

Axioms for root data

Friday, April 24, 2020 11:35 AM

as in notes on web page roots.pdf

X^*, Y^* lattices
X_*, Y_* duals

$T: X^* \to Y^*$ lattice map

${}^tT: Y_* \to X_*$

Like transpose for matrices:
${}^t({}^tT) = T$
${}^t(ST) = {}^tT\,{}^tS$
etc.

$\mathcal{R} = (X^*, \Delta, X_*, \Delta^\vee)$ is a root datum if

numbers related to axioms for ROOT SYSTEM

RD0 X^* and X_* are dual lattices, with finite subsets Δ and $\Delta^\vee$ in bijection.

less complicated than root data.

see Humphreys Intro to Lie algs and repn theory

RD1.5 For all α in Δ, $\langle \alpha, \alpha^\vee \rangle = 2$.

true for compact group by $\mathfrak{su}(2)$ calc.

Define $s_\alpha : X^* \to X^*$, $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. Lattice automorphism, order 2.

"reflection in α"

by RD 1.5

easy calculation

Transpose is $s_{\alpha^\vee} : X_* \to X_*$, $s_{\alpha^\vee}(\ell) = \ell - \langle \alpha, \ell \rangle \alpha^\vee$.

Next Monday: explain why true for ~~root~~ (K,T)

RD3 s_α permutes Δ, $s_{\alpha^\vee}$ permutes $\Delta^\vee$, $[s_\alpha(\beta)]^\vee = s_{\alpha^\vee}(\beta^\vee)$.

The root datum is called reduced if it satisfies either of the equivalent axioms

RD4 If α and $c.\alpha$ both belong to Δ, then $c = \pm 1$.

Monday: explain why true for $\mathcal{R}(K,T)$.

RD4$^\vee$ If $\alpha^\vee$ and $c.\alpha^\vee$ both belong to $\Delta^\vee$, then $c = \pm 1$.

$W(\mathcal{R})$ acts on $\mathcal{R}$, by "automorphisms"

RD3

The Weyl group of $W(\mathcal{R})$ is the subgroup of $\mathrm{Aut}(X^*)$ generated by all s_α.

It is isomorphic (by the inverse transpose isomorphism $\mathrm{Aut}(X^*) \simeq \mathrm{Aut}(X_*)$)

to the group generated by all $s_{\alpha^\vee}$.

EXAMPLE

Friday, April 24, 2020 3:54 PM

G_2 ← 14-dim'l compact ⟷ "Octonions"

8-dim'l non-assoc algebra/$\mathbb{R}$

dual

$X^* = \{\underbrace{\lambda \in \mathbb{Z}^3 \mid \sum \lambda_i = 0}_{\text{2-dim'l}}\}$

$X_* = \{ l \in \mathbb{Q}^3 \mid$ [all $l_i \in \mathbb{Z}$ OR all $l_i \in \mathbb{Z} + 1/3$ OR all $l_i \in \mathbb{Z} + 2/3$] AND $\sum l_i = 0 \}$

2-dim'l

$\Delta = \{e_i - e_j \mid i \neq j\}$
$\cup \{2e_i - e_j - e_k\} \cup \{-2e_i + e_j + e_k\}$
$\{i,j,k\} = \{1,2,3\}$

12 roots

Try at home:
$X^* \cong \mathbb{Z}^2$
$X_* \cong \mathbb{Z}^2$

$\Delta^\vee$: $e_i - e_j$, $\frac{1}{3}(2/3, -1/3, -1/3)$, etc

12 coroots

NOT LIKE $U(n)$, $O(n)$, $Sp(n)$ exs

Axioms for root data

Sunday, April 26, 2020 10:05 PM

α roots: nonzero weights of T on $\mathfrak{k}_{\mathbb{C}}$; in X^*

defined using $SU(2)_\alpha$; in X_* $\alpha^\vee \in X_*$

$\mathcal{R} = (X^* , \Delta , X_* , \Delta^\vee)$ is a root datum if

RD0 X^* and X_* are dual lattices, with finite subsets Δ and $\Delta^\vee$ in bijection.

RD1.5 For all α in Δ, $\langle \alpha , \alpha^\vee \rangle = 2$. ← LOOK AT SU(2): $\mathrm{Ad}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}(X_\alpha) = z^2 X_\alpha$

Define $s_\alpha : X^* \to X^*$, $s_\alpha(\lambda) = \lambda - \langle \lambda , \alpha^\vee \rangle \alpha$. Lattice automorphism, order 2.

Transpose is $s_{\alpha^\vee} : X_* \to X_*$, $s_{\alpha^\vee}(\ell) = \ell - \langle \alpha , \ell \rangle \alpha^\vee$. ← main topic today

RD3 s_α permutes Δ, $s_{\alpha^\vee}$ permutes $\Delta^\vee$, $[s_\alpha(\beta)]^\vee = s_{\alpha^\vee}(\beta^\vee)$.

The root datum is called reduced if it satisfies either of the equivalent axioms

RD4 If α and $c.\alpha$ both belong to Δ, then $c = \pm 1$.

RD4$^\vee$ If $\alpha^\vee$ and $c.\alpha^\vee$ both belong to $\Delta^\vee$, then $c = \pm 1$.

TODAY: WHY DOES THE ROOT SYSTEM OF A COMPACT GROUP SATISFY THESE? Root SU(2)

Main tool: realize s_α inside $\phi_\alpha(SU(2))$ inside K.

How to spot a root SU(2)

Sunday, April 26, 2020 10:05 PM

Suppose we have maximal torus T in compact K. When is $\phi: SU(2) \to K$ a root SU(2)?

Answer: whenever PRESERVED by σ_α on next page

1. ϕ(diagonal) is a nontrivial subgroup of T, and
2. $T \subset N_K\phi(SU(2))$; that is, T normalizes the image of ϕ.

"obviously" true for φ_α constructed last week

Once these conditions are true, can define

$\mathfrak{k}_\alpha = \mathbf{C}d\phi_{\mathbf{C}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\alpha \in X^*(T)$ character by which T acts on $\mathfrak{k}_\alpha$.

(Need to *prove* that T preserves $\mathfrak{k}_\alpha$, but that's not difficult.)

Now it's easy to see that $\phi = \phi_\alpha$.

1-dim'l torus

Record here an easy fact about T: $T = (\ker\alpha)(\alpha^\vee(S^1))$. Usually not a *direct* product.

id comp is $(n-1)$-dim'l torus $\boxed{n = \dim T}$

Corresponding facts about lattices:

(kernel of α on X_*) + $\mathbf{Z}\alpha^\vee$ has finite index in X_* ; equivalently

$X_* = \mathrm{Hom}(S^1, T)$

$\alpha : X_* \to \mathbb{Z}$ $(\alpha \in X^*)$

(kernel of $\alpha^\vee$ on X^*) + $\mathbf{Z}\alpha$ has finite index in X^* .

$\boxed{\text{NONZERO}}$

Try at home: suppose $\lambda \in X^*$ and $\ell \in X_*$ are λ elements of lattice, dual lattice. TFAE:

$\lambda : X_* \to \mathbb{Z}$

(kernel of λ on X_*) + $\mathbf{Z}\ell$ has finite index in X_*

(kernel of ℓ on X^*) + $\mathbf{Z}\lambda$ has finite index in X^*

$\langle \lambda , \ell \rangle$ is a nonzero integer.

V vector space, V^* dual

$v \in V$, $\lambda \in V^*$ BOTH NONZERO

TFAE:

1) $\ker\lambda + kv = V$
2) $\ker v_{\subset V^*} + k\lambda = V^*$
3) $\lambda(v) \neq 0$

How are these three integers (what three integers?) related?

Finding s_α in $\phi_\alpha(SU(2))$

Sunday, April 26, 2020 11:03 PM

$X_* = \mathrm{Hom}(S^1, T)$ $\quad s_\alpha : X_* \to X_*$ SAME AS $s_\alpha : T \to T$

Action of s_α on X_* is $s_\alpha(\ell) = \ell - \langle \alpha, \ell\rangle \alpha^\vee$ ('integer'). On a point $\ell(z)$ of T (z in ~~$\mathbb{C}^\times$~~ S^1) action is

$z \in S^1$ $\ell(z) \in T$

$$s_\alpha(\ell(z)) = \ell(z) \cdot [\alpha^\vee(z)]^{-\langle\alpha,\ell\rangle}$$ (Integer)

$s_\alpha(\alpha^\vee(z)) = \alpha^\vee(z)\,[\alpha^\vee(z)]^{-\langle\alpha, \alpha^\vee\rangle} = \alpha^\vee(z)\,[\alpha^\vee(z)]^{-2} = \alpha^\vee(z)^{-1}$ $\quad$ s_α = inverse on $\alpha^\vee(S^1)$.

Similarly, we find s_α = identity on $(\ker\alpha)_0$.

So I know how s_α acts on all of T.

Recall that $T = (\ker\alpha)_0(\alpha^\vee(S^1))$. Since the $\pm\alpha$ root spaces generate $\mathfrak{su}(2)_{\mathbf{C}}$,

ker α ~~centralizer in T of $\phi_\alpha(SU(2))$~~ ELEMENT OF ROOT SU(2) $\alpha^\vee(S^1) = T \cap \phi_\alpha(SU(2))$ $\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ NORMALIZES diagonal matrices in SU(2), sends $\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix} \to \begin{pmatrix} z^{-1} & 0 \\ 0 & z\end{pmatrix}$

Define $\sigma_\alpha = \phi_\alpha\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$. Then

$\quad$ $\mathrm{Ad}(\sigma_\alpha)$ = identity on centralizer of $\phi_\alpha(SU(2)) \supset \ker\ \alpha$

$\quad$ $\mathrm{Ad}(\sigma_\alpha)$ = inverse on ϕ_α(diagonal in SU(2)) = $\alpha^\vee(S^1)$

Theorem. Suppose ϕ_α is a root SU(2); define σ_α as above. Then $\mathrm{Ad}(\sigma_\alpha)$ normalizes T, and acts on T by the automorphism s_α .
The automorphism $\mathrm{Ad}(\sigma_\alpha)$ of K permutes the root SU(2)s; so s_α permutes the roots.

Proof. First assertion follows from the descriptions in red earlier on the page. Second assertion follows from the characterization of root SU(2)s on the previous page. QED. ROOT DATUM

We have now established that the root datum of (K,T) ~~satisfies~~ reduced root datum RD0, RD1.5, and RD3.
Remains to check RD4: twice a root is never a root.

root $SU(2)_\beta$ conjugated by s_α to new SU(2)

2nd slide $\Longrightarrow$ new one is $SU(2)_\gamma$ also root SU(2)

$$\boxed{\gamma = s_\alpha(\beta)}$$

Representations of SU(2)

Monday, April 27, 2020 9:02 AM

All we have proven so far about the structure of compact groups is based on theorem from
April 6 describing complex representations of a torus. To prove that the root system of a compact group is reduced, and to prove that the root system determines the compact group, we need to understand the representations of SU(2).

Now example why it isn't easy that twice a root is never a root.

$K = SU(3)$ 3×3 complex unitary, det 1

$T = SO(2) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset K$ torus $X^*(T) = \mathbb{Z}$, $X_*(T) = \mathbb{Z}$

$T \subset SO(3) \subset SU(3)$

$\uparrow$ 2-1

$SU(2)$ ← shows that $1 \in X^*$ is a root of T in K

coroot is $2 \in X_*$

CONJUGATE T in $SU(3)$ to $\left\{\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} = T'$ ← easy: diagonalize $\begin{pmatrix} \cos & \sin \\ \vdots & \end{pmatrix}$

$T' \subset \begin{pmatrix} SU(2) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \subset SU(3)$ Shows that $2 \in X^*$ is a root of T' (so also T) in K

$1 \in X_*$ corr coroot

TWICE A ROOT IS A ROOT.

$\boxed{\text{RD4}}$ supposed to be true for ANY MAXIMAL T.

$SO(2) \subset T'' = \begin{pmatrix} e^{i\varphi}\cos\theta & e^{i\varphi}\sin\theta & 0 \\ -e^{i\varphi}\cos\theta & e^{i\varphi}\cos\theta & 0 \\ 0 & 0 & e^{-2i\varphi} \end{pmatrix}$ ← 2-dim'l

$SO(2)$ isn't maximal

(More obvious for $T' \subset$ diag)

Representations of SU(2)

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$su(2)_{\mathbb{C}} = sl(2, \mathbb{C})$ basis $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$[H, E] = 2E$, $[H, F] = -2F$

$[E, F] = H$

Thm In any finite-dim'l complex repn of $su(2)_{\mathbb{C}}$

[SAME THING : Lie grp hom $\pi : SU(2) \to GL(V)$]

complex vec space

a) $d\pi(H)$ has integer eigenvalues :

$$V = \bigoplus_{p \in \mathbb{Z}} V_p \qquad V_p = \{ v \in V \mid d\pi(H) v = pv \}$$

b) $d\pi(E)\, V_p \subset V_{p+2}$; EQUAL if $p \geq -1$

c) $d\pi(F)\, V_q \subset V_{q-2}$; EQUAL if $q \leq 1$

SKIP PROOF : a) follows from torus repn theorem

b), c) easy except for EQUAL

see Humphreys, "Intro to Lie algs and repn theory"

Application to $k_\alpha = \mathrm{Lie}(\ |$

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Fix root α

Theorem: all roots appear in α STRINGS

$$\boxed{\beta,\ \beta+\alpha,\ \ldots,\ \beta+m\alpha}$$

$$\beta(\alpha^\vee) = -m$$
$$(\beta+\alpha)(\alpha^\vee) = -m+2$$
$$(\beta+m\alpha)(\alpha^\vee) = m$$

$m+1$ roots

$$[X_\alpha, X_{\beta+p\alpha}] = c_{\alpha,\beta+p\alpha} X_{\beta+(p+1)\alpha}$$

$$[X_{-\alpha}, X_{\beta+p\alpha}] = d_{\alpha,\beta+p\alpha} X_{\beta+(p-1)\alpha}$$

Wednesday: look at α strings through all multiples of α. [Tuesday: clean up this slide]

DEDUCE $\pm 2\alpha$ can't be roots

Root data and SU(2)

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K compact Lie $\supset T$ maximal torus $\alpha \in \Delta(K,T)$ weight of T in adjoint

$\rightsquigarrow \varphi_\alpha : \boxed{SU(2) \to K}$ Lie group hom $\boxed{d\varphi_\alpha : su(2)_\mathbb{C} \to k_\mathbb{C}}$ Lie alg hom

$[H_\alpha, X_\alpha] = 2X_\alpha$ etc.

$X_\alpha = d\varphi_\alpha\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $H_\alpha = d\varphi_\alpha\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $Y_\alpha = d\varphi_\alpha\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

"Picture" of $k_\mathbb{C}$

Dot for each root space

$ad(X_\alpha) = [X_\alpha, \cdot]$ moves from $k_{\mathbb{C},\beta}$ to $k_{\mathbb{C},\beta+\alpha}$

γ $[X_\alpha$ $\gamma-\alpha$ $[X_\alpha$ $\gamma-2\alpha$

$k_{\mathbb{C},\alpha} \ni X_\alpha$ $[X_\alpha$

$t_\mathbb{C} = \text{Lie}(T)_\mathbb{C}$ ← ZERO weight space

$[X_\alpha$ $k_{\mathbb{C},-\alpha} \ni Y_\alpha$

$k_{\mathbb{C},\beta}$ $[X_\alpha$ $k_{\mathbb{C},\beta-\alpha}$

Concentrate on weight spaces $0, \pm\alpha, \pm 2\alpha, \pm 3\alpha \ldots$

Zero weight space $t_\mathbb{C}$
$= \mathbb{C}H_\alpha \oplus \underline{\ker(\alpha)}$ inside CENT of $SU(2)_\alpha$

SO $[X_\alpha, t_\mathbb{C}] = [X_\alpha, \mathbb{C}H_\alpha] = \mathbb{C}X_\alpha$

rest is zero

Result stated Monday last slide:

$[X_\alpha, k_{\mathbb{C},m\alpha}] \to k_{\mathbb{C},(m+1)\alpha}$ ONTO for $m \geq -\frac{1}{2}$

CONCLUDE:

$k_{\mathbb{C},\alpha} = [X_\alpha, t_\mathbb{C}] = \mathbb{C}X_\alpha$

$k_{\mathbb{C},2\alpha} = [X_\alpha, k_{\mathbb{C},\alpha}] = \mathbb{C}[X_\alpha, X_\alpha] = 0$

THEOREM Each root space has dim 1. If α is a root, 2α is not a root.

COR Root datum of (K,T) is reduced.

Structure of $\mathfrak{k}_{\mathbb{C}}$

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<u>BASIS</u> of $\mathfrak{k}_{\mathbb{C}}$ = basis of $\mathfrak{t}_{\mathbb{C}}$ $\cup$ one nice root vector X_α for each root α

$\dim T$ = rank of lattice X_* or X^*

$$\boxed{\dim K = \dim T + \# \Delta(K,T)}$$

<u>Ex</u> $K = U(n) = n \times n$ complex unitary T = diagonal $\cong U(1)^n$

$\underbrace{X^*(T) = X_*(T) \cong \mathbb{Z}^n}_{\text{basis } e_1 \dots e_n}$ $\Delta(K,T) = \underbrace{\{ e_i - e_j \mid i \neq j \}}_{n(n-1) \text{ of these}}$

$\Delta^\vee(K,T) = \{ e_i - e_j \mid i \neq j \}$

$\dim U(n) = n + n(n-1) = n^2 \leftarrow$ knew ~~$\mathfrak{k}_{\mathbb{C}}$~~ $\underbrace{\mathfrak{u}(n)_{\mathbb{C}}}_{\text{cplxified Lie alg of } U(n)} \cong n \times n$ matrices $\boxed{\dim = n^2}$

O(n)

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$O(2m)$ $T = SO(2)^m$ $X^* = X_* = \mathbb{Z}^m$

roots $\pm e_p \pm e_q$ $p \neq q$ $\leftarrow 2m(m-1) \leftarrow \frac{m(m-1)}{2}$ pairs (p,q)

4 sign choices/pair

$\dim(O(2m)) = \underbrace{m}_{T} + 2m(m-1) = \cancel{2n}\ m(2m-2) + m = m(2m-1)$

$= \binom{2m}{2} \leftarrow$ knew before: $\mathfrak{o}(2m)$ = skew-symm $2m \times 2m$ real matrices

$O(2m+1)$ $T = SO(2)^m$ $\pm e_p \pm e_q$, $\pm e_p \leftarrow 2m^2$ (2m of new kind)

coroots $\pm e_p \pm e_q$, $\pm 2e_p$

$\dim O(2m+1) = m + 2m^2 = (2m+1) \cdot m = \binom{2m+1}{2} \leftarrow$ knew that

$Sp(n)$ = $n \times n$ quaternionic, preserve form $T = (S^1)^n$ (diag complex)

Roots $\boxed{\begin{matrix} \pm e_p \pm e_q, & \pm 2e_p \\ \pm e_p \pm e_q, & \pm e_p \end{matrix}} \leftarrow 2n^2$ pairs; $\boxed{\dim Sp(n) = 2n^2 + n} \leftarrow$ knew that

coroots

Recall: compact groups ⇝ reduced root

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Thm ① If T maxl torus in connected compact Lie K, then

$$R(K,T) = \big(\underbrace{X^*}_{\text{chars of } T},\ \underbrace{\Delta(K,T)}_{\text{roots}},\ \underbrace{X_*}_{\substack{\text{cochars of } T \\ = \operatorname{Hom}(S^1, T)}},\ \underbrace{\Delta^\vee(K,T)}_{\text{coroots}}\big)$$

is a reduced root datum. ~~Any~~ isom

② Any reduced root datum arises from a compact conn Lie.

③ Any isom. of reduced root data induces an isom. of corr. compact Lie groups.

Combinatorial, computable description of all compact conn. Lie groups

Problem sets - help you think about classification of root data

EXCEPTIONAL COMPACT GROUPS

Wednesday, April 29, 2020 3:37 PM

There are exactly seven compact simple groups that are not essentially SU(n), SO(n), Sp(n)

up to finite covering groups

simple: no proper closed normal subgroups except finite center

Saw SO(n) has a 2-1 cover Spin(n), and PSO(2m) quotient
SU(n) has $\varphi(n)$ quotients by finite central
Euler φ

Sp(n) has 2 quotients by finite central
Sp(n), Sp(n)/{±I}

This is ~~ALL~~ COMPACT "SIMPLE"

allow finite central normal subgroups

SO(n) story a bit more complicated

Finite simple groups: bunch of nice ∞ families
(A_n, $n \geq 5$, $PSL(n, \mathbb{F}_q)$ most n, q
...

+ 26 EXCEPTIONS

LIE CASE IS EASIER

Reason Lie groups are easier:
"Lie" means "manifold" means approximately $\mathbb{R}^n$
means mult is SMOOTH means approximately addition on $\mathbb{R}^n$

CLASS: Lie group structure ⟶ Lie alg structure; supposed to be "infinitesimal mult"
1st order approx to group = + in Lie alg ⟵ makes Lie groups easier than finite
[,] is 2nd order approx

Exceptional root data

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7: G_2, F_4, $E_{6,ad}$, $E_{6,sc}$, $E_{7,ad}$, $E_{7,sc}$, E_8

$\underline{F_4:}$ $X^* = \{\lambda \in \mathbb{Z}^4 \mid \lambda_1+\lambda_2+\lambda_3+\lambda_4 \text{ even}\}$

roots $\underbrace{\pm 2e_p}_{8}$, $\underbrace{\pm e_p \pm e_q}_{24}$, $\underbrace{\pm e_1 \pm e_2 \pm e_3 \pm e_4}_{16}$ ← 48 roots

coroots $\pm e_p$, $\pm e_p \pm e_q$ $\pm \frac{1}{2}e_1 \pm \frac{1}{2}e_2 \pm \frac{1}{2}e_3 \pm \frac{1}{2}e_4$

$X_* = \{l \in \mathbb{Q}^4 \mid \text{all } l_i \in \mathbb{Z} \text{ or all } l_i \in \mathbb{Z} + \frac{1}{2}\}$

FRIDAY describe G_2, E.

start on classification

Simple roots and Dynkin diagrams

Friday, May 1, 2020 2:28 PM

So far we know that a compact Lie group gives rise to the combinatorial structure of a reduced root datum.

Stated (but did not prove) that isomorphic compact groups come from isomorphic root data.

Today: start toward classification of reduced root data.

Method: define an even simpler invariant of the root datum, the Dynkin diagram: finite graph in which some edges are double or triple, and those edges are directed.

First tool: notion of positive roots. Recall that roots come in pairs $(\alpha,-\alpha)$. A set of positive roots for R is $R^+ \subset R$ so that

1. Each pair $(\alpha,-\alpha)$ has exactly one positive root.
2. If α and β are positive roots and $\alpha+\beta$ is a root, then $\alpha+\beta$ is positive.

To a set of positive roots R^+ we attach simple roots $\Pi = \Pi(R^+)$:

$\Pi = \{\alpha$ in R^+ so that α is not of the form $\beta+\gamma$ for β and γ in $R^+\}$.

Dynkin diagram: graph, vertices are simple roots

Try to define "positive vectors" in $\mathbb{R}^3$: require

1) ~~at~~ exactly one of v, $-v$ positive

2) if v, w pos, so is $v+w$

[3] if $r>0$ real, v positive, then rv pos'

One way to achieve: LEXICOGRAPHICAL

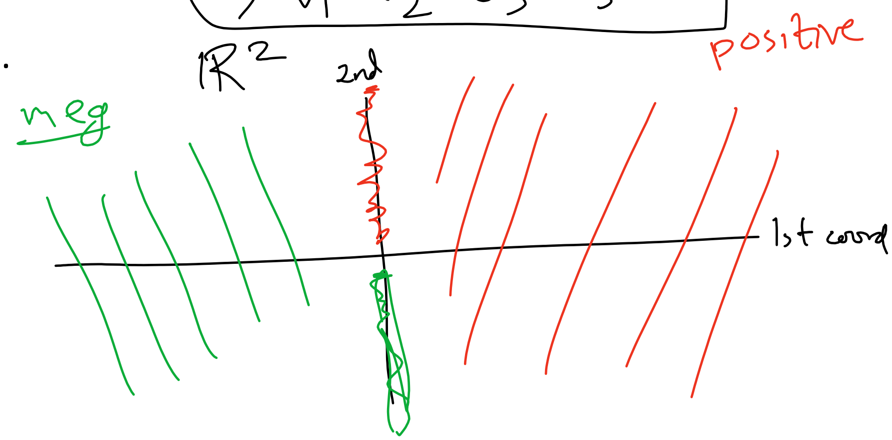
(v_1,v_2,v_3) called POSITIVE if either

a) $v_1>0$, or

b) $v_1=0$, $v_2>0$ or

c) $v_1=v_2=0$, $v_3>0$

$\mathbb{R}^2$



Why do positive root systems exist?

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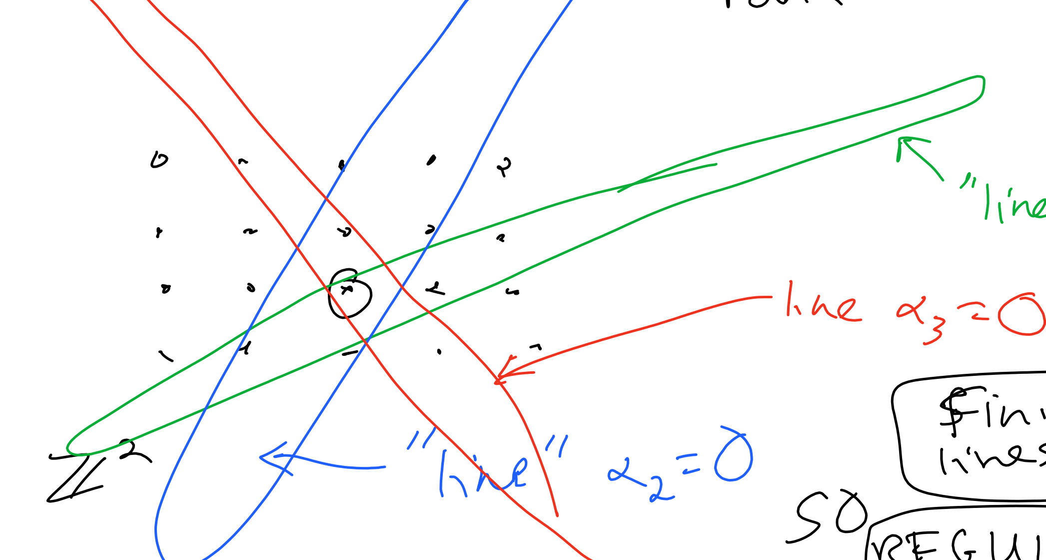
Definition. Recall $R \subset X^*$ lattice

$X_* =$ dual lattice $= \mathbb{Z}$-valued "linear functions" on X^*

Say $\ell \in X_*$ REGULAR if $\boxed{\alpha(\ell) \neq 0, \text{ all } \alpha \in R}$

NON REGULAR ℓ = finite union of sublattices, rank one less than rank of X_*



Finite number of such lines can't fill $\mathbb{Z}^2$

Same for $\mathbb{Z}^n$

SO REGULAR ℓ exist

Construct R^+: pick a regular ℓ

Define $\boxed{R^+ = \{\alpha \in R \mid \alpha(\ell) > 0\}}$

Easy: exactly one of $\{\alpha, -\alpha\}$ is in R^+

If $\alpha, \beta, \alpha+\beta$ roots and $\alpha, \beta \in R^+$ THEN $\alpha + \beta \in R^+$

Example: U(n)

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$R = \{ e_p - e_q \mid 1 \le p \ne q \le n \} \subset \mathbb{Z}^n = X^*$ $X_* = \mathbb{Z}^n$

$\ell \in \mathbb{Z}^n$ regular means $\boxed{\langle e_p - e_q, \ell \rangle = \ell_p - \ell_q}$

$\boxed{\ell_p - \ell_q \ne 0, \text{ all } p \ne q}$ Regular: all n coords of ℓ are DISTINCT

"Typical" regular ℓ: $\boxed{(n, n-1, n-2, \ldots, 3, 2, 1)}$

$R^+ = \{ e_p - e_q \mid p < q \} = \{ e_1 - e_2, e_1 - e_3, \ldots e_1 - e_n$ ($n-1$)
$e_2 - e_3, e_2 - e_4, \ldots e_2 - e_n$ ($n-2$)
$e_3 - e_4, e_3 - e_5 \ldots e_3 - e_n$
$\vdots$
$e_{n-1} - e_n$ (1)

$\# R^+ = 1 + 2 + \ldots + n-1 = \binom{n}{2} = \frac{1}{2} |R|$

SIMPLE for R^+?

$\underset{\text{pos}}{e_3 - e_6} = \underbrace{(e_3 - e_4) + (e_4 - e_6)}_{\text{both pos}}$ so $e_3 - e_6$ not simple

SIMPLE $\Pi(R^+) = \{ e_p - e_{p+1} \mid 1 \le p \le n-1 \}$

Example: Sp(n)

Friday, May 1, 2020 3:25 PM

$T = (S^1)^n \quad X^* = \mathbb{Z}^n \cong X_*$ roots $\pm 2e_p, \pm e_p \pm e_q$

$l = (l_1, \dots l_n)$ REGULAR $\Longleftrightarrow$ $\boxed{2l_p \neq 0, \ \pm l_p \pm l_q \neq 0}$

REGULAR: all l_i nonzero, distinct in absolute value

Ex $(n, n-1, \dots 1)$ still regular.

$$\boxed{R^+ = 2e_p, \quad e_p \pm e_q \quad (1 \le p < q \le n)}$$

Simple:

$$\underbrace{2e_3}_{\text{NOT SIMPLE}} = \underbrace{(e_3+e_2) + (e_3-e_2)}_{\text{pos}}$$

$p<q$

$$\underbrace{e_p+e_q}_{\text{not simple}} = \underbrace{(e_p-e_q)}_{\text{pos}} + \underbrace{2e_q}_{\text{pos}}$$

$$\Pi(R^+) = \{e_p - e_{p+1} \mid 1 \le p \le n-1\} \cup \{2e_n\}$$

Try at home: find exs of R^+, $\Pi(R^+)$ for $SO(2m)$, $SO(2m+1)$

What do simple roots tell you?

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Homework If α, β are two roots, $\alpha \neq \pm\beta$, then

a) $\langle \alpha, \beta^\vee \rangle$ and $\langle \beta, \alpha^\vee \rangle$ are either

0) both zero

$1)_\pm$ one is ± 1, other ± 1 (same sign)

$2)_\pm$ one is ± 1, other ± 2 (same sign)

$3)_\pm$ one is ± 1, other ± 3 (same sign)

proved as Prop in solns to PSET 11??

LEMMA If α, β are simple in R^+, then only possibilities are 0), $1)_-$, $2)_-$, $3)_-$

Sketch of proof Suppose we had $1)_+$. ~~Means~~ (Want CONTRADICTION)

$\langle \alpha, \beta^\vee \rangle = +1 \quad \langle \beta, \alpha^\vee \rangle = +1$

(HW) $\alpha - \beta$ is a root. If negative, then $\beta - \alpha$ pos

$\beta = \underbrace{(\beta - \alpha) + \alpha}_{\text{pos}}$ SO β NOT SIMPLE

ETC. QED

Definition of Dynkin diagram

Friday, May 1, 2020 3:39 PM

Pick $R^+ \to \Pi$ simple roots of R^+

Dynkin diagram is a graph, vertices $= \Pi$

edges: if $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 0$, no edge $\alpha \quad \beta$

if $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1$ single edge $\alpha - \beta$

if $\langle \alpha, \beta^\vee \rangle = -1$, $\langle \beta, \alpha^\vee \rangle = -2$ double edge $\alpha \Leftarrow \beta$

with arrow from β to α (I hope) (in roots.pdf in text)

if $\langle \alpha, \beta^\vee \rangle = -1$, $\langle \beta, \alpha^\vee \rangle = -3$ triple edge $\alpha \Lleftarrow \beta$

SAY α is "shorter" than β

(I haven't used an inner product, so terminology sounds undefined.)

Diagrams of U(n) and Sp(n)

Friday, May 1, 2020 3:48 PM

Dynkin diagram for U(n); called A_{n-1}

subscript = number of simple roots

e_1-e_2 — e_2-e_3 — e_3-e_4 — . . . . $e_{n-1}-e_n$

since

$\langle e_1-e_2, e_2-e_3\rangle = -1$

$= \langle e_2-e_3, e_1-e_2\rangle$

Sp(n) called C_n

e_1-e_2 — e_2-e_3 — . . . — $e_{n-1}-e_n$ $\Leftarrow$ $2e_n$

$(2e_n)^\vee$

$\langle e_{n-1}-e_n, e_n\rangle = -1$

$\langle 2e_n, e_{n-1}-e_n\rangle = -2$

$(e_{n-1}-e_n)^\vee$

(all) other diagrams written in roots.pdf

What's the Dynkin diagram tell you?

Friday, May 1, 2020 2:57 PM

We're going to classify Dynkin diagrams. How it helps

Prop 1) Simple roots are a $\mathbb{Z}$-basis of $\boxed{\mathbb{Z}\cdot R \subset X^*}$

2) Every positive root is $\sum_{i=1}^{\ell} m_i \alpha_i$ $\{\alpha_i\} = \Pi$

$m_i \geq 0$ integers

3) $W(R)$ generated by $\{s_{\alpha_i} \mid i = 1 \dots \ell\}$

4) Get every positive root by...

a) start with $\{\alpha_1, \dots \alpha_\ell\} = \Pi$

b) at step ___ pick root β on your list, find simple α_i so $\langle \beta, \alpha_i^\vee \rangle < 0$. ADD to list $\boxed{s_{\alpha_i}(\beta) = \beta + \text{pos mult of } \alpha_i}$

~~When you can no longer find new~~ β

Eventually stop getting new roots, have all R^+

COMPUTER PROGRAM to calculate R^+ from Dynkin diagram

Next week: how to classify diags

What's the Dynkin diagram tell you?

Monday, May 4, 2020 12:48 PM

$\mathscr{R} = (R, X^*, R^\vee, X_*)$ reduced root datum

Fix regular $l \in X_*$

Define $R^+ := \{\alpha \in R \mid \alpha(l) > 0\}$

R^+ choice of positive roots, Π simple roots of R^+

Any two choices of R^+ differ by action of $W(\mathscr{R})$ ← proved in roots.pdf

ALSO text, Ch 7?

Γ = Dynkin diagram: vertices Π, edge α to β when $\langle\alpha,\beta^\vee\rangle$ not zero

Conclusion: Dynkin diagram "independent of choice of R^+"

$W = W(\mathscr{R})$ Weyl group (inside $\mathrm{Aut}(X^*) \simeq \mathrm{Aut}(X_*)$)

~~First goal today: clarify algorithm from Friday to get W and R from Γ.~~

Write any s_β as integer matrix in "basis" $\alpha_1 \ldots \alpha_l$

Just basis for sublattice of X^*.

On X_*: get formula in "basis" $\alpha_1^\vee \ldots \alpha_l^\vee$

Second goal: understand possible Dynkin diagrams in ~~simply laced case~~ to be defined.

Positive and simple roots

Monday, May 4, 2020 1:09 PM

Write $\Pi = \{ \alpha_1 , \alpha_2 , \dots , \alpha_\ell \}$ for the simple roots. (Use this notation a lot.)

Don't claim all R_r are disjoint

Define $R_1 = \Pi \subset R^+$. For $r > 1$, define more subsets of R^+ recursively:

$R_{r+1} = \{ s_\alpha(\beta) \mid \beta \text{ in } R_r , \alpha \text{ in } \Pi , \langle \beta , \alpha^\vee \rangle < 0 \}$: $\beta = \beta + m\,\alpha$ (m = 1 or 2 or 3).

Note that $s_\alpha(\beta)^\vee = s_\alpha(\beta^\vee) = \beta^\vee = \beta^\vee + m'\,\alpha^\vee$ (m' = 1 or 2 or 3).

At each stage, we have explicit formulas

$(\beta \in R_r)$

$\beta = \Sigma\, n_j \alpha_j \quad \beta^\vee = \Sigma\, n_j' \alpha_j^\vee$ (n_j, n_j' nonnegative integers)

for all β in R_r, so we can compute R_{r+1} always just using the Dynkin diagram.

Shows: define W_0 = group gen by $\{ s_{\alpha_i} \mid \alpha_i \text{ simple} \} \subset W$

1) $W_0 \cdot \Pi = R$

2) reflection in any root is in W_0

Reason: $\boxed{s_{s_\alpha(\beta)} = s_\alpha s_\beta s_\alpha^{-1}}$

so $W_0 = W$

3) Every pos root is nonnegative integer comb of simple roots.

Example of G_2

Monday, May 4, 2020 1:52 PM

$\alpha_1 \Lleftarrow \alpha_2$ $\quad R_1$

$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \qquad \lambda \in X^*$

$s_\alpha(l) = l - \langle \alpha, l \rangle \alpha^\vee \qquad l \in X_*$

(RD axiom)

$$\langle \alpha_1, \alpha_2^\vee \rangle = -1 \qquad \langle \alpha_1, \alpha_1^\vee \rangle = 2$$
$$\langle \alpha_2, \alpha_1^\vee \rangle = -3 \qquad \langle \alpha_2, \alpha_2^\vee \rangle = 2$$

R_2

$s_{\alpha_2}(\alpha_1) = \alpha_1 + \alpha_2 =_{def} \gamma_1 \qquad \langle \gamma_1, \alpha_1^\vee \rangle = -1 \qquad \langle \gamma_1, \alpha_2^\vee \rangle = +1$

$\langle \alpha_1, \alpha_1^\vee \rangle = 2$
~~$\langle \alpha_1, \alpha_2^\vee \rangle = -1$~~
$\langle \alpha_2, \alpha_1^\vee \rangle = -3$

RD axiom $\longrightarrow$ $\gamma_1^\vee = s_{\alpha_2}(\alpha_1^\vee) = \alpha_1^\vee + 3\alpha_2^\vee \qquad \langle \alpha_1, \gamma_1^\vee \rangle = -1 \qquad \langle \alpha_2, \gamma_1^\vee \rangle = +3$

highest δ_2 adj to α_2 (only)

$s_{\alpha_1}(\alpha_2) = \alpha_2 + 3\alpha_1 =_{def} \gamma_2 \qquad \langle \gamma_2, \alpha_1^\vee \rangle = +3 \qquad \langle \gamma_2, \alpha_2^\vee \rangle = -1$

$\gamma_2^\vee = s_{\alpha_1}(\alpha_2^\vee) = \alpha_2^\vee + \alpha_1^\vee \qquad \langle \alpha_1, \gamma_2^\vee \rangle = +1 \qquad \langle \alpha_2, \gamma_2^\vee \rangle = -1$

R_3

$s_{\alpha_1}(\gamma_1) = \gamma_1 + \alpha_1 = 2\alpha_1 + \alpha_2 =_{def} \delta_1 \qquad \langle \delta_1, \alpha_1^\vee \rangle = +1 \qquad \langle \delta_1, \alpha_2^\vee \rangle = 0$

$\delta_1^\vee = s_{\alpha_1}(\gamma_1^\vee) = \gamma_1^\vee + \alpha_1^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee \qquad \langle \alpha_1, \delta_1^\vee \rangle = +1 \qquad \langle \alpha_2, \delta_1^\vee \rangle = 0$

ONLY HIGHEST

$s_{\alpha_2}(\gamma_2) = \gamma_2 + \alpha_2 = 2\alpha_2 + 3\alpha_1 =_{def} \delta_2 \qquad \langle \delta_2, \alpha_1^\vee \rangle = 0 \qquad \langle \delta_2, \alpha_2^\vee \rangle = +1$

$\delta_2^\vee = s_{\alpha_2}(\gamma_2^\vee) = \gamma_2^\vee + \alpha_2^\vee = 2\alpha_2^\vee + \alpha_1^\vee \qquad \langle \alpha_1, \delta_2^\vee \rangle = 0 \qquad \langle \alpha_2, \delta_2^\vee \rangle = +1$

NO MINUS SIGNS: ALGORITHM STOPS

Lowest roots and the extended Dynkin diagram

Monday, May 4, 2020 2:41 PM

A highest root is a positive root γ such that $\gamma + \alpha$ is not a root for any simple α.

A lowest root is a negative root γ such that $\gamma - \alpha$ is not a root for any simple α.

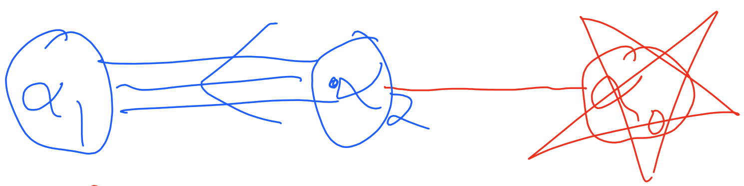
Lowest roots are negatives of highest roots. CLEAR

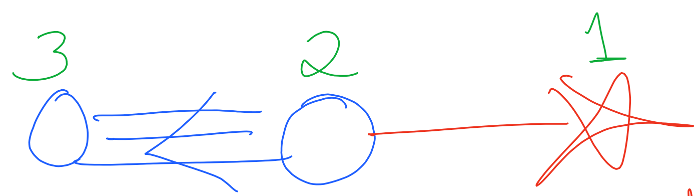
Every root appearing first in the last R_r is a highest root. (So highest/lowest exist.)

If γ lowest and α simple, then $\langle \gamma, \alpha^\vee \rangle \le 0$. (otherwise $\gamma - \alpha$ would be a root.)

Extended Dynkin diagram: ADD a new vertex for each lowest root; make extra vertex a ~~$*$~~ instead

G_2





$\alpha_0 = -\delta_2$

Extended Dynkin diagram for G_2

LABEL extended diagram:
extra vertex $\alpha_0 = -\delta = -n_1\alpha_1 \cdots - n_\ell\alpha_\ell$

G_2: $-\delta_2 = -3\alpha_1 - 2\alpha_2$

Label extra vertex 1
Label other vertex α_i by n_i from formula $-\alpha_0 = n_1\alpha_1 + \cdots + n_\ell\alpha_\ell$

$U(2m+1)$

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$X^* = \mathbb{Z}^{2m+1}$ $\quad X_* = \mathbb{Z}^{2m+1}$

$\Pi = \underbrace{\{e_1 - e_2, e_2 - e_3, \ldots e_{2m} - e_{2m+1}\}}_{2m \text{ simple}}$ " = " $\Pi^\vee$

all coeffs 1

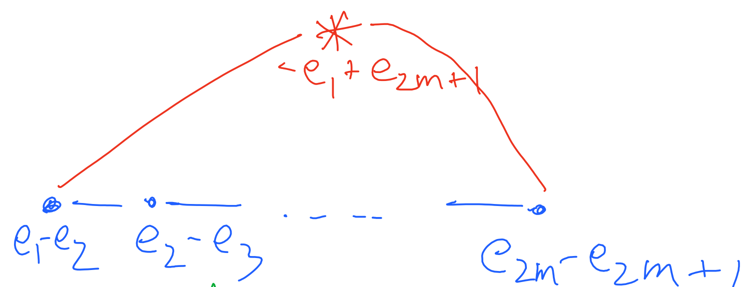
Highest root $e_1 - e_{2m+1} = (e_1 - e_2) + (e_2 - e_3) + \ldots + (e_{2m} - e_{2m+1})$

Lowest root $-e_1 + e_{2m+1}$

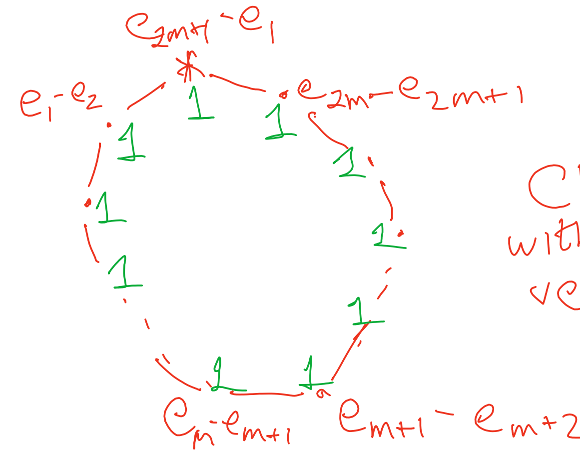
$\langle -e_1 + e_{2m+1}, e_1 - e_2 \rangle = -1$

$\langle -e_1 + e_{2m+1}, e_{2m} - e_{2m+1} \rangle = -1$

$\langle -e_1 + e_{2m+1}$, other simple coroots$\rangle = 0$



Classical tendency: draw diagram as line. Extended diagram suggests: circle picture is "more natural"



CIRCLE with ~~2m~~ 2m+1 vertices

$O(2m)$

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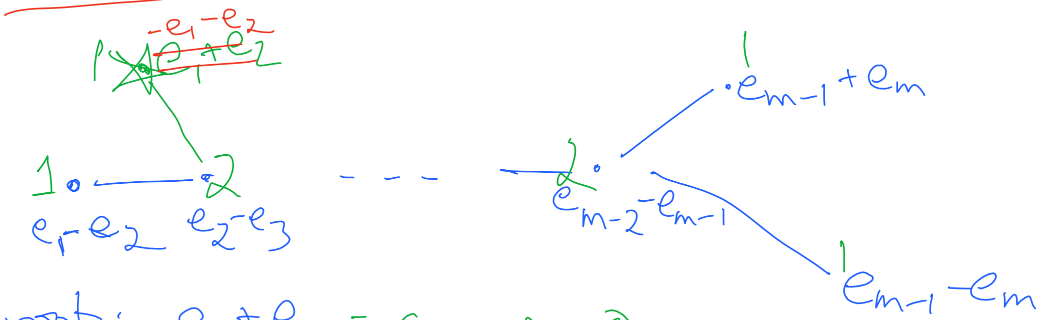
$X^* = \mathbb{Z}^m$

Π $\quad e_1-e_2,\ e_2-e_3 \ldots e_{m-1}-e_m,\ e_{m-1}+e_m$

coroots "same"

lines record when $\boxed{\langle \alpha, \alpha^{\vee} \rangle < 0}$



D_m

subgraph: $D_p \times \boxed{\text{bunch of lines}}$

$O(2p) \times (\text{bunch of } U(q))$

subgroups of $O(2m)$

highest root: $e_1+e_2 = (e_1-e_2) + 2(e_2-e_3) + 2(e_3-e_4) \ldots + 2(e_{m-2}-e_{m-1}) + (e_{m-1}+e_m) + (e_{m-1}-e_m)$

lowest: $-e_1-e_2$

adj to e_2-e_3 ONLY

(labelled) extended diagram

Compute $\langle e_1+e_2, \text{simple coroot} \rangle = 0$ $\boxed{\text{except } e_2-e_3}$

codim m

$O(2m) \supset O(2m-1)$

dim $= m(2m-1) = 2m^2-m$

dim $= 2(m-1)^2 + (m-1) = 2(m-1)(m) = 2m^2-2m$

What's special about labelled diagrams?

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Defn R is simply laced if $\boxed{\langle \alpha, \beta^\vee \rangle = 0 \text{ or } \pm 1 \text{ all } \alpha \neq \pm \beta}$

"one root length"

Said if $\langle \alpha, \beta^\vee \rangle = \pm 1$

$\langle \beta, \alpha^\vee \rangle = \pm m$

$m = 2$ or 3

Problem set roots $(1,0)$, $(0,1)$ coroots $(2,a)$ $(b,2)$

$a,b = \ldots$ very few

SAY α is "short"

"shorter than β by factor of m"

refers to squared length in inner product

(people might say "$\sqrt{2}$ or $\sqrt{3}$")

PROP In a simply laced extended diagram,

$$\boxed{\text{label}(\alpha) = \frac{1}{2} \sum (\alpha' \text{ adjacent } \cancel{\text{to}}\ \alpha)(\text{label of } \alpha')}$$

REALLY EASY: proof Wednesday

Wednesday this property determines all simply laced Γ.

Coxeter graphs

Wednesday, May 6, 2020 1:52 PM

$n_0\alpha_0 + n_1\alpha_1 + \cdots + n_\ell\alpha_\ell = 0$

Last time: given reduced root datum $\mathcal{R} = (R, X^*, R^\vee, X_*)$,

positive roots R^+ , simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$, lowest root α_0 (one for each simple factor).

Had highest root $\alpha_0 = \Sigma_p\, n_p\, \alpha_p$, $n_0 = 1$.

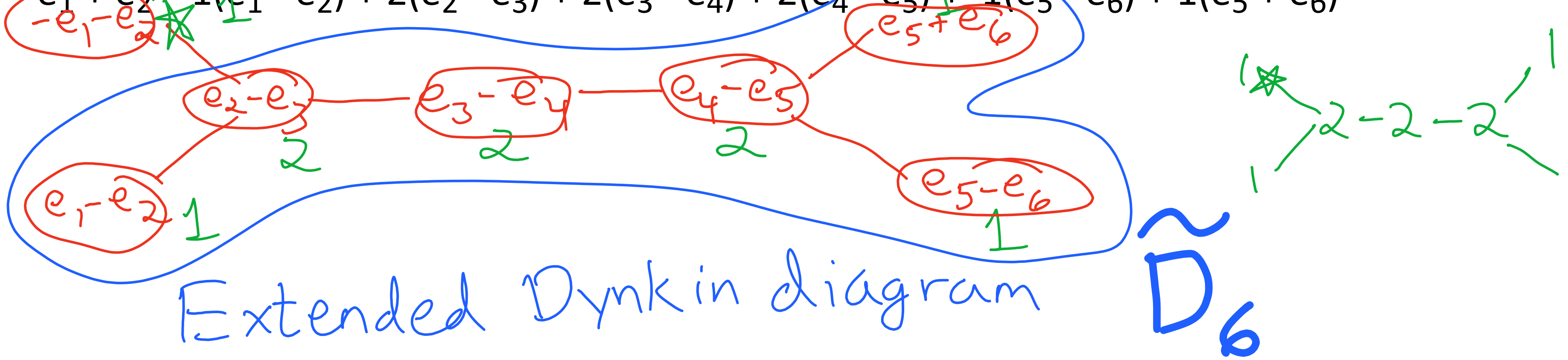
highest + simple never a root

vertices of Dynkin diagram

Example: D_6 in $\mathbf{Z}^6$ $R^+ = \{ e_p \pm e_q \mid 1 \leq p < q \leq 6 \}$,

$\Pi = \{ e_1 - e_2 ,\ e_2 - e_3 ,\ e_3 - e_4 ,\ e_4 - e_5 ,\ e_5 - e_6 ,\ e_5 + e_6 \}$ $-\alpha_0 = e_1 + e_2$

$-\alpha_0 = e_1 + e_2 = 1(e_1 - e_2) + 2(e_2 - e_3) + 2(e_3 - e_4) + 2(e_4 - e_5) + 1(e_5 - e_6) + 1(e_5 + e_6)$



Extended Dynkin diagram...

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...is Dynkin diagram (vertices simple roots α_p) with one extra vertex α_0 for lowest root.

Mark extra vertex ✻. Label each vertex with n_p, $n_0 = 1$.

Γ = Dynkin diagram

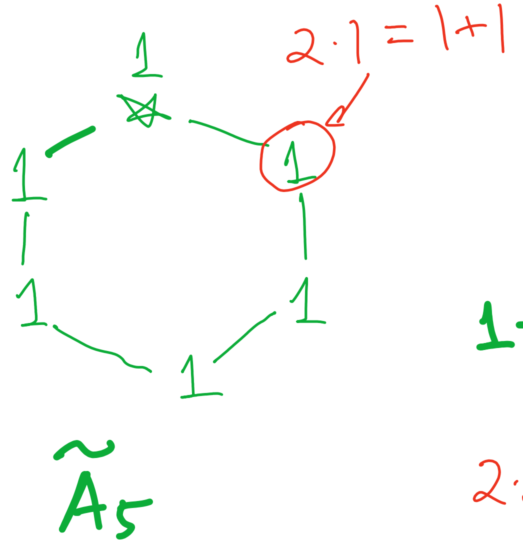
$\tilde{\Gamma}$ = extended Dynkin diagram

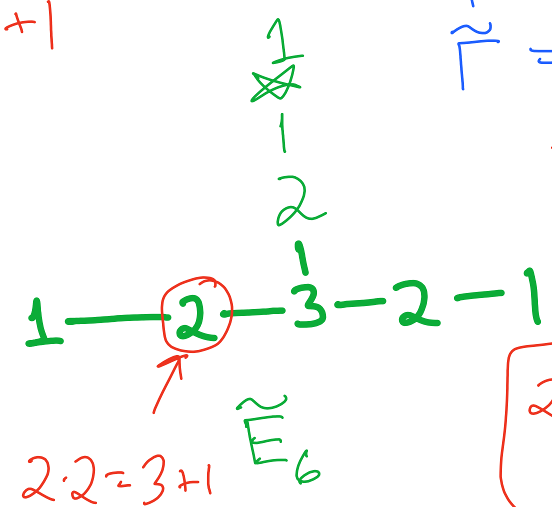
HERE'S MAIN PROP OF DYNKIN DIAGRAMS

THM Suppose all edges are simple — (no double or triple). THEN

$$\boxed{2\cdot \text{label}(\alpha) = \text{sum of labels of adjacent vertices}}$$

(Proof ~~on next slide~~.) below





Proof of thm $\boxed{0 = n_0\alpha_0 + n_1\alpha_1 + \dots + n_\ell\alpha_\ell}$

Pair with simple coroot $\alpha^\vee$:

$$0 = n_0\langle\alpha_0,\alpha^\vee\rangle + n_1\langle\alpha_1,\alpha^\vee\rangle + \dots + n_\ell\langle\alpha_\ell,\alpha^\vee\rangle$$

$\alpha=\alpha_i$ That term is $n_i\cdot\langle\alpha_i,\alpha_i^\vee\rangle = 2n_i$

Other nonzero terms are α_j adjacent to α_i: $\boxed{\langle\alpha_j,\alpha_i^\vee\rangle = -1}$

So eqn is $\boxed{0 = 2n_i + \sum_{j \text{ adj to } i} (-1)\cdot n_j}$

Coxeter graphs

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Defn A Coxeter graph is a finite connected graph, special vertex labelled 1, >0 integer labels on vertices AND 2·label(any vertex) = sum of labels on adj. vertices.

Proved: An extended Dynkin diagram without multiple edges is a Coxeter graph

Thm Any Coxeter graph is Extended Dynkin for root system without multiple edges

Can prove this without classification. roots.pdf ?

Won't do in class

Next : CLASSIFY Coxeter graphs.

suitable for clever elementary school students.

How does any Coxeter graph look?

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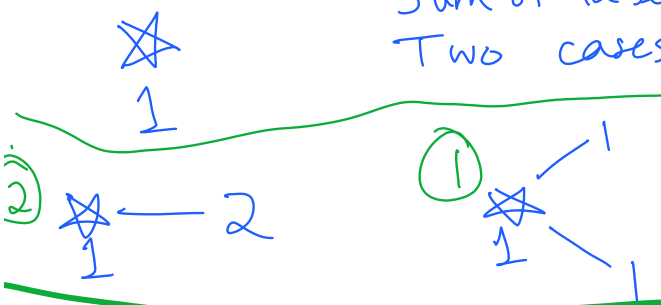
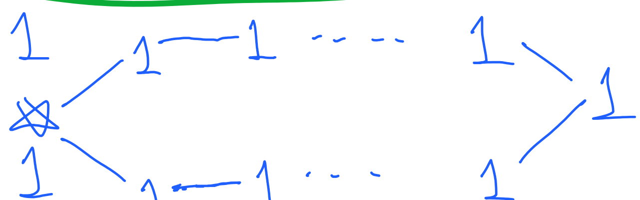
Sum of labels of adj vertices must be 2.

Two cases:



Case 1



n+1 vertices: call this $\tilde{A}_n$ (n unstarred)

sum of labels of adj vertices is 2. SO there's exactly one more adjacent, labelled 1

HAS TO CLOSE to be finite.

CONTINUE.

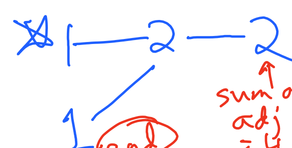
Case 2

sum of labels of adj = 4; already have 1, need 3

3 subcases

CASE 2 3

CASE 2(2)



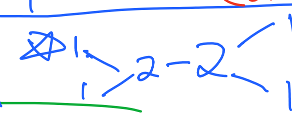
end

sum of adj labels = 4

CONTINUE: analyze new vertices

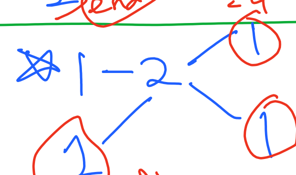
CONTINUE

$\tilde{D}_n$: 4 1's, n-3 2's



DONE: $\tilde{D}_5$

CASE 2(11)



DONE

sum of adj labels = 2; END

Called $\tilde{D}_4$

Case 23

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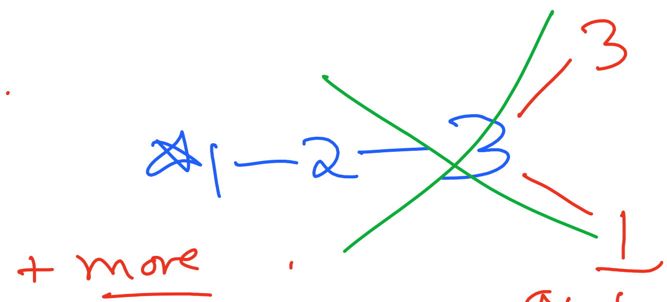
CASE 23 ☆1 — 2 — 3

sum of labels of adj = 6; need 4 more

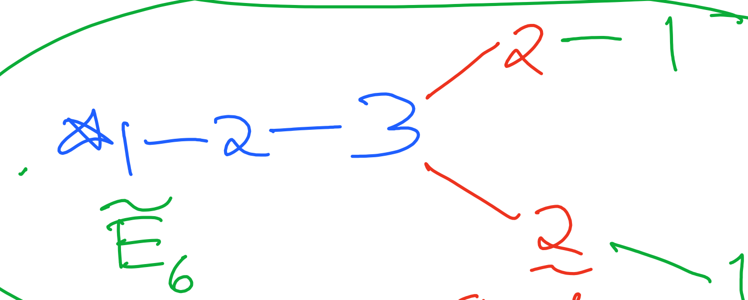
Remains: see how to extend ☆1 — 2 — 3 — 4

☆1 — 2 — 3 — 4



analyze this; sum of adj labels has to be 2 IMPOSSIBLE

Can't attach label smaller than n/2 to label n



$\tilde{E}_6$

sum of adj labels = 4

More type E graphs

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$\tilde{E}_7$

1 — 2 — 3 — 4 — 3 — 2 — 1

4 — 2 end

sum of labels of adj = 8; already have 3, need 5 more. Can't use smaller than 2

OR

1 — 2 — 3 — 4 — 5

sum = 10; need 6 more, using at least 3

1 - 2 - 3 - 4 - 5 — 3 — 1 sum = 6

5 — 3

impossible

1 — 2 — 3 — 4 — 5 — 6 need 7 more, using ≥ 3

1 — 2 — 3 — 4 — 5 — 6 — 4 — 2 end

6 — 3 end

$\tilde{E}_8$

1 — 2 — 3 — 4 — 5 — 6 — 7 — ~~4~~ 1

7 — 4

need 8, use at least 4

IMPOSSIBLE

1 — 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9 — 10 — 11 — 12 — . . .

FORMALLY OK, but infinite

CONTINUE

Pictures of all Coxeter graphs

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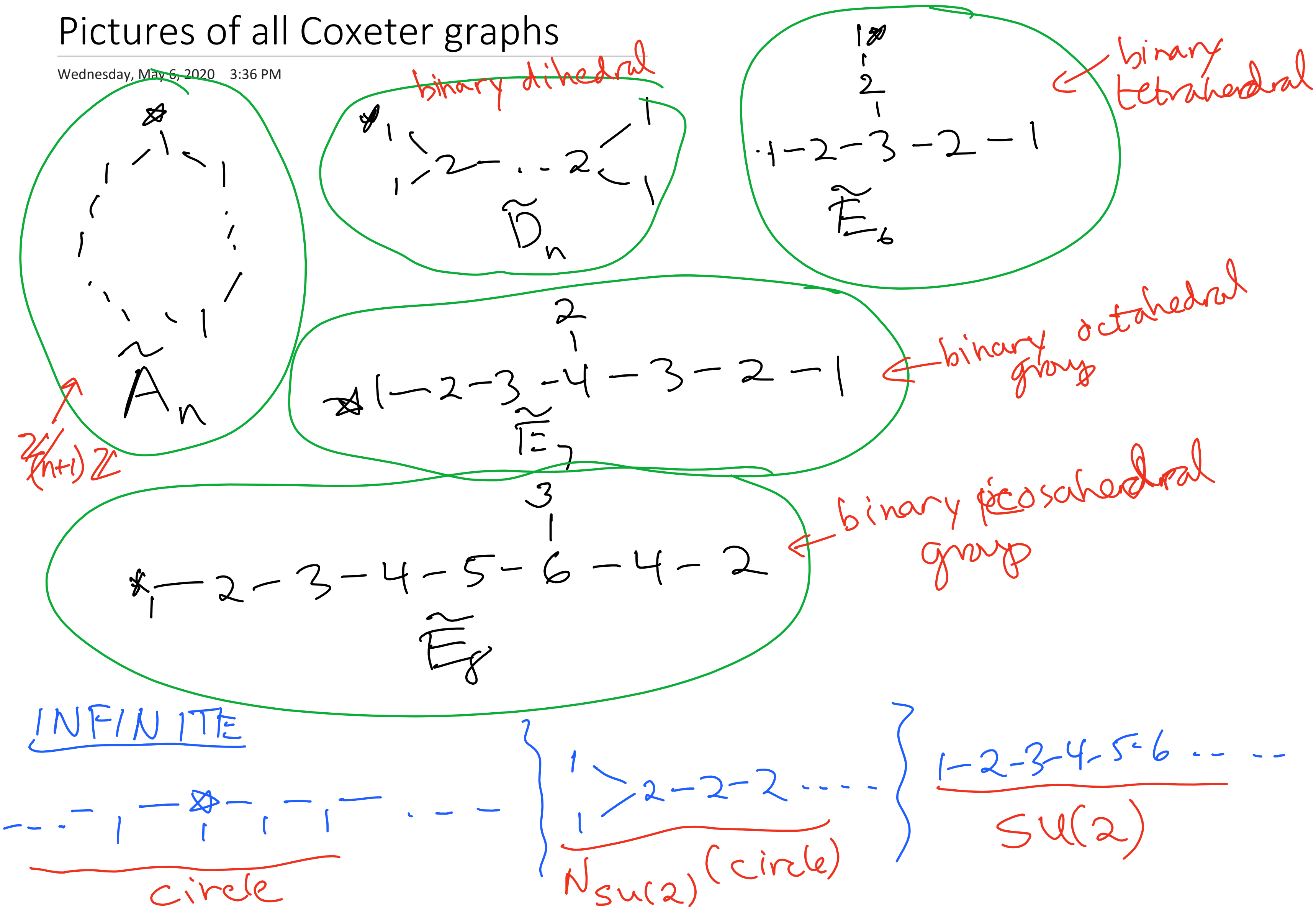

~~ALL~~ COMPACT SUBGROUPS OF SU(2)

FINITE: 18.701 → text "Algebra"

Friday

Wednesday, May 6, 2020 3:53 PM

How to get diagrams with "two root length" (some multiple edges) from (one length diagram) + automorphism — subject of pset

Explain why this gives all two root length things.

Where else in math do Coxeter graphs arise?

Two root lengths

Friday, May 8, 2020 2:06 PM

$\mathcal{R} = (R, X^*, R^\vee, X_*)$ reduced root datum

R^+ choice of positive roots, Π simple roots of R^+

Γ = Dynkin diag: verts $\Pi = \{\alpha_1,\ldots,\alpha_\ell\}$, edge α_p to α_q if $\langle \alpha_p, \alpha_q^\vee \rangle$ not zero

extended diag: add vertex α_0 lowest root.

Positive labels n_p, $n_0 = 0$, $\Sigma_p\, n_p\, \alpha_p = 0$.

Emphasized one root length: $2n_p = \Sigma_{p\text{--}q}\, n_q$ (definition of Coxeter graph).

Today: lengths differ by m $2n_p = \Sigma_{p\text{--}q \text{ same or shorter}}\, n_q + m(\Sigma_{p\text{--}q' \text{ longer}}\, n_{q'})$

Theorem. Suppose we have an extended diagram with two lengths differing by m. Then the short labels are all divisible by m. We can therefore "unfold" the diagram, replacing each arm of short roots by m disjoint arms with labels divided by m, to get a one-length diagram with an automorphism of order m.

Roots α & β have lengths "differing by m" if $\langle \alpha, \beta^\vee \rangle = -1$, $\langle \beta, \alpha^\vee \rangle = -m$

SHORT LONG

2 or 3

Examples G_2

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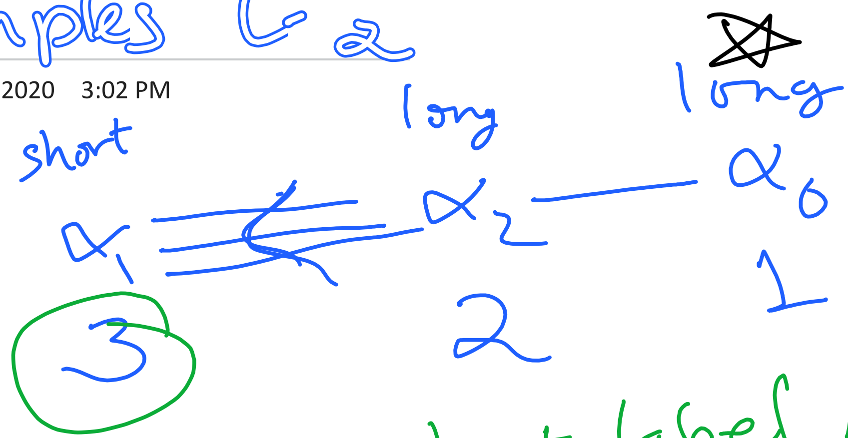
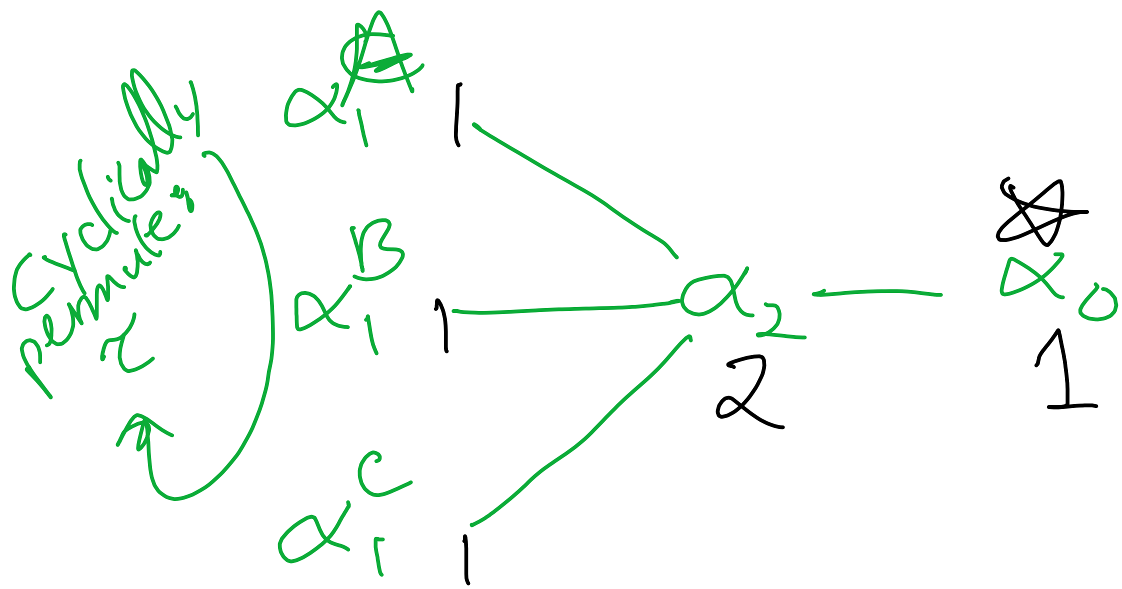


$\tilde{G}_2$

Theorem: short label had to be divisible by 3 → "unfold"



Replace shorter-by-3 simple by 3 different

Example: B_n

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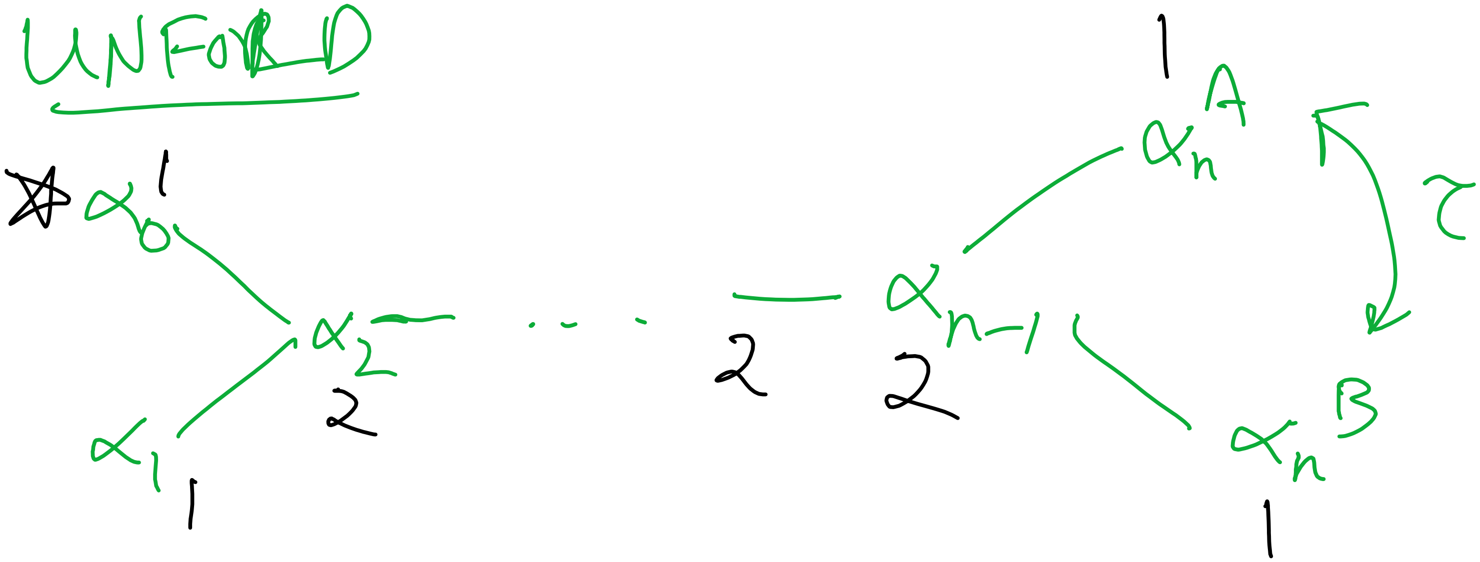
only short; label div. by 2

$\alpha_p = e_p - e_{p+1}$ $1 \leq p < n$

$\alpha_n = e_n$

$\alpha_0 = -e_1 - e_2$

UNFOLD



Example: F_4

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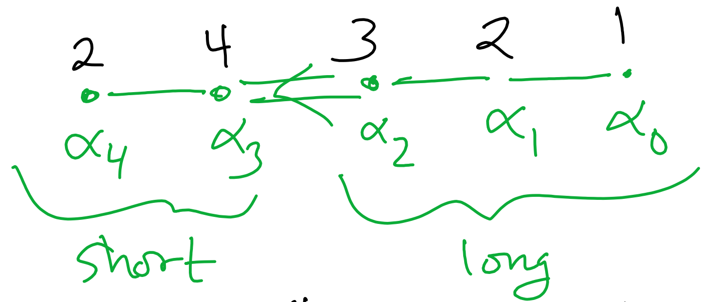


Roots in $\mathbb{Z}^4$, sum of coords even

$\pm 2e_p$ 8

$\pm e_p \pm e_q$ 24

$\pm e_1 \pm e_2 \pm e_3 \pm e_4$ 16

48 roots

POSITIVE: 1st coord is +

1, 2, 3 . .

$\alpha_0 = (-2\ 0\ 0\ 0)$

$-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$

simple: coroots

$\alpha_2 = (0\ 0\ 0\ 2)$ $(0\ 0\ 0\ 1)$

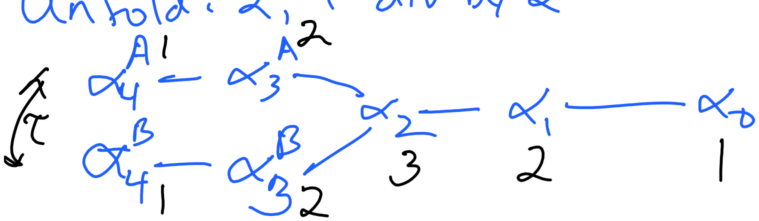
$\alpha_3 = (0\ 0\ 1\ -1)$ $(0\ 0\ 1\ -1)$

$\alpha_4 = (0\ 1\ -1\ 0)$ $(0\ 1\ -1\ 0)$

$\alpha_1 = (1\ -1\ -1\ -1)$ $(\tfrac12\ -\tfrac12\ -\tfrac12\ -\tfrac12)$

CHECK: can't be written as sum of two pos.

Unfold: 2, 4 div by 2



To do at home: C_n

Where are we?

Friday, May 8, 2020 3:24 PM

Finishes sketch of classification of simple root systems

TODAY

1) other occurrences of Coxeter graphs in math

[Monday: Outline classification of root data

Coxeter graphs

Friday, May 8, 2020 3:26 PM

Labelled-by-pos ints finite graph,

$2\cdot \text{label}(p) = $ sum of labels of $p' - p$

Special vertex labelled 1

2 constructions of such things

① Start with compact subgroups $K \subset SU(2)$. (Think of K as "~~kompact~~" KLEIN (Felix) finite if you like; that's very interesting. Stated list of such K last time

"binary octahedral" not explained...

EXPLANATION Quaternions

$Sp(1) \cong$ unit quaternions $\cong SU(2)$

ACTS on "purely imag quaternions" $ai+bj+ck \cong \mathbb{R}^3$ by conjugation: $SU(2) \to SO(3)$ double cover

"octahedral group" means = rotation symmetries of OCTAHEDRON

O_{12} (order 12)

2-1

BINARY OCTAHEDRAL = preimage in $SU(2)$

$\tilde{O}_{24}$ (order 24 since map is 2-1)

Make from K GRAPH: one vertex for each irr. rep. of K. Label vertex by dim(rep)

Special vertex: TRIVIAL rep of K: label 1

EDGES join vertex p to vertex q IF

$$p \otimes \mathbb{C}^2 \supseteq q$$

p: irr rep; $\mathbb{C}^2$: action of $K \subset SU(2)$ on $\mathbb{C}^2$; q: irr rep

$$\boxed{p \otimes \mathbb{C}^2 = \bigoplus_{q-p} q}$$ as repns of K

Take dimensions

$$2\cdot \dim p = \sum_{q-p} \dim q$$

Easy to show this makes a Coxeter graph. ← NEEDS non-classif. PROOF

Every Coxeter graph arises thus: only proof known is by classification of Coxeter graphs and of subgroups of $SU(2)$.

Root data

Friday, May 8, 2020 3:41 PM

Why does every coxeter graph arise from a root datum?

· EASY PROOF: classify Coxeter graphs, construct a root datum for each.

DID

did for A, D ; leaves E_6, E_7, E_8

maybe return to this?

GOOD PROOF: use defn of Coxeter graph, PROVE you can make a root datum...

SKETCH $\alpha_1 \ldots \alpha_\ell$ vertices of Γ

X^* LATTICE = free rank ℓ, BASIS $\alpha_1 \ldots \alpha_\ell$

Define inner product on X^* by

$$\langle \alpha_p, \alpha_p \rangle = 2, \quad \langle \alpha_p, \alpha_q \rangle = \begin{cases} -1 & p \neq q,\ p - q \\ 0 & \text{else} \end{cases}$$

LEMMA $(\,,\,)$ is positive definite

Proof ~~use~~ $\alpha_0 = -\sum_{p=1}^{\ell} n_p \alpha_p$ ~~use (α_0, α_0)~~ =

Coxeter graph def gives properties of this particular vector/lattice elt.

Bourbaki, Lie grps/Lie algs volume

Exercise 7 for Chapter 5, §3

EXERCISE: these properties imply form is definite

Define R = all vectors in X^* of length 2 (obviously finite by definiteness)

Define "$R^\vee$" $\alpha^\vee = \alpha$, all $\alpha \in R$

Root reflections respect inner product: they're orthogonal, preserve $R, R^\vee$.

GET ROOT DATUM!

(Then $\alpha_1 \ldots \alpha_\ell$ are simple)

MONDAY: classify root data.

Ex D_4

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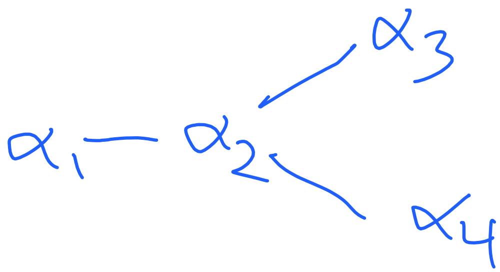

matrix of inner product in basis

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

LEMMA SAYS: POS DEF (all eig vals > 0)

$tr = 8$ $\det = 4$ ... now it's hard

Everything else

COMPLETE CLASSIFICATION OF COMPACT GROUPS

Friday, May 8, 2020 4:01 PM

SLIDE 2

So far we have a list of all possible simple root data.

How do you understand ALL root data $\mathcal{R} = (R, X^*, R^\vee, X_*)$?

lattice dual lattice

Interesting parts correspond to roots and coroots: lattice generated by roots

root lattice = $\mathbf{Z}\cdot R \subset X^*$

coroot lattice = $\mathbf{Z}\cdot R^\vee \subset X_*$

Boring parts correspond to CENTER of compact group: FIXED by W Related to center of compact group

$X_*(\text{cent}) = \{\, \ell \in X_* \mid \langle \alpha , \ell \rangle = 0, \text{ all } \alpha \in R \,\}$

$X^*(\text{cent}) = \{\, \lambda \in X^* \mid \langle \lambda , \alpha^\vee \rangle = 0, \text{ all } \alpha^\vee \in R^\vee \,\}$

Would be nice if these added up to X^*, but that isn't quite true:

$\mathbf{Z}\cdot R \oplus X^*(\text{cent})$ has finite index in X^*

$\mathbf{Z}\cdot R^\vee \oplus X_*(\text{cent})$ has finite index in X_* (K)

R(i) simple

simple root system simple $\alpha_1, \dots, \alpha_l$ coroots $(\alpha_1^\vee, \dots, \alpha_l^\vee)$

index = det $(\langle \alpha_i, \alpha_j^\vee \rangle)$

M n×n integer matrix of a quad form on matrix

span(rows of M) $\subset \mathbb{Z}^n$ sublattice

index = det M (if not = 0)

$(\alpha_1, \dots, \alpha_n)$

Bourbaki Liegps (i.e. alg) $\langle \, , \, \rangle$ is nondeg so $\det(\langle \alpha_i, \alpha_j^\vee \rangle) \neq 0$

$\mathbf{Z}\cdot R^\vee$ has finite index in $P^\vee =_{\text{def}}$ dual lattice of $\mathbf{Z}\cdot R$, and vice versa

$X_*(\text{cent})$ has finite index in dual lattice of $X^*(\text{cent})$, and vice versa.

Define $X^*_{\mathbf{Q}} = X^* \otimes_{\mathbf{Z}} \mathbf{Q}$, rational vector space generated by X^*.

Dual vector space is $X_{*\mathbf{Q}} = X_* \otimes_{\mathbf{Z}} \mathbf{Q}$. In rational vector spaces, all is good:

V rational vec space $\supset L$ lattice $\operatorname{rk} L = \dim V$

Then $L^* = \{ \lambda \in V^* \mid \lambda|_L \ \mathbb{Z}\text{-valued} \}$

$\mathbf{Q}\cdot R^\vee \oplus X_*(\text{cent})_{\mathbf{Q}} = X_{*\mathbf{Q}}$ $\qquad$ $\mathbf{Q}\cdot R \oplus X^*(\text{cent})_{\mathbf{Q}} = X^*_{\mathbf{Q}}$

$\mathbf{Z}\cdot R^\vee \subset X_* \cap \mathbf{Q}\cdot R^\vee \subset P^\vee = \{\, \ell \in \mathbf{Q}\cdot R^\vee \mid \langle \alpha , \ell \rangle \in \mathbf{Z}, \text{ all } \alpha \in R \,\}$

dual lattice to $\mathbb{Z}R$; hard to understand

G group $\quad G^d =_{\text{def}}$ subgroup gen. by $xyx^{-1}y^{-1}$ NORMAL

G^d = derived gp commutator subgroup

G/G^d biggest commutative quotient

$\mathfrak{g}$ Lie alg $\quad \mathfrak{g}^d = \text{span}([X,Y])$ ideal $\quad \mathfrak{g}/\mathfrak{g}^d$ abelian Lie alg

Decomposing into simple factors

Monday, May 11, 2020 5:33 AM

$\mathbb{Q}$ = rational numbers ↖ {mathbb Q}

$\mathscr{R} = (R, X^*, R^\vee, X_*)$ root datum (possibly corresponding to (K, T)).

Let $\sim$ be the equivalence relation on R generated by

$$\alpha \sim \beta \text{ if } \langle \alpha, \beta^\vee \rangle \neq 0.$$

(same as $\langle \beta, \alpha^\vee \rangle \neq 0$; same relation on coroots)

Then R is the disjoint union of equivalence classes $R(1), \ldots, R(s)$.

How do we get corresponding "sub root data?" Two natural ways ...

1. $\mathscr{R}_i = (R(i), X^* / (\text{kernels of all } R(i)^\vee), R(i)^\vee, X_* \cap (\mathbf{Q}\, R(i)^\vee))$.

   Root datum of subgroup K_i generated by all $\phi_\alpha(SU(2))$, α in $R(i)$.

2. $\mathscr{R}^i = (R(i), X^* \cap (\mathbf{Q}\, R(i)), R(i)^\vee, X_* / (\text{kernels of all } R(i)))$.

   Root datum of quotient K^i by id comp of cent(all $\phi_\alpha(SU(2))$, α in $R(i)$).

   $K / \langle$group gen by all $\varphi_\beta(SU(2))$, $\beta \notin R(i)\rangle$ (not in)

Similarly, two kinds of "center":

1. $\mathscr{R}_0 = (\emptyset, X^* / X^* \cap (\mathbf{Q}\, R(i)), \emptyset$, ~~$X_* \cap$ (kernels of~~ all α in R)) ← one way to get torus from K

   Root datum of identity component of center of K.

2. $\mathscr{R}^0 = (\emptyset, X^* \cap (\text{kernels of all } \alpha^\vee \text{ in } R^\vee), \emptyset, X_* / X_* \cap (\mathbf{Q}\, R(i)^\vee))$ also torus ← $K/[K,K]$

   Root datum of quotient of K by its derived group.

Root datum R is not "direct sum" of either of these versions.

Example of U(2)

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$T_d = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \middle| e^{i(\theta+\varphi)} = 1 \right\}$ subtorus of T

K = U(2), T=U(1) x U(1), $\mathcal{R} = (\mathbf{Z}^2 , \{ \pm(1, -1)\} , \mathbf{Z}^2 , \{ \pm(1, -1)\})$ coords sum to 0

derived group K_d = SU(2), $\mathcal{R}_d = (\mathbf{Z}^2 / \mathbf{Z}(1,1) , \{ \pm(1, -1)\} , S\mathbf{Z}^2 , \{ \pm(1, -1)\})$ not isom

commutator subgroup

divide by chars triv on coroots rational span of coroots

quotient by conn center K^q = PU(2), $\mathcal{R}^q = (S\mathbf{Z}^2 , \{ \pm(1, -1)\} , \mathbf{Z}^2 / \mathbf{Z}(1,1), \{ \pm(1, -1)\})$

PU(2) = U(2)/center

rational span of roots divide by cochars triv on roots

max central torus Z_0 = U(1) $\mathcal{R}_0 = (\mathbf{Z}^2 / S\mathbf{Z}^2), \emptyset , \mathbf{Z}(1,1) , \emptyset)$

divide by chars triv on central cochars central cochars

quo by derived grp Z^0 = U(2)/SU(2) $\mathcal{R}^0 = (\mathbf{Z}(1,1) , \emptyset , \mathbf{Z}^2 / S\mathbf{Z}^2) , \emptyset)$

chars triv on coroots divide by rational span of coroots

SL(2) ⊂ GL(2)

PGL(2) ↞ GL(2) ← algebraic groups versions

Reason $\mathcal{R}^q \not\cong \mathcal{R}_d$

$S\mathbb{Z}^2 = \mathbb{Z}\cdot(1,-1)$

$\mathbb{Z}^2/\mathbb{Z}\cdot(1,1) \neq \mathbb{Z}\cdot(1,-1)$

Structure theorem

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MAIN THEOREM: suppose K is a compact connected Lie group with maximal torus T, and root datum (X^*, R , X_* , $R^\vee$). Write

maybe not connected

connected

finite ← F

Z = center of K $\quad$ K_d = derived group of K $\quad$ $T_d = K_d \cap T$ $\quad$ Z_d = center of K_d.

X_*(cent) = kernel of all roots in X_* = cocharacters of Z_0 connected torus in cent(K)

X_{*d} $=_{def}$ $X_* \cap \mathbf{Q}\cdot R^\vee$ = cocharacters of T_d

1. Z_d is isomorphic to the quotient $P^\vee$ / (X_{*d}) , a finite abelian group F.
2. $Z_d \cap Z_0$ is isomorphic to the quotient ($P^\vee \cap X_*$) / (X_{*d}) , a finite abelian subgroup F_0. fin gen abelian, not nec free
3. K is isomorphic to the quotient (K x Z_0) / ($Z_d \cap Z_0$).
4. The group of characters of the center X^*(Z) is isomorphic to X^* / $\mathbf{Z}\cdot R$. fin gen abelian, not nec free
5. The fundamental group π_1(K) is isomorphic to X_* / $\mathbf{Z}\cdot R^\vee$.

EASY: $\mathfrak{k} = [\mathfrak{k},\mathfrak{k}] \oplus \mathfrak{z}$

$\mathrm{Lie}(K^d)$ $\quad$ $\mathrm{Lie}(Z)$

$\Rightarrow$ K_d, Z_0 are normal, $K_d \cdot Z_0$ ~~open~~ contains nbhd of e

$\therefore$ $\boxed{K = K_d \cdot Z_0}$; product map $K_d \times Z_0 \to K$ ONTO

Kernel = $\{(x,z) \in K_d \times Z_0 \mid x = z^{-1}\}$

x has to be in Z(K); $Z(K) \cap K_d = Z_d$

$\cong Z_d \cap Z_0$ emb by $x \mapsto (x, x^{-1}) \in K_d \times Z_0$

$K = U(n)$

$K_d = SU(n)$

$Z = S^1 \cdot (I_n)$ scalar matrices

$\left\{\begin{pmatrix} e^{i\theta} & & \\ & \ddots & \\ & & e^{i\theta}\end{pmatrix}\right\} \cong S^1$

CONN

center of K_d

$Z \cap K_d$ = ~~nth~~ nth roots of 1

$F = F_0$

$U(n) \cong [S^1 \times SU(n)] / $ nth roots of 1

Recipe for a compact group / reduced root datum

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$\boxed{\text{van der Waerden}}$

CLASSIFIED!

First ingredient: s simple root systems $R_1, \ldots, R_s$ (from Dynkin diagrams $\Gamma_1 \ldots \Gamma_s$)

Get lattices $\mathbf{Z}R_1, \ldots, \mathbf{Z}R_s$ (bases $\Pi_1, \ldots, \Pi_s$ sets of simple roots)

Dual lattices $P_1^\vee, \ldots, P_s^\vee$ (bases ~~fundamental coweights~~) ← value 1 on one simple root, 0 on all other simple

Automatically $\mathbf{Z}R_i^\vee \subset P_i^\vee$, similarly $\mathbf{Z}R_i \subset P_i$

Second: lattice X_{*d}, $\mathbf{Z}R_1^\vee \oplus \ldots \oplus \mathbf{Z}R_s^\vee \subset X_{*d} \subset \mathbf{Z}P_1^\vee \oplus \ldots \oplus \mathbf{Z}P_s^\vee$ ← dual

gives dual lattice $\mathbf{Z}P_1 \oplus \ldots \oplus \mathbf{Z}P_s \supset X^*_d \supset \mathbf{Z}R_1 \oplus \ldots \oplus \mathbf{Z}R_s$

Same as choice of finite abelian quotient $F \twoheadleftarrow [\mathbf{Z}P_1^\vee/\mathbf{Z}R_1^\vee] \oplus \ldots \oplus [\mathbf{Z}P_s^\vee/\mathbf{Z}R_1^\vee]$ (finite; finite groups; center of ...)

Same as specification of derived group K_d — $\subset$

Third ingredient: (central) lattice C^*, dual lattice C_* — To specify all 6-dim'l compact groups

Fourth ingredient: choice of subgroup $F_0 \subset F$, inclusion $F_0 \hookrightarrow (C_* \otimes_{\mathbf{Z}} \mathbf{Q} / C_*)$ — and all remaining dims:

(1) $A_1 + A_1$, A_1, $\emptyset$ — 3 dims, 0 dims

rk 2, 4 roots accounts for 6 dims; no more room

(3) C^*:

$A_1 + A_1 + 0$ C^*

$A_1 + \mathbb{Z}^3$ C^*

$\emptyset + \mathbb{Z}^6$ C^*

If $A \subset C$ abelian, subgroup B between A & C SAME as quotient of C/A (same as subgp of C/A)

Any dim: FINITE set of possibilities for R_i, C^*

A_1: $P_1^\vee / \mathbb{Z}R_1^\vee \simeq \mathbb{Z}/2$

$A_1 + A_1$ case: F = subgp of $\mathbb{Z}/2 \times \mathbb{Z}/2$ — 5 possible: triv, 3 ord 2, $\mathbb{Z}/2 \times \mathbb{Z}/2$

5 groups — F_0: has to be related to trivial C_*; $F_0 = \text{triv}$

$A_1 + \mathbb{Z}^3$: 2 choices of F. $F_0 = \text{triv}$: no more choices

$F = F_0 = \mathbb{Z}/2$: choose $\boxed{\mathbb{Z}/2 \hookrightarrow \mathbb{Q}^3/\mathbb{Z}^3}$ — HARD — ∞ many: ~~finitely~~ up to $GL(3,\mathbb{Z})$

$\simeq 3$ groups

$\mathbb{Z}^6$: $F = F_0$ trivial: no more choices

1 group = 6-dim'l torus.

Can write a computer program to list all possible K of dim 1, 2, 3, ... (finitely many for each dim. Fokko du Cloux → software "atlas" ← work with arb root data.

B_2: 8 roots, rank 2: dim 10

A_2: 6 roots, rank 2: dim 8

G_2: 12 roots, rank 2: dim 14

only appear in K of dim ≥ 8

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$$Z(SU(2)) = \pm I$$

$$SU(2)\times SU(2) \leftarrow \text{center} = \mathbb{Z}/2 \times \mathbb{Z}/2$$

5

$$SU(2)/\pm 1 \times SU(2) \quad * \quad \text{isom}$$

$$SU(2)\times SU(2)/\pm 1$$

$$[SU(2)\times SU(2)]/\text{diag}\,(I,I),(-I,-I)$$

$$SU(2)/\pm 1 \times SU(2)/\pm 1$$

SO(8)

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What are all root data related to  D_4?

Roots inside $\mathbb{Z}^4$: $\{\pm e_i \pm e_j \mid 1 \leq i \neq j \leq 4\} = R$

coroots $\{\ \cdots\ \} = R^\vee$

① $\mathbb{Z}R = \{(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{Z}^4 \mid \sum \lambda_i \text{ even}\}$

$P = \{\lambda \in \mathbb{Q}^4 \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \text{ all } \alpha^\vee \in R^\vee\}$

$\mathbb{Q}^4$: $\mathbb{Q} \cdot R$

$= \{\lambda \in \mathbb{Q}^4 \mid \pm\lambda_p \pm \lambda_q \in \mathbb{Z}, \text{ all } p \neq q\}$

$= \mathbb{Z}^4 \cup (\mathbb{Z} + \tfrac{1}{2})^4.$

② $P/\mathbb{Z}R \cong \mathbb{Z}/2 \times \mathbb{Z}/2$

generators: $(1,0,0,0)$

$(\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2})$

5 compact groups with this Dynkin diag:

$PSO(8)$ $\quad F = \text{triv.}$

$Spin(8)$ $\quad F = \mathbb{Z}/2 \times \mathbb{Z}/2$

3 more

$SO(8)$, $Spin^+(8)$, $Spin^-(8)$

"half spin ~~groups~~"

In F^2, $\underline{\text{ALL}}$ 1-dim'l subspaces are

$F \cdot \begin{pmatrix} 1 \\ x \end{pmatrix}$, $F \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$x \in F$

← GAUSSIAN ELIM.

$\boxed{F = \mathbb{Z}/2\mathbb{Z} : 3}$

L-groups

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