18.755 sixth problems, due Monday, October 26, 2015

This problem set is mostly about the groups and Lie algebras preserving various bilinear forms. We can say a lot of things over any field k, and that's worth doing. Recall that a *bilinear form* on a k vector space V is a map

$$\beta\colon V \times V \to k$$

satisfying

$$\beta(au + bv, w) = a\beta(u, w) + b\beta(v, w), \qquad \beta(x, cy + dz) = c\beta(x, y) + d\beta(x, z);$$

here $u, v, w, x, y, z \in V$ are vectors and $a, b, c, d \in k$ are scalars.

The form is called *symmetric* if

$$\beta(v, w) = \beta(w, v),$$

and *skew-symmetric* if

$$\beta(v, w) = -\beta(w, v).$$

The form is *nondegenerate* if for every nonzero $v \in V$ there is a $w \in V$ so that $\beta(v, w) \neq 0$, and for every nonzero w' there is a v' so that $\beta(v', w') = 0$.

If $V = k^n$, then bilinear forms β can be identified with $n \times n$ matrices B by the equation

$$\beta(v, w) = {}^t wBv, \qquad B_{ij} = \beta(e_j, e_i).$$

Clearly symmetric (respectively skew-symmetric) forms correspond to symmetric (respectively skew-symmetric) matrices. Nondegenerate forms correspond to invertible matrices.

The group $GL(V)$ of invertible linear transformations acts on bilinear forms by change of variable

$$(g \cdot \beta)(v, w) = \beta(g^{-1} \cdot v, g^{-1} \cdot w).$$

The corresponding action of $GL(n, k)$ on matrices is

$$g \cdot B = {}^t g^{-1} B g^{-1}.$$

This action (which is *not* quite conjugation on matrices) preserves the property of being symmetric or skew symmetric.

1. Describe all possible symmetric bilinear forms on $\mathbb{R}^n$ up to the action of $GL(n, \mathbb{R})$. That is, describe the orbits of $GL(n, \mathbb{R})$ on symmetric $n \times n$ matrices. (Hint: there are $(n + 1)(n + 2)/2$ orbits.)

2. If β is a symmetric bilinear form on V, the *orthogonal group of* β is by definition

$$O_\beta = \{g \in GL(V) \mid g \cdot \beta = \beta\}.$$

(If the characteristic of k is two, this definition still makes sense but it is not correct—that is, it is not what is called an orthogonal group in characteristic two.)

If $V = \mathbb{R}^n$, and β corresponds to a symmetric matrix B, then the definition says that

$$O_\beta = \{g \in GL(n, \mathbb{R}) \mid {}^t g^{-1} B g^{-1} = B\}.$$

Find a simple description for the set of matrices X in the Lie algebra $\mathfrak{o}_\beta$ of O_β. (Hint: you can use the fact that the Lie algebra of a Lie subgroup H of G consists of all $X \in \mathfrak{g}$ such that $\exp(tX) \in H$ for every real t.)

3. Let $k = \mathbb{R}$, $n = p + q$, and let $B = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ correspond to the symmetric form β of signature (p, q). The corresponding orthogonal group O_β (preserving the "length" function

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)$$

is called $O(p, q)$. How many connected components does $O(p, q)$ have? (Hint: this is not so easy. You should look carefully at some small values of p and q to try to formulate the answer.)

4. Describe the Lie algebra $\mathfrak{o}(p, q)$ by saying exactly which matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (A\ p \times p, \quad B\ p \times q, \quad C\ q \times p, \quad D\ q \times q)$$

belong to the Lie algebra. (This means writing some conditions on and relations among the four matrices A, B, C, and D. A good answer is one that can be understood just by knowing about matrices, without knowing about bilinear forms.) Use your description to calculate the dimension of $O(p, q)$.