Frobenius theorem

is a theorem about manifolds and submanifolds.

N.B. USED theorem to construct 1-param subgroups of a Lie group!

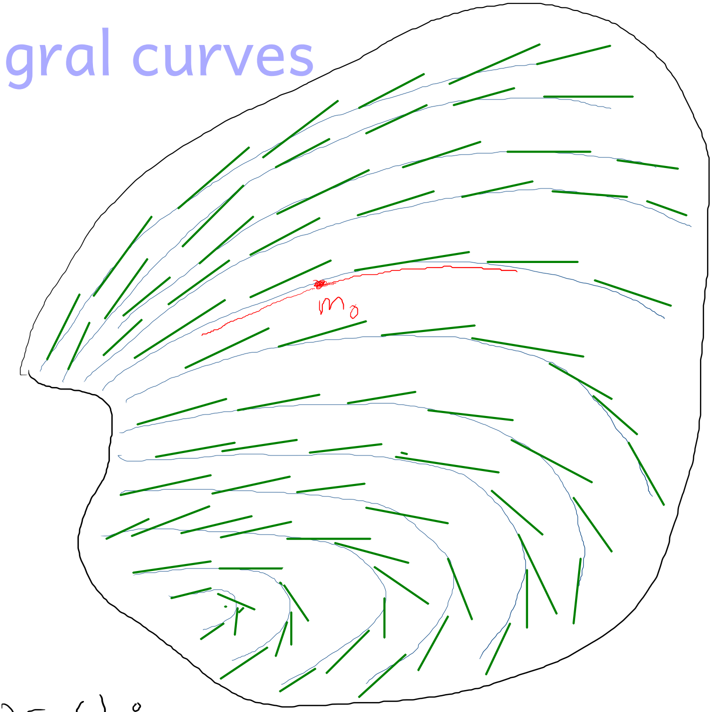
Statement in a moment. First, what question is it addressing?

Theorem. X vector field on manifold M;
for each m0 in M there is a unique γ_{X,m0}:
(a,b) --> M with γ_{X,m0}(0) = m0,
γ_{X,m0}'(t) = X(γ_{X,m0}(t)).

Manifold is nicely covered (foliated) by 1-dimensional submanifolds (as long as X(m) is never zero).

vector field on a manifold,

integral curves



$X(m_1) = 0$:

$\boxed{\gamma_{\{X,m_1\}}(t) = m_1}$ ← POINT, not curve

What about TWO vector fields X and Y?

X(m) and Y(m) define a 2-dimensional subspace of the tangent space T_m(M) at each m in M. (unless they don't!)

Hope: through each m0 in M passes unique 2-diml submanifold γ_{X,Y,m0}

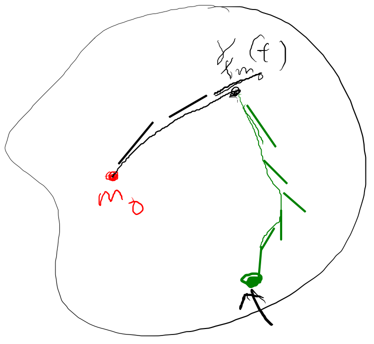
$\gamma\_{\{X,Y,m0\}}(s,t)=\gamma\_{\{Y,(\gamma\_{\{X,m0\}}(t))\}}\ (s)$

Works: this <u>is</u> 2-diml submanifold (as long as $X(m_0)$, $Y(m_0)$ lin ind and s,t small)

FAILURE: tangent space at $\gamma_c(s,t)$ is <u>NOT</u> (necessarily) spanned by $\underset{\text{LOST}}{\underline{X(\gamma(s,t))}}$, $\underbrace{Y(\gamma(s,t))}_{\text{IS in tangent space to subman.}}$

That is, start at m0, follow curve defined by X for a while, then follow curve defined by Y for a while.

X: black
Y: green



$\gamma_{X,Y,m_0}(s,t)$

s, t small real

Frobenius theorem

Definition M: a $\boxed{d\text{-dimensional smooth distrib.}}$ $\mathcal{D}$ on M is an assignment $\underset{\underset{M}{\cap}}{m} \longrightarrow \underbrace{\mathcal{D}_m}_{d\text{-dimensional subspace of } T_mM}$

SUCH THAT [smooth]: near every $m_1 \in M$, there are d smooth vector fields $D_1 \dots D_d$ defined near m_1,

$$\mathcal{D}_m = \operatorname{span}\left(D_1(m), \dots D_d(m)\right), \text{ all } m \text{ near } m_1$$

Smoothly varying family of subspaces $\{S_\alpha \subset V \mid \alpha \in \underset{\text{manifold}}{\underline{A}}\}$ should mean: near each $\alpha_0 \in A$, have $\underline{d}$ ~~vectors~~ smooth functions $V_1(\alpha), \dots V_d(\alpha)$, $S_\alpha = \langle V_1(\alpha), \dots V_d(\alpha) \rangle$, $\boxed{\text{all } \alpha \text{ near } \alpha_0}$

When is $\mathcal{D}$ family of tangent spaces to nice d-dim'l submanifolds of M?

1-dim'l $\mathcal{D}$: ALWAYS

2-dim'l $\mathcal{D}$: NOT ALWAYS

Vector field X on M "belongs to $\mathcal{D}$" if $\underline{X(m)} \in \underline{\mathcal{D}_m}$, all m

tangent vector at m

A-diml subspace of $T_m(M)$

Defn $\mathcal{D}$ is <u>INVOLUTIVE</u> if whenever X, Y belong to $\mathcal{D}$, $[X, Y]$ also belongs to $\mathcal{D}$

<u>(NON) EXAMPLE</u>

Has to have dimension at least 2

If M has dimension 2, <u>ANY</u> distribution is involutive [WHY?]

need 2-diml $\mathcal{D}$ in $\mathbb{R}^3$

$$\mathcal{D} = \text{~~all~~ span of } \Big\langle \frac{\partial}{\partial x},\ \underbrace{\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}}_{\text{lin comb of}} \Big\rangle \subseteq \underbrace{T_{x,y,z}\mathbb{R}^3}_{\text{span}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)}$$

$\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$; $x \in \mathbb{R}$

EASY: $\mathcal{D}$ is smooth 2-diml

[illegible] Spanned by

$$X = \frac{\partial}{\partial x}$$

$$Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$$

not this

hard

solve $\frac{\partial f}{\partial x} = 0$

$\frac{\partial f}{\partial y} = 0$

<u>EASY</u>

$$[X, Y] = \frac{\partial}{\partial z} \notin \text{smooth} \cdot X + \text{smooth} \cdot Y$$

NOT INVOLUTIVE

follow X, then follow Y → DIFFERENT surfaces in $\mathbb{R}^3$ from follow Y, then follow X

Thm FROBENIUS $\mathcal{D}$ involutive d-dim'l distn on M; through any $m_0 \in M$ $\exists$ unique connect d-dim'l embedded submanifold $\Gamma_{\mathcal{D},m_0}$ such that

$$\boxed{\underbrace{T_m(\Gamma_{\mathcal{D},m_0})}_{\overset{\parallel}{\mathcal{D}(m)}} \subset T_m M}$$

$d=1$: $\mathcal{D}$ = span of 1 vector field X (locally). Involutive automatic

Thm follows from "1-param group of diffeos" thm (1st page)

Higher: thm about differential equations There are $\boxed{\frac{\partial}{\partial \ldots}}$ in it; - PDE? ← harder

INVOLUTIVE lets you reduce to ~~d~~ ODE's.

SKIP proof: try to add a reference to proof accessible online

1-dim'l version ⟶

<u>Thm</u> G Lie group, $X \in \mathfrak{g}$ ⇝ get one-parameter subgroup

$$\boxed{t \mapsto \exp(tX) \in G}$$

$$\boxed{\mathbb{R} \xrightarrow{\gamma_X} G}$$

<u>Lie group hom.</u>

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d-dim'l:

<u>Thm</u> G Lie gp, $\mathfrak{h} \subset \mathfrak{g}$ d-dimensional Lie subalgebra, get embedded Lie subgroup

$$\boxed{\gamma_{\mathfrak{h}} : H \to G}$$

of dim d

<u>FRIDAY</u>