## 18.034 Problem Set 9

Due Wednesday, May 10 in class.

**1.** This problem (taken from Hirsch, Smale, and Devaney) is about the system

$$x' = (\cos x)(.1 \sin x - \sin y), \qquad y' = (\cos y)(\sin x + .1 \sin y).$$

a) Find all the equilibrium points. Decide (as completely as you can) which ones are stable and which are unstable.

b) Explain why an orbit which meets the horizontal line $y = (2k+1)\pi/2$ (for $k$ any integer) has to remain on that line for all time. Describe as completely as you can the orbits on this line.

c) Same question for the vertical line $x = (2j+1)\pi/2$.

Pause for exposition: suppose $y(t)$ is a solution of the autonomous differential equation system $y' = F(y)$ in $\mathbb{R}^n$, and that $y(t)$ is defined for all $t$. The *positive limit set for* $y$, written $\omega(y)$, by definition consists of all points $x \in \mathbb{R}^n$ with the following property: there is a sequence $t_j \to \infty$ so that

$$\lim_{j\to\infty} y(t_j) = x.$$

In the same way we can define the *negative limit set* $\alpha(y)$. These are introduced on page 531 of the text, but with no precise definition.

d) Suppose $(x(t), y(t))$ is a solution of the differential equation above with $x(0) = y(0) = \pi/4$. Find the positive limit set $\omega(x, y)$. (I'm looking for a very precise answer, but not necessarily a very precise proof: just try to figure out what seems to be happening, make some pictures, and formulate a reasonable guess.)

**2.** This problem is about the same system of differential equations as in problem **1**. Consider the function

$$V(x, y) = (\cos x)(\cos y).$$

a) Find all the critical points of $V$ (points where both partial derivatives are zero). Which ones are local maxima? Which ones are local minima? Which ones are neither?

b) In some parts of the plane $V$ is increasing along solution curves to the differential equation; in some parts it is constant; and in some parts it is decreasing. Say as completely as you can what happens where.

c) Your answers to (b) should say that $V$ (or some very simple modification of it) can serve as a Lyapunov function for the differential equation in certain regions. Explain how that works, and what conclusions you can draw about stability of certain equilibrium points.

**3.** This problem is about the van der Pol system

$$x' = y - f(x), \qquad y' = -x$$

described in the text on page 525. Here $f$ is a function with a continuous derivative satisfying the requirements

(1) $f$ is odd—that is, $f(x) = -f(-x)$;

(2) there is a positive number $a$ so that $f$ is (strictly) negative on $(0, a)$ and (strictly) positive on $(a, \infty)$.

The text also asks that $f(x) \to \infty$ as $x \to \infty$, but that doesn't seem to matter for the questions I want to ask. (If you look up the van der Pol equation, you may find $f(x) = cx^3 - dx$, with $c$ and $d$ positive constants. You may also find it written as a single second-order equation for $y$; that fits better with some important applications to electrical circuits.)

a) Prove that $(0, 0)$ is the only equilibrium point.

b) Prove that the function $x^2 + y^2$ is strictly increasing along solutions for $0 < |x| < a$, and strictly decreasing along solutions for $|x| > a$.

c) Prove that the equilibrium point $(0, 0)$ cannot be a positive limit point of any solution. Hint: if

$$\lim_{j\to\infty} (x(t_j), y(t_j)) = (0, 0),$$

then $|x(t_j)| < a$ for all large enough $j$.

I want you to use the idea explained in Example 9.2.2 to prove that the van der Pol system above must have a limit cycle. Fix a positive number $M$ so that

$$|f(x) - 2x| < M, \qquad (|x| \leq 2a).$$

Now consider the region $R$ in the $(x, y)$ plane bounded by four curves: two arcs of the circle $x^2 + y^2 = M^2 + 4aM + 8a^2$, a segment of the line $y - x = M$, and a segment of the line $y - x = -M$. (The four "corners" of $R$ are the points $(-2a, \pm M - 2a)$, and $(2a, \pm M + 2a)$.)

d) Prove that a solution crossing the circle $x^2 + y^2 = c$ at a point where $|x| \geq 2a$ must be crossing from outside to inside.

e) Prove that a solution crossing the line $y - x = M$ where $|x| \leq 2a$ must be crossing from above to below (in the direction of decreasing $y - x$). In exactly the same way, it follows that a solution crossing the line $y - x = -M$ where $|x| \leq 2a$ must be crossing from below to above.

f) Explain why a solution starting inside $R$ must remain inside $R$ for all positive time. Then explain how Theorem 9.2.1 proves that there must be a cycle inside $R$.