18.034 Problem Set 7

Due Wednesday, April 19 in class.

1. Text page 404, problem 19. (First write an explicit formula for e^{tA}, with A the matrix appearing in the problem.)

2. Text page 404, problem 24.

3. Same as previous problem, but replace the driving function $\begin{pmatrix} \cos t \\ 2t \end{pmatrix}$ with $\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$.

4. Text page 404, problem 28(d).

5. Same as previous problem, but add a driving function $F(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$. (This is more or less the same differential equation as in problem 3, but you're supposed to solve it by a very different method.)

6. You may know that the ratio of the nth to the $(n+1)$st Fibonacci numbers approaches $(\sqrt{5}-1)/2$ as n approaches infinity. This problem is a continuous version of the same fact. Consider the initial value system

$$y_1'(t) = y_2(t), \qquad y_2'(t) = y_1(t) + y_2(t), \qquad y_1(0) = 0, \qquad y_2(0) = 1.$$

Prove that

$$\lim_{t\to\infty} y_1(t)/y_2(t) = (\sqrt{5}-1)/2.$$

Can you find a generalization that applies to a large family of systems $y' = Ay$? (It's reasonable to look only at 2×2 matrices A.)

7. (This one has almost nothing to do with differential equations, but it's fun.) Let A be the matrix from the preceding problem:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Consider the successive powers of A:

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}.$$

Find a closed formula for the entries of A^n. (Hint: some of the methods discussed in recitation for calculating e^{tA} apply equally well to this problem.)