Lipschitz properties of transport maps under a log-Lipschitz condition

Dan Mikulincer

MIT

Joint work with Max Fathi and Yair Shenfeld
Transportation of measure

Given probability measures $\nu$ and $\mu$, for now in $\mathbb{R}^d$, we seek a transport map $T$ from $\nu$ to $\mu$ with “good properties”.

Drawing of a train leaving a high-dimensional city (DALLL·E)
Transportation of measure

**Transportation:** $T$ transports $\nu$ to $\mu$ if

\[ X \sim \nu \quad \Rightarrow \quad T(X) \sim \mu, \]

\[ \mu(A) = \nu(T^{-1}(A)) \]

and in terms of densities

\[ d\nu(x) = d\mu(T(x))|\det DT(x)| \]

**Good properties:** $T$ should be $L$-Lipschitz:

\[ |T(x) - T(y)| \leq L|x - y|, \]

and in terms of derivatives

\[ |\nabla T(x)| \leq L. \]
Transportation of measure

Transportation: \( T \) transports \( \nu \) to \( \mu \) if

\[
X \sim \nu \quad \Rightarrow \quad T(X) \sim \mu,
\]

\[
\mu(A) = \nu(T^{-1}(A))
\]

and in terms of densities

\[
d\nu(x) = d\mu(T(x))|\det DT(x)|
\]

Good properties: \( T \) should be \( L \)-Lipschitz:

\[
|T(x) - T(y)| \leq L|x - y|,
\]

and in terms of derivatives

\[
|\nabla T(x)| \leq L.
\]
Transportation of functional inequalities

* log-Sobolev: \( \int f^2 \log(f) d\nu =: \text{Ent}_\nu(f^2) \leq C_{\text{LS}}(\nu) \mathbb{E}_\nu [ |\nabla f|^2 ] \)

**Claim**

Suppose \( \nu \) satisfies a log-Sobolev\(^*\) inequality with constant \( C_{\text{LS}}(\nu) \).

Suppose there exist an \( L \)-Lipschitz map \( T : \mathbb{R}^d \to \mathbb{R}^d \) which transports \( \nu \) to \( \mu \). Then, \( \mu \) satisfies a log-Sobolev inequality with constant \( C_{\text{LS}}(\mu) \leq C_{\text{LS}}(\nu)L^2 \)

**Proof.**

\[
\text{Ent}_\mu(f^2) = \text{Ent}_\nu((f \circ T)^2) \leq C_{\text{LS}}(\nu) \mathbb{E}_\nu [ |\nabla(f \circ T)|^2 ] \\
\leq C_{\text{LS}}(\nu) \mathbb{E}_\nu [ |DT|^2 |\nabla f(T)|^2 ] \leq C_{\text{LS}}(\nu)L^2 \mathbb{E}_\nu [ |\nabla f(T)|^2 ] \\
= C_{\text{LS}}(\nu)L^2 \mathbb{E}_\mu [ |\nabla f|^2 ] .
\]
Transportation of functional inequalities

\* log-Sobolev: \( \int f^2 \log(f) d\nu =: \text{Ent}_\nu(f^2) \leq C_{LS}(\nu) \mathbb{E}_\nu [ |\nabla f|^2] \)

**Claim**

Suppose \( \nu \) satisfies a log-Sobolev* inequality with constant \( C_{LS}(\nu) \). Suppose there exist an \( L \)-Lipschitz map \( T : \mathbb{R}^d \rightarrow \mathbb{R}^d \) which transports \( \nu \) to \( \mu \). Then, \( \mu \) satisfies a log-Sobolev inequality with constant \( C_{LS}(\mu) \leq C_{LS}(\nu)L^2 \)

**Proof.**

\[
\text{Ent}_\mu(f^2) = \text{Ent}_\nu((f \circ T)^2) \leq C_{LS}(\nu) \mathbb{E}_\nu [ |\nabla (f \circ T)|^2]
\]

\[
\leq C_{LS}(\nu) \mathbb{E}_\nu [ |DT|^2 |\nabla f(T)|^2] \leq C_{LS}(\nu)L^2 \mathbb{E}_\nu [ |\nabla f(T)|^2]
\]

\[
= C_{LS}(\nu)L^2 \mathbb{E}_\mu [ |\nabla f|^2].
\]
Transportation of functional inequalities

\[ \log\text{-Sobolev: } \int f^2 \log(f) d\nu =: \text{Ent}_\nu(f^2) \leq C_{LS}(\nu) \mathbb{E}_\nu [ |\nabla f|^2 ] \]

**Claim**

Suppose \( \nu \) satisfies a log-Sobolev\( ^* \) inequality with constant \( C_{LS}(\nu) \).

Suppose there exist an \( L \)-Lipschitz map \( T : \mathbb{R}^d \to \mathbb{R}^d \) which transports \( \nu \) to \( \mu \). Then, \( \mu \) satisfies a log-Sobolev inequality with constant \( C_{LS}(\mu) \leq C_{LS}(\nu)L^2 \)

**Proof.**

\[
\begin{align*}
\text{Ent}_\mu(f^2) &= \text{Ent}_\nu((f \circ T)^2) \\
&\leq C_{LS}(\nu) \mathbb{E}_\nu [ |\nabla(f \circ T)|^2 ] \\
&\leq C_{LS}(\nu) \mathbb{E}_\nu [ |DT|^2 |\nabla f(T)|^2 ] \\
&\leq C_{LS}(\nu) L^2 \mathbb{E}_\nu [ |\nabla f(T)|^2 ] \\
&= C_{LS}(\nu) L^2 \mathbb{E}_\mu [ |\nabla f|^2 ].
\end{align*}
\]
Transportation of functional inequalities

* log-Sobolev: \( \int f^2 \log(f) d\nu =: \text{Ent}_\nu(f^2) \leq C_{LS}(\nu) \mathbb{E}_\nu [ |\nabla f|^2 ] \)

Claim

Suppose \( \nu \) satisfies a log-Sobolev* inequality with constant \( C_{LS}(\nu) \). Suppose there exist an \( L \)-Lipschitz map \( T : \mathbb{R}^d \to \mathbb{R}^d \) which transports \( \nu \) to \( \mu \). Then, \( \mu \) satisfies a log-Sobolev inequality with constant \( C_{LS}(\mu) \leq C_{LS}(\nu)L^2 \)

Proof.

\[
\text{Ent}_\mu(f^2) = \text{Ent}_\nu((f \circ T)^2) \leq C_{LS}(\nu) \mathbb{E}_\nu [ |\nabla(f \circ T)|^2 ] \\
\leq C_{LS}(\nu) \mathbb{E}_\nu [ |DT|^2 |\nabla f(T)|^2 ] \leq C_{LS}(\nu)L^2 \mathbb{E}_\nu [ |\nabla f(T)|^2 ] \\
= C_{LS}(\nu)L^2 \mathbb{E}_\mu [ |\nabla f|^2 ].
\]
**Brenier 87’**: For reasonable $\mu, \nu$ there exists an optimal transport map $\psi^{opt} : \mathbb{R}^d \to \mathbb{R}^d$, satisfying:

$$
\psi^{opt} = \arg \min_{\psi \ast \nu = \mu} \mathbb{E}_\nu [ \| \psi(x) - x \|^2 ] .
$$

**Caffarelli 00’**: If $\nu = \gamma_d$ is the standard Gaussian and $\mu$ is more log-concave, $\psi^{opt}$ is 1-Lipschitz.

(strong log-concavity: $-\nabla^2 \log \left( \frac{d\mu}{d\lambda}(x) \right) \succeq \text{Id}.$)
Brenier 87': For reasonable $\mu, \nu$ there exists an optimal transport map $\psi^{\text{opt}} : \mathbb{R}^d \to \mathbb{R}^d$, satisfying:

$$
\psi^{\text{opt}} = \arg \min_{\psi \ast \nu = \mu} \mathbb{E}_\nu \left[ \| \psi(x) - x \|^2 \right].
$$

Caffarelli 00': If $\nu = \gamma_d$ is the standard Gaussian and $\mu$ is more log-concave, $\psi^{\text{opt}}$ is 1-Lipschitz.

(strong log-concavity: $-\nabla^2 \log \left( \frac{d\mu}{dx}(x) \right) \succeq \text{Id}$.)
**Brenier 87’**: For reasonable $\mu, \nu$ there exists an optimal transport map $\psi^{\text{opt}} : \mathbb{R}^d \to \mathbb{R}^d$, satisfying:

$$\psi^{\text{opt}} = \operatorname{arg\,min}_{\psi_*\nu = \mu} \mathbb{E}_\nu \left[ \| \psi(x) - x \|^2 \right].$$

**Caffarelli 00’**: If $\nu = \gamma_d$ is the standard Gaussian and $\mu$ is more log-concave, $\psi^{\text{opt}}$ is 1-Lipschitz.

(strong log-concavity: $-\nabla^2 \log \left( \frac{d\mu}{dx}(x) \right) \succeq \text{Id}.$)
Gaussian log-Sobolev inequality (Gross 75’): For \( \gamma_d \) the standard Gaussian and any test function \( f \),

\[
\text{Ent}_{\gamma_d}(f^2) \leq \mathbb{E}_{\gamma_d}[\|\nabla f\|^2].
\]

**Theorem (Bakry-Emery 85’)**

If \( \mu \) is more log-concave than \( \gamma_d \), then \( C_{LS}(\mu) \leq 1 \).
Gaussian log-Sobolev inequality (Gross 75’): For $\gamma_d$ the standard Gaussian and any test function $f$, 

$$ \text{Ent}_{\gamma_d}(f^2) \leq \mathbb{E}_{\gamma_d} [\|\nabla f\|^2] . $$

Theorem (Bakry-Emery 85’)

If $\mu$ is more log-concave than $\gamma_d$, then $C_{LS}(\mu) \leq 1$. 

Further results

Caffarelli’s original result was extended in several directions, mostly when $\nu = \gamma_d$, and

- $\mu$ is a structured perturbation. Colombo, Figali, Jhaveri (2017), Colombo, Fathi (2019), and Neeman (2022)
- $\mu$ is log-concave with bounded support. Kolesnikov (2011) and M., Shenfeld (2021)
- $\mu$ is a Gaussian mixture. M., Shenfeld (2021) and Klartag, Putterman (2021)
- $\mu$ is isotropic and log-concave.* M., Shenfeld (2021)

All the above examples were known to satisfy log-Sobolev inequalities.
Caffarelli’s original result was extended in several directions, mostly when $\nu = \gamma d$, and

- $\mu$ is a structured perturbation. Colombo, Figali, Jhaveri (2017), Colombo, Fathi (2019), and Neeman (2022)
- $\mu$ is log-concave with bounded support. Kolesnikov (2011) and M., Shenfeld (2021)
- $\mu$ is a Gaussian mixture. M., Shenfeld (2021) and Klartag, Putterman (2021)
- $\mu$ is isotropic and log-concave.* M., Shenfeld (2021)

All the above examples were known to satisfy log-Sobolev inequalities.
Further results

Caffarelli’s original result was extended in several directions, mostly when $\nu = \gamma^d$, and

- $\mu$ is a structured perturbation. Colombo, Figali, Jhaveri (2017), Colombo, Fathi (2019), and Neeman (2022)
- $\mu$ is log-concave with bounded support. Kolesnikov (2011) and M., Shenfeld (2021)
- $\mu$ is a Gaussian mixture. M., Shenfeld (2021) and Klartag, Putterman (2021)
- $\mu$ is isotropic and log-concave.* M., Shenfeld (2021)

All the above examples were known to satisfy log-Sobolev inequalities.
Caffarelli’s original result was extended in several directions, mostly when $\nu = \gamma d$, and

- $\mu$ is a structured perturbation. Colombo, Figali, Jhaveri (2017), Colombo, Fathi (2019), and Neeman (2022)
- $\mu$ is log-concave with bounded support. Kolesnikov (2011) and M., Shenfeld (2021)
- $\mu$ is a Gaussian mixture. M., Shenfeld (2021) and Klartag, Putterman (2021)
- $\mu$ is isotropic and log-concave.* M., Shenfeld (2021)

All the above examples were known to satisfy log-Sobolev inequalities.
Caffarelli’s original result was extended in several directions, mostly when $\nu = \gamma_d$, and

- $\mu$ is a structured perturbation. Colombo, Figali, Jhaveri (2017), Colombo, Fathi (2019), and Neeman (2022)
- $\mu$ is log-concave with bounded support. Kolesnikov (2011)
- $\mu$ is a Gaussian mixture. M., Shenfeld (2021) and Klartag, Putterman (2021)
- $\mu$ is isotropic and log-concave.* M., Shenfeld (2021)

All the above examples were known to satisfy log-Sobolev inequalities.
If $\nu$ and $\mu$ are measures on a Riemannian manifold $(M, d)$ much less is known.

McCann 2001': For reasonable $\mu, \nu$ there exists an optimal transport map $\psi^{\text{opt}} : M \to M$, satisfying:

$$
\psi^{\text{opt}} = \arg \min_{\psi \ast \nu = \mu} \mathbb{E}_\nu \left[ d(\psi(x), x)^2 \right].
$$

Moreover, if $\mu, \nu$ have full support, then $\psi^{\text{opt}}$ is Lipschitz.

**Question**

Is there an analogue of Caffarelli’s theorem for manifolds?
If $\nu$ and $\mu$ are measures on a Riemannian manifold $(M, d)$ much less is known.

McCann 2001': For reasonable $\mu, \nu$ there exists an optimal transport map $\psi^{\text{opt}} : M \to M$, satisfying:

$$
\psi^{\text{opt}} = \arg \min_{\psi^* \nu = \mu} \mathbb{E}_\nu \left[ d(\psi(x), x)^2 \right].
$$

Moreover, if $\mu, \nu$ have full support, then $\psi^{\text{opt}}$ is Lipschitz.

**Question**

Is there an analogue of Caffarelli’s theorem for manifolds?
If $\nu$ and $\mu$ are measures on a Riemannian manifold $(M, d)$, much less is known.

McCann 2001': For reasonable $\mu$, $\nu$ there exists an optimal transport map $\psi^{opt} : M \rightarrow M$, satisfying:

$$\psi^{opt} = \arg \min_{\psi^* \nu = \mu} \mathbb{E}_{\psi} \left[ d(\psi(x), x)^2 \right].$$

Moreover, if $\mu$, $\nu$ have full support, then $\psi^{opt}$ is Lipschitz.

**Question**

Is there an analogue of Caffarelli’s theorem for manifolds?
Beyond Euclidean spaces - example

Consider $M$ the round sphere with $\nu$ as its uniform probability measure. Let $\mu$ be uniform on a hemisphere, $\{(x_1, \ldots x_d) \in M | x_1 > 0\}$. The optimal transport map should not be Lipschitz in this case.
Consider $M$ the round sphere with $\nu$ as its uniform probability measure. Let $\mu$ be uniform on a hemisphere, $$\{(x_1, \ldots x_d) \in M|x_1 > 0\}.$$ The optimal transport map should not be Lipschitz in this case.
When should Lipschitz transport maps exist?

**Question**

For a given \( \nu \), for which target \( \mu \) should we expect to have Lipschitz transport maps?

Rough intuition: the target measure \( \mu \) should be more “concentrated” than the source measure \( \nu \).

- \( \mu \) is more log-concave than \( \nu \).
- \( \mu \) is supported on a smaller set than \( \nu \).
- \( \mu \) is a mixture of \( \nu \).
- \( \mu \) is a bounded perturbation of \( \nu \).
- \( \mu \) is a log-Lipschitz perturbation of \( \nu \). **Today** (i.e., \( d\nu = e^{-W} d\mu \) with \( W \) Lipschitz).
When should Lipschitz transport maps exist?

Question

For a given $\nu$, for which target $\mu$ should we expect to have Lipschitz transport maps?

Rough intuition: the target measure $\mu$ should be more “concentrated” than the source measure $\nu$.

- $\mu$ is more log-concave than $\nu$.
- $\mu$ is supported on a smaller set than $\nu$.
- $\mu$ is a mixture of $\nu$.
- $\mu$ is a bounded perturbation of $\nu$.
- $\mu$ is a log-Lipschitz perturbation of $\nu$. Today (i.e., $d\nu = e^{-W} d\mu$ with $W$ Lipschitz).
Question

For a given $\nu$, for which target $\mu$ should we expect to have Lipschitz transport maps?

Rough intuition: the target measure $\mu$ should be more “concentrated” than the source measure $\nu$.

- $\mu$ is more log-concave than $\nu$.
- $\mu$ is supported on a smaller set than $\nu$.
- $\mu$ is a mixture of $\nu$.
- $\mu$ is a bounded perturbation of $\nu$.
- $\mu$ is a log-Lipschitz perturbation of $\nu$. Today (i.e., $d\nu = e^{-W} d\mu$ with $W$ Lipschitz).
When should Lipschitz transport maps exist?

**Question**
For a given $\nu$, for which target $\mu$ should we expect to have Lipschitz transport maps?

Rough intuition: the target measure $\mu$ should be more “concentrated” than the source measure $\nu$.

- $\mu$ is more log-concave than $\nu$.
- $\mu$ is supported on a smaller set than $\nu$.
- $\mu$ is a mixture of $\nu$.
- $\mu$ is a bounded perturbation of $\nu$.
- $\mu$ is a log-Lipschitz perturbation of $\nu$. Today (i.e., $d\nu = e^{-W} d\mu$ with $W$ Lipschitz).
When should Lipschitz transport maps exist?

**Question**

For a given \( \nu \), for which target \( \mu \) should we expect to have Lipschitz transport maps?

Rough intuition: the target measure \( \mu \) should be more “concentrated” than the source measure \( \nu \).

- \( \mu \) is more log-concave than \( \nu \).
- \( \mu \) is supported on a smaller set than \( \nu \).
- \( \mu \) is a mixture of \( \nu \).
- \( \mu \) is a bounded perturbation of \( \nu \).
- \( \mu \) is a log-Lipschitz perturbation of \( \nu \). **Today** (i.e., \( d\nu = e^{-W} d\mu \) with \( W \) Lipschitz).
When should Lipschitz transport maps exist?

**Question**

For a given $\nu$, for which target $\mu$ should we expect to have Lipschitz transport maps?

Rough intuition: the target measure $\mu$ should be more “concentrated” than the source measure $\nu$.

- $\mu$ is more log-concave than $\nu$.
- $\mu$ is supported on a smaller set than $\nu$.
- $\mu$ is a mixture of $\nu$.
- $\mu$ is a bounded perturbation of $\nu$.
- $\mu$ is a log-Lipschitz perturbation of $\nu$. Today (i.e., $d\nu = e^{-W} d\mu$ with $W$ Lipschitz).
When should Lipschitz transport maps exist?

Question

For a given \( \nu \), for which target \( \mu \) should we expect to have Lipschitz transport maps?

Rough intuition: the target measure \( \mu \) should be more “concentrated” than the source measure \( \nu \).

- \( \mu \) is more log-concave than \( \nu \).
- \( \mu \) is supported on a smaller set than \( \nu \).
- \( \mu \) is a mixture of \( \nu \).
- \( \mu \) is a bounded perturbation of \( \nu \).
- \( \mu \) is a log-Lipschitz perturbation of \( \nu \).  **Today** (i.e., \( d\nu = e^{-W} d\mu \) with \( W \) Lipschitz).  

What to expect?

Suppose that $\nu = \gamma_d$ and that $\frac{d\mu}{d\gamma_d} = e^{-W}$, with $W$ $L$-Lipschitz. 

**Miclo’s trick:** $\mu$ satisfies a log-Sobolev inequality with constant $e^{\sqrt{d}L^2}$. The proof decomposes $W = \text{bounded} + \text{concave}$ and then invokes Holley-Stroock.

**Lower bound:** If $W(x) = L|x|$ is straightforward to show that $\mu$ satisfies a log-Sobolev inequality with constant $e^{L^2}$. 
What to expect?

Suppose that \( \nu = \gamma_d \) and that \( \frac{d\mu}{d\gamma_d} = e^{-W} \), with \( W \) \( L \)-Lipschitz.

**Miclo’s trick:** \( \mu \) satisfies a log-Sobolev inequality with constant \( e^{\sqrt{d}L^2} \). The proof decomposes \( W = \text{bounded} + \text{concave} \) and then invokes Holley-Stroock.

**Lower bound:** If \( W(x) = L|x| \) is is straightforward to show that \( \mu \) satisfies a log-Sobolev inequality with constant \( e^{L^2} \).
Theorem

Theorem (informal)

Let $\nu$ and $\mu$ be two measures on a Riemannian manifold $(M, d)$. Assume that $(M, d, \nu)$ satisfies an appropriate curvature assumption and that $\mu$ is an $L$-log-Lipschitz perturbation of $\nu$. Then, there exists a transport with Lipschitz constant $e^{eL^2}$.

Moreover, if $M \in \{\mathbb{R}^d, \mathbb{S}^d\}$ then the Lipschitz constant can be improved to $e^{L^2}$.

Theorem (Improved Miclo’s trick)

Let $\nu = \gamma_d$ and $\mu$ as above. Then, $C_{LS}(\mu) \leq e^{L^2}$.
Theorem (informal)

Let $\nu$ and $\mu$ be two measures on a Riemannian manifold $(M, d)$. Assume that $(M, d, \nu)$ satisfies an appropriate curvature assumption and that $\mu$ is an $L$-log-Lipschitz perturbation of $\nu$. Then, there exists a transport with Lipschitz constant $e^{e^{L^2}}$.

Moreover, if $M \in \{\mathbb{R}^d, S^d\}$ then the Lipschitz constant can be improved to $e^{L^2}$.

Theorem (Improved Miclo’s trick)

Let $\nu = \gamma_d$ and $\mu$ as above. Then, $C_{LS}(\mu) \leq e^{L^2}$.
Theorem (informal)

Let $\nu$ and $\mu$ be two measures on a Riemannian manifold $(M, d)$. Assume that $(M, d, \nu)$ satisfies an appropriate curvature assumption and that $\mu$ is an $L$-log-Lipschitz perturbation of $\nu$. Then, there exists a transport with Lipschitz constant $e^{L^2}$.

Moreover, if $M \in \{\mathbb{R}^d, \mathbb{S}^d\}$ then the Lipschitz constant can be improved to $e^{L^2}$.

Theorem (Improved Miclo’s trick)

Let $\nu = \gamma_d$ and $\mu$ as above. Then, $C_{LS}(\mu) \leq e^{L^2}$. 
Kim and E. Milman (2012) were the first to consider transportation along Langevin dynamics, building on the work of Otto, Villani (2000). In particular, they were the first to consider Lipschitz properties.

Rather than constructing the transport map at once as a solution to an optimization problem, the map is constructed infinitesimally along the Langevin dynamics.
Kim and E. Milman (2012) were the first to consider transportation along Langevin dynamics, building on the work of Otto, Villani (2000). In particular, they were the first to consider Lipschitz properties.

Rather than constructing the transport map at once as a solution to an optimization problem, the map is constructed infinitesimally along the Langevin dynamics.
Our approach - Transportation along Langevin dynamics

Kim and E. Milman (2012) were the first to consider transportation along Langevin dynamics, building on the work of Otto, Villani (2000). In particular, they were the first to consider Lipschitz properties.

Rather than constructing the transport map at once as a solution to an optimization problem, the map is constructed infinitesimally along the Langevin dynamics.
Transportation along Langevin dynamics

Let \((X_t)_{t \geq 0}\) be the Langevin process:

\[
dX_t = \nabla \log \left( \frac{d\nu}{dx} \right) (X_t)dt + \sqrt{2}dB_t, \quad X_0 \sim \mu,
\]

with \((B_t)_{t \geq 0}\) a Brownian motion.

\[P_t \eta(x) := E[\eta(X_t)|X_0 = x]\]

Langevin semigroup.

\[
\rho_t := P_t \left( \frac{d\mu}{d\nu} \right) d\nu = \text{Law}(X_t)
\]

is a path of measures interpolating between \(\rho_0 = \mu\) to \(\rho_\infty = \nu\).
Let \((X_t)_{t \geq 0}\) be the Langevin process:

\[ dX_t = \nabla \log \left( \frac{d\nu}{dx} \right) (X_t) dt + \sqrt{2} dB_t, \quad X_0 \sim \mu, \]

with \((B_t)_{t \geq 0}\) a Brownian motion.

\[ P_t \eta(x) := E[\eta(X_t)|X_0 = x] \]  
Langevin semigroup.

\[ \rho_t := P_t \left( \frac{d\mu}{d\nu} \right) d\nu = \text{Law}(X_t) \]

is a path of measures interpolating between \(\rho_0 = \mu\) to \(\rho_\infty = \nu\).
Transportation along Langevin dynamics

Let \((X_t)_{t \geq 0}\) be the Langevin process:

\[ dX_t = \nabla \log \left( \frac{d\nu}{dx} \right) (X_t) dt + \sqrt{2} dB_t, \quad X_0 \sim \mu, \]

with \((B_t)_{t \geq 0}\) a Brownian motion.

\[ P_t \eta(x) := E[\eta(X_t)|X_0 = x] \quad \text{Langevin semigroup.} \]

\[ \rho_t := P_t \left( \frac{d\mu}{d\nu} \right) d\nu = \text{Law}(X_t) \]

is a path of measures interpolating between \(\rho_0 = \mu\) to \(\rho_\infty = \nu\).
The continuity equation

Recall

\[ \rho_t := P_t \left( \frac{d\mu}{d\nu} \right) d\nu = \text{Law}(X_t). \]

The Langevin path \((\rho_t)\) satisfies the continuity equation

\[ \partial_t \rho_t + \nabla \cdot (-V_t \rho_t) = 0. \]

\[ \partial_t \rho_t + \nabla \cdot \left( -\nabla \log P_t \left( \frac{d\mu}{d\nu} \right) \rho_t \right) = 0. \]

\[ = V_t \]
The continuity equation

Recall

\[ \rho_t := P_t \left( \frac{d\mu}{d\nu} \right) d\nu = \text{Law}(X_t). \]

The Langevin path \((\rho_t)\) satisfies the continuity equation

\[ \partial_t \rho_t + \nabla \cdot (-V_t \rho_t) = 0. \]

\[ \partial_t \rho_t + \nabla \cdot \left( -\nabla \log P_t \left( \frac{d\mu}{d\nu} \right) \rho_t \right) = 0. \]

\[ = V_t \]
Define the family of diffeomorphisms $S_t : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\partial_t S_t(x) = V_t(S_t(x)), \quad S_0(x) = x.$$

$S_t$ transports $\mu = \rho_0$ to $\rho_t$ and $T_t := S_t^{-1}$ transports $\rho_t$ to $\rho_0 = \mu$.

The transport map along Langevin dynamics is

$$T_{LVN} := \lim_{t \to \infty} T_t$$

transporting $\nu = \rho_{\infty}$ to $\rho_0 = \mu$. 
Define the family of diffeomorphisms \( S_t : \mathbb{R}^d \to \mathbb{R}^d \) by

\[
\partial_t S_t(x) = V_t(S_t(x)), \quad S_0(x) = x.
\]

\( S_t \) transports \( \mu = \rho_0 \) to \( \rho_t \) and \( T_t := S_t^{-1} \) transports \( \rho_t \) to \( \rho_0 = \mu \).

The transport map along Langevin dynamics is

\[
T_{LVN} := \lim_{t \to \infty} T_t \quad \text{transporting} \quad \nu = \rho_\infty \quad \text{to} \quad \rho_0 = \mu.
\]
Lipschitz properties of $T_{LVN}$

Recall

$$\partial_t S_t(x) = V_t(S_t(x)), \quad S_0(x) = x$$

so

$$\partial_t \nabla S_t(x) = \nabla V_t(S_t(x)) \nabla S_t(x).$$

Lemma

The Lipschitz constant of $T_{LVN}$ is at most

$$\exp \left( \int_0^\infty \sup_x \lambda_{\text{max}} (-\nabla V_t) \, dt \right).$$

Hence, the key point is to bound $-\nabla V_t = \nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$. 
Lipschitz properties of $T_{LVN}$

Recall

$$\partial_t S_t(x) = V_t(S_t(x)), \quad S_0(x) = x$$

so

$$\partial_t \nabla S_t(x) = \nabla V_t(S_t(x)) \nabla S_t(x).$$

**Lemma**

*The Lipschitz constant of $T_{LVN}$ is at most $\exp \left( \int_0^\infty \sup_x \lambda_{\text{max}} (-\nabla V_t) \, dt \right).$*

Hence, the **key point** is to bound $-\nabla V_t = \nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right).$
Recall

\[ \partial_t S_t(x) = V_t(S_t(x)), \quad S_0(x) = x \]

so

\[ \partial_t \nabla S_t(x) = \nabla V_t(S_t(x)) \nabla S_t(x). \]

**Lemma**

The Lipschitz constant of \( T_{\text{LVN}} \) is at most

\[ \exp \left( \int_0^\infty \sup_x \lambda_{\text{max}} \left( -\nabla V_t \right) dt \right). \]

Hence, the key point is to bound \(-\nabla V_t = \nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)\).
Lipschitz properties of $T_{LVN}$

Recall

$$\partial_t S_t(x) = V_t(S_t(x)), \quad S_0(x) = x$$

so

$$\partial_t \nabla S_t(x) = \nabla V_t(S_t(x)) \nabla S_t(x).$$

**Lemma**

The Lipschitz constant of $T_{LVN}$ is at most

$$\exp \left( \int_0^\infty \sup_x \lambda_{\text{max}} \left( -\nabla V_t \right) dt \right).$$

Hence, the key point is to bound $-\nabla V_t = \nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$. 
Lipschitz properties of $T_{LVN}$

Recall

$$\partial_t S_t(x) = V_t(S_t(x)), \quad S_0(x) = x$$

so

$$\partial_t \nabla S_t(x) = \nabla V_t(S_t(x)) \nabla S_t(x).$$

**Lemma**

The Lipschitz constant of $T_{LVN}$ is at most

$$\exp \left( \int_0^\infty \sup_x \lambda_{\text{max}} (-\nabla V_t) \, dt \right).$$

Hence, the **key point** is to bound $-\nabla V_t = \nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$. 
Known bounds on $\nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$:

- $\mu$ is more log-concave than $\nu = \text{Gaussian}$ [Kim and E. Milman (2012)]. The Ornstein-Uhlenbeck semigroup $(P_t)$ preserves log-concavity.

- $\nu = \text{Gaussian}$ and $\mu = \text{log-concave with compact support}$ [M., Shenfeld (2022)]. $\nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$ can be written as a covariance matrix. Deduce two bounds: one for compact support and one for log-concavity. Optimize bounds.
Known bounds on $\nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$:

- $\mu$ is more log-concave than $\nu = \text{Gaussian}$ [Kim and E. Milman (2012)]. The Ornstein-Uhlenbeck semigroup $(P_t)$ preserves log-concavity.

- $\nu = \text{Gaussian}$ and $\mu = \text{log-concave with compact support}$ [M., Shenfeld (2022)]. $\nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$ can be written as a covariance matrix. Deduce two bounds: one for compact support and one for log-concavity. Optimize bounds.
Known bounds on $\nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$:

- $\mu$ is more log-concave than $\nu = \text{Gaussian}$ [Kim and E. Milman (2012)]. The Ornstein-Uhlenbeck semigroup $(P_t)$ preserves log-concavity.

- $\nu = \text{Gaussian}$ and $\mu = \text{log-concave with compact support}$ [M., Shenfeld (2022)]. $\nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$ can be written as a covariance matrix. Deduce two bounds: one for compact support and one for log-concavity. Optimize bounds.
Examples of upper bounds

Known bounds on $\nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$:

- $\mu$ is more log-concave than $\nu = \text{Gaussian}$ [Kim and E. Milman (2012)]. The Ornstein-Uhlenbeck semigroup $(P_t)$ preserves log-concavity.

- $\nu = \text{Gaussian}$ and $\mu = \text{log-concave with compact support}$ [M., Shenfeld (2022)]. $\nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$ can be written as a covariance matrix. Deduce two bounds: one for compact support and one for log-concavity. Optimize bounds.
Examples of upper bounds

Known bounds on $\nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$:

- $\mu$ is more log-concave than $\nu = \text{Gaussian}$ [Kim and E. Milman (2012)]. The Ornstein-Uhlenbeck semigroup $(P_t)$ preserves log-concavity.

- $\nu = \text{Gaussian}$ and $\mu = \text{log-concave with compact support}$ [M., Shenfeld (2022)]. $\nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$ can be written as a covariance matrix. Deduce two bounds: one for compact support and one for log-concavity. Optimize bounds.
Examples of upper bounds

Known bounds on $\nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$:

- $\mu$ is more log-concave than $\nu = \text{Gaussian}$ [Kim and E. Milman (2012)]. The Ornstein-Uhlenbeck semigroup ($P_t$) preserves log-concavity.

- $\nu = \text{Gaussian}$ and $\mu = \text{log-concave with compact support}$ [M., Shenfeld (2022)]. $\nabla^2 \log P_t \left( \frac{d\mu}{d\nu}(x) \right)$ can be written as a covariance matrix. Deduce two bounds: one for compact support and one for log-concavity. Optimize bounds.
Theorem (Fathi, M., Shenfeld (Work in progress))

Let $\nu$ and $\mu$ be two measures on $\mathbb{R}^d$.

Assumptions for the source:

- Convexity: $\nu$ is $\kappa$ log-concave, $-\nabla^2 \log(\frac{d\nu}{dx}) \geq \kappa \text{Id}$.
- Third order regularity: $|\nabla^3 \log(\frac{d\nu}{dx})| \leq K$

Assumptions for the target:

- Log-Lipschitz: $\mu$ is an $L$-log-Lipschitz, $|\nabla \log \frac{d\mu}{d\nu}| \leq L$.

Then: $T_{LVN}$ is $O \left( \exp \left( \frac{L^2}{\kappa} + \frac{L}{\sqrt{\kappa}} + \frac{LK}{\kappa^2} \right) \right)$-Lipschitz.
Theorem (Fathi, M., Shenfeld (Work in progress))

Let \( \nu \) and \( \mu \) be two measures on \( \mathbb{R}^d \).

Assumptions for the source:

- **Convexity:** \( \nu \) is \( \kappa \) log-concave, \(-\nabla^2 \log \left( \frac{d\nu}{dx} \right) \geq \kappa \text{Id.} \)
- **Third order regularity:** \( \left| \nabla^3 \log \left( \frac{d\nu}{dx} \right) \right| \leq K \)

Assumptions for the target:

- **Log-Lipschitz:** \( \mu \) is an \( L \)-log-Lipschitz, \( \left| \nabla \log \frac{d\mu}{d\nu} \right| \leq L \).

Then: \( T_{LVN} \) is \( O \left( \exp \left( \frac{L^2}{\kappa} + \frac{L}{\sqrt{\kappa}} + \frac{LK}{\kappa^2} \right) \right) \)-Lipschitz.
Let $\nu$ and $\mu$ be two measures on $\mathbb{R}^d$.

Assumptions for the source:

- Convexity: $\nu$ is $\kappa$ log-concave, $-\nabla^2 \log \left( \frac{d\nu}{dx} \right) \geq \kappa \text{Id}$.
- Third order regularity: $|\nabla^3 \log \left( \frac{d\nu}{dx} \right)| \leq K$

Assumptions for the target:

- Log-Lipschitz: $\mu$ is an $L$-log-Lipschitz, $|\nabla \log \frac{d\mu}{d\nu}| \leq L$.

Then: $T_{\nu\mu}^{LVN}$ is $O \left( \exp \left( \frac{L^2}{\kappa} + \frac{L}{\sqrt{\kappa}} + \frac{LK}{\kappa^2} \right) \right)$-Lipschitz.
Bismut’s formula

For \( f = \frac{d\mu}{d\nu} \), integration by parts on Wiener space (Malliavin calculus) ⇒

\[
\nabla^2 P_t f(x) = \nabla^2 \mathbb{E} [f(X_t^x)] = \frac{1}{t\sqrt{2}} \mathbb{E} \left[ \nabla f(X_t^x) \nabla X_t^x \int_0^t \langle \nabla X_s, dB_s \rangle \right]
\]

\[+ \frac{1}{t} \int_0^t \mathbb{E} \left[ \nabla P_{t-s} f(X_s^x) \nabla^2 X_s^x \right] ds.
\]

where

\[
\nabla_u X_t^x := \lim_{\varepsilon \downarrow 0} \frac{X_t^{x+\varepsilon u} - X_t^x}{\varepsilon} \in \mathbb{R}^d,
\]

\[
\nabla^2_{u,v} X_t^x := \lim_{\varepsilon \downarrow 0} \frac{\nabla_v X_t^{x+\varepsilon u} - \nabla_v X_t^x}{\varepsilon} \in \mathbb{R}^d.
\]
Bismut’s formula

For $f = \frac{d\mu}{d\nu}$, integration by parts on Wiener space (Malliavin calculus) \Rightarrow

$$\nabla^2 P_t f(x) = \nabla^2 \mathbb{E} [f(X_t^x)] = \frac{1}{t\sqrt{2}} \mathbb{E} \left[ \nabla f(X_t^x) \nabla X_t^x \int_0^t \langle \nabla X_s, dB_s \rangle \right]$$

$$+ \frac{1}{t} \int_0^t \mathbb{E} \left[ \nabla P_{t-s} f(X_s^x) \nabla^2 X_s^x \right] ds.$$  

where

$$\nabla_u X_t^x := \lim_{\varepsilon \downarrow 0} \frac{X_t^{x+\varepsilon u} - X_t^x}{\varepsilon} \in \mathbb{R}^d,$$

$$\nabla^2_{u, v} X_t^x := \lim_{\varepsilon \downarrow 0} \frac{\nabla_v X_t^{x+\varepsilon u} - \nabla_v X_t^x}{\varepsilon} \in \mathbb{R}^d.$$
Under the hood

We need to upper bound: $\nabla f(X_t^x), \nabla X_t^x, \nabla^2 X_t^x$ and $\int_0^t \langle \nabla X_s, dB_s \rangle$.

**Relative density** $\nabla f(X_t^x)$: Use $L$-log-Lipschitz assumption.

**First variation** $\nabla X_s$: Use $\kappa$-log-concavity.

**Second variation** $\nabla^2 X_s$: Use $\kappa$-log-concavity + $K$ bound on 3rd derivative of $\log \frac{d\nu}{dx}$.

**Martingale** $\int_0^t \langle \nabla X_s, dB_s \rangle$: The correct bound is the key for sharp result.
We need to upper bound: $\nabla f(X_t^x), \nabla X_t^x, \nabla^2 X_t^x$ and $\int_0^t \langle \nabla X_s, dB_s \rangle$.

**Relative density $\nabla f(X_t^x)$:** Use $L$-log-Lipschitz assumption.

**First variation $\nabla X_s$:** Use $\kappa$-log-concavity.

**Second variation $\nabla^2 X_s$:** Use $\kappa$-log-concavity + $K$ bound on 3rd derivative of log $\frac{d\nu}{dx}$.

**Martingale $\int_0^t \langle \nabla X_s, dB_s \rangle$:** The correct bound is the key for sharp result.
We need to upper bound: $\nabla f(X_t^x), \nabla X_t^x, \nabla^2 X_t^x$ and $
abla \int_0^t \langle \nabla X_s, dB_s \rangle$.

Relative density $\nabla f(X_t^x)$: Use $L$-log-Lipschitz assumption.

First variation $\nabla X_s$: Use $\kappa$-log-concavity.

Second variation $\nabla^2 X_s$: Use $\kappa$-log-concavity + $K$ bound on 3rd derivative of log $d\nu dx$.

Martingale $\int_0^t \langle \nabla X_s, dB_s \rangle$: The correct bound is the key for sharp result.
We need to upper bound: $\nabla f(X^x_t)$, $\nabla X^x_t$, $\nabla^2 X^x_t$ and $\int_0^t \langle \nabla X_s, dB_s \rangle$.

**Relative density** $\nabla f(X^x_t)$: Use $L$-log-Lipschitz assumption.

**First variation** $\nabla X_s$: Use $\kappa$-log-concavity.

**Second variation** $\nabla^2 X_s$: Use $\kappa$-log-concavity + $K$ bound on 3rd derivative of log $\frac{d\nu}{dx}$.

**Martingale** $\int_0^t \langle \nabla X_s, dB_s \rangle$: The correct bound is the key for sharp result.
Under the hood

We need to upper bound: $\nabla f(X^x_t)$, $\nabla X^x_t$, $\nabla^2 X^x_t$ and $\int_0^t \langle \nabla X_s, dB_s \rangle$.

**Relative density $\nabla f(X^x_t)$**: Use $L$-log-Lipschitz assumption.

**First variation $\nabla X_s$**: Use $\kappa$-log-concavity.

**Second variation $\nabla^2 X_s$**: Use $\kappa$-log-concavity + $K$ bound on 3rd derivative of log $\frac{d\nu}{dx}$.

**Martingale $\int_0^t \langle \nabla X_s, dB_s \rangle$**: The correct bound is the key for sharp result.
Recall

\[ dX_t^x = \nabla \log \left( \frac{d\nu}{dx} \right) (X_t^x) dt + \sqrt{2} dB_t, \quad X_0^x = x. \]

Differentiate to get

\[ \frac{d}{dt} \nabla X_t^x = \nabla^2 \log \left( \frac{d\nu}{dx} \right) (X_t^x) \nabla X_t^x, \quad \nabla X_0^x = \text{Id}. \]

So, \( \nabla X_t^x \) can be controlled since \( -\nabla^2 \log \left( \frac{d\nu}{dx} \right) \geq \kappa \text{Id}. \)
First variation

Recall

\[ dX_t^x = \nabla \log \left( \frac{d\nu}{dx} \right) (X_t^x) dt + \sqrt{2} dB_t, \quad X_0^x = x. \]

Differentiate to get

\[ \frac{d}{dt} \nabla X_t^x = \nabla^2 \log \left( \frac{d\nu}{dx} \right) (X_t^x) \nabla X_t^x, \quad \nabla X_0^x = \text{Id}. \]

So, \( \nabla X_t^x \) can be controlled since \(-\nabla^2 \log \left( \frac{d\nu}{dx} \right) \geq \kappa \text{Id} \).
Second variation

Recall

\[
\frac{d}{dt} \nabla X_t^x = \nabla^2 \log \left( \frac{d\nu}{dx} \right) (X_t^x) \nabla X_t^x, \quad \nabla X_0^x = \text{Id}.
\]

Differentiate to get

\[
\frac{d}{dt} \nabla^2 X_t^x = \nabla^3 \log \left( \frac{d\mu}{dx} \right) (X_t^x)(\nabla X_t^x, \nabla X_t^x) + \nabla^2 \log \left( \frac{d\mu}{dx} \right) (X_t^x) \nabla^2 X_t^x.
\]

So, \( \nabla^2 X_t^x \) can be controlled since \( |\nabla^3 \log \left( \frac{d\mu}{dx} \right)| \leq K \) and \(-\nabla^2 \log \left( \frac{d\mu}{dx} \right) \geq \kappa \text{Id.}\)
Recall

\[ \frac{d}{dt} \nabla X_t^x = \nabla^2 \log \left( \frac{d\nu}{dx} \right) (X_t^x) \nabla X_t^x, \quad \nabla X_0^x = \text{Id}. \]

Differentiate to get

\[ \frac{d}{dt} \nabla^2 X_t^x = \nabla^3 \log \left( \frac{d\mu}{dx} \right) (X_t^x)(\nabla X_t^x, \nabla X_t^x) \]
\[ + \nabla^2 \log \left( \frac{d\mu}{dx} \right) (X_t^x) \nabla^2 X_t^x. \]

So, \( \nabla^2 X_t^x \) can be controlled since \( \left| \nabla^3 \log \left( \frac{d\mu}{dx} \right) \right| \leq K \) and \( -\nabla^2 \log \left( \frac{d\mu}{dx} \right) \geq \kappa \text{Id}. \)
Recall Bismut’s formula:

\[
\nabla^2 P_t f(x) = \frac{1}{t\sqrt{2}} \mathbb{E} \left[ \nabla f(X_t^x) \nabla X_t^x \int_0^t \langle \nabla X_s, dB_s \rangle \right] \\
+ \frac{1}{t} \int_0^t \mathbb{E} \left[ \nabla P_{t-s} f(X_s^x) \nabla^2 X_s^x \right] ds.
\]

Need to control \( \mathbb{E} \left[ \nabla f(X_t^x) \nabla X_t^x \int_0^t \langle \nabla X_s, dB_s \rangle \right] \). If we control \( \nabla f(X_t^x) \nabla X_t^x \) and \( \int_0^t \langle \nabla X_s, dB_s \rangle \) separately we get sub-optimal results. Instead, a more refined analysis is needed to get the sharp results.
Recall Bismut’s formula:

\[ \nabla^2 P_t f(x) = \frac{1}{t \sqrt{2}} \mathbb{E} \left[ \nabla f(X_t^x) \nabla X_t^x \int_0^t \langle \nabla X_s, dB_s \rangle \right] + \frac{1}{t} \int_0^t \mathbb{E} \left[ \nabla P_{t-s} f(X_s^x) \nabla^2 X_s^x \right] ds. \]

Need to control \( \mathbb{E} \left[ \nabla f(X_t^x) \nabla X_t^x \int_0^t \langle \nabla X_s, dB_s \rangle \right] \). If we control \( \nabla f(X_t^x) \nabla X_t^x \) and \( \int_0^t \langle \nabla X_s, dB_s \rangle \) separately we get sub-optimal results. Instead, a more refined analysis is needed to get the sharp results.
The martingale

Recall Bismut’s formula:

\[
\nabla^2 P_t f(x) = \frac{1}{t \sqrt{2}} \mathbb{E} \left[ \nabla f(X_t^x) \nabla X_t^x \int_0^t \langle \nabla X_s, dB_s \rangle \right] \\
+ \frac{1}{t} \int_0^t \mathbb{E} \left[ \nabla P_{t-s} f(X_s^x) \nabla^2 X_s^x \right] ds.
\]

Need to control \( \mathbb{E} \left[ \nabla f(X_t^x) \nabla X_t^x \int_0^t \langle \nabla X_s, dB_s \rangle \right] \). If we control \( \nabla f(X_t^x) \nabla X_t^x \) and \( \int_0^t \langle \nabla X_s, dB_s \rangle \) separately we get sub-optimal results. Instead, a more refined analysis is needed to get the sharp results.
Let $\nu$ and $\mu$ be two measures on a Riemannian manifold $(M, d)$. 

Assumptions for the source:

- **Convexity**: $(M, d, \nu)$ is $\text{CD}(\kappa, \infty)$,
  $$\text{Ric}_M - \nabla^2 \log(\frac{d\nu}{d\text{Vol}}) \geq \kappa \text{Id}.$$
- **Third order regularity**: $|\nabla^3 \log(\frac{d\nu}{d\text{Vol}}) + \text{curvature}| \leq K$

Assumptions for the target:

- **Log-Lipschitz**: $\mu$ is an $L$-log-Lipschitz, $|\nabla \log\frac{d\mu}{d\nu}| \leq L$.

Then: $T_{LVN}$ is 

$$O \left( \exp \left( \frac{L^2}{\kappa} + \frac{L}{\sqrt{\kappa}} + \frac{LK}{\kappa^2} + e^{\frac{L^2}{\kappa}} \|\text{Riem}\|_{\infty} \right) \right)$$ - Lipschitz.

Curvature terms := $\nabla \text{Ric} + d^* \text{Riem} + \text{Riem}(\nabla \log(\frac{d\nu}{d\text{Vol}}))$. 
Theorem (Fathi, M., Shenfeld (Work in progress))

Let $\nu$ and $\mu$ be two measures on a Riemannian manifold $(M, d)$.

Assumptions for the source:

- **Convexity**: $(M, d, \nu)$ is $CD(\kappa, \infty)$,
  \[
  \text{Ric}_M - \nabla^2 \log \left( \frac{d\nu}{d\text{Vol}} \right) \geq \kappa \text{Id}.
  \]

- **Third order regularity**: $|\nabla^3 \log \left( \frac{d\nu}{d\text{Vol}} \right) + \text{curvature}| \leq K$

Assumptions for the target:

- **Log-Lipschitz**: $\mu$ is an $L$-log-Lipschitz, $|\nabla \log \frac{d\mu}{d\nu}| \leq L$.

Then: $T_{LVN}$ is

\[
O \left( \exp \left( \frac{L^2}{\kappa} + \frac{L}{\sqrt{\kappa}} + \frac{LK^2}{\kappa} + e^{\frac{L^2}{\kappa} \|\text{Riem}\|_{\infty}} \right) \right) - \text{Lipschitz}.
\]

Curvature terms := $\nabla \text{Ric} + d^* \text{Riem} + \text{Riem}(\nabla \log \left( \frac{d\nu}{d\text{Vol}} \right))$. 
Theorem (Fathi, M., Shenfeld (Work in progress))

Let \( \nu \) and \( \mu \) be two measures on a Riemannian manifold \((M, d)\).

Assumptions for the source:

- **Convexity**: \((M, d, \nu)\) is \(\text{CD}(\kappa, \infty)\),
  \[ \text{Ric}_M - \nabla^2 \log \left( \frac{d\nu}{d\text{Vol}} \right) \geq \kappa \text{Id}. \]
- **Third order regularity**: \( |\nabla^3 \log \left( \frac{d\nu}{d\text{Vol}} \right) + \text{curvature}| \leq K \)

Assumptions for the target:

- **Log-Lipschitz**: \( \mu \) is an \(L\)-log-Lipschitz,
  \[ |\nabla \log \frac{d\mu}{d\nu}| \leq L. \]

Then: \( T_{LVN} \) is
\[
O \left( \exp \left( \frac{L^2}{\kappa} + \frac{L}{\sqrt{\kappa}} + \frac{LK}{\kappa^2} + e^{\frac{L^2}{\kappa} \|\text{Riem}\|_\infty} \right) \right) - \text{Lipschitz}.
\]

Curvature terms := \( \nabla \text{Ric} + d^* \text{Riem} + \text{Riem}(\nabla \log \frac{d\nu}{d\text{Vol}}) \).
Theorem (Fathi, M., Shenfeld (Work in progress))

Let \( \nu \) and \( \mu \) be two measures on a Riemannian manifold \((M, d)\).

Assumptions for the source:

- Convexity: \((M, d, \nu)\) is \(CD(\kappa, \infty)\),
  \[ \text{Ric}_M - \nabla^2 \log \left( \frac{d\nu}{d\text{Vol}} \right) \geq \kappa \text{Id}. \]
- Third order regularity: \( |\nabla^3 \log \left( \frac{d\nu}{d\text{Vol}} \right) + \text{curvature} | \leq K \)

Assumptions for the target:

- Log-Lipschitz: \( \mu \) is an \( L \)-log-Lipschitz, \( |\nabla \log \frac{d\mu}{d\nu}| \leq L \).

Then: \( T_{LVN} \) is

\[
O \left( \exp \left( \frac{L^2}{\kappa} + \frac{L}{\sqrt{\kappa}} + \frac{L K}{\kappa^2} + e^{\frac{L^2}{\kappa}} \| \text{Riem} \|_{\infty} \right) \right) \text{-Lipschitz.}
\]

Curvature terms := \( \nabla \text{Ric} + d^* \text{Riem} + \text{Riem}(\nabla \log \left( \frac{d\nu}{d\text{Vol}} \right)) \).
Bismut’s formula on manifolds

A similar Bismut formula (properly interpreted), due to Cheng Thalmaier, and Wang also applies on manifolds:

\[ \nabla^2 P_t f(x) = \frac{1}{t \sqrt{2}} \mathbb{E} \left[ \nabla f(X^x_t) \nabla X^x_t \int_0^t \langle \nabla X_s, dB_s \rangle \right] \]
\[ + \frac{1}{t} \int_0^t \mathbb{E} \left[ \nabla P_{t-s} f(X^x_s) \nabla^2 X^x_s \right] ds \]
\[ + \text{curvature terms.} \]

Better control of the curvature terms, as in the sphere, can lead to better bounds.
A similar Bismut formula (properly interpreted), due to Cheng Thalmaier, and Wang also applies on manifolds:

\[ \nabla^2 P_t f(x) = \frac{1}{t\sqrt{2}} \mathbb{E} \left[ \nabla f(X^x_t) \nabla X^x_t \int_0^t \langle \nabla X_s, dB_s \rangle \right] \]

\[ + \frac{1}{t} \int_0^t \mathbb{E} \left[ \nabla P_{t-s} f(X^x_s) \nabla^2 X^x_s \right] ds \]

+ curvature terms.

Better control of the curvature terms, as in the sphere, can lead to better bounds.
Further Questions

- Is third order regularity necessary?
- Is the double exponential necessary?
- More generally, when should we expect the existence of Lipschitz transport maps on manifolds?
- Even more generally, can we characterize which measures can be coupled by Lipschitz maps? What about when the source measure is Gaussian?
Further Questions

- Is third order regularity necessary?
- Is the double exponential necessary?
- More generally, when should we expect the existence of Lipschitz transport maps on manifolds?
- Even more generally, can we characterize which measures can be coupled by Lipschitz maps? What about when the source measure is Gaussian?
Further Questions

- Is third order regularity necessary?
- Is the double exponential necessary?
- More generally, when should we expect the existence of Lipschitz transport maps on manifolds?
- Even more generally, can we characterize which measures can be coupled by Lipschitz maps? What about when the source measure is Gaussian?
Further Questions

- Is third order regularity necessary?
- Is the double exponential necessary?
- More generally, when should we expect the existence of Lipschitz transport maps on manifolds?
- Even more generally, can we characterize which measures can be coupled by Lipschitz maps? What about when the source measure is Gaussian?
Further Questions

- Is third order regularity necessary?
- Is the double exponential necessary?
- More generally, when should we expect the existence of Lipschitz transport maps on manifolds?
- Even more generally, can we characterize which measures can be coupled by Lipschitz maps? What about when the source measure is Gaussian?
Thank You