Lipschitz properties of transport maps under a log-Lipschitz condition

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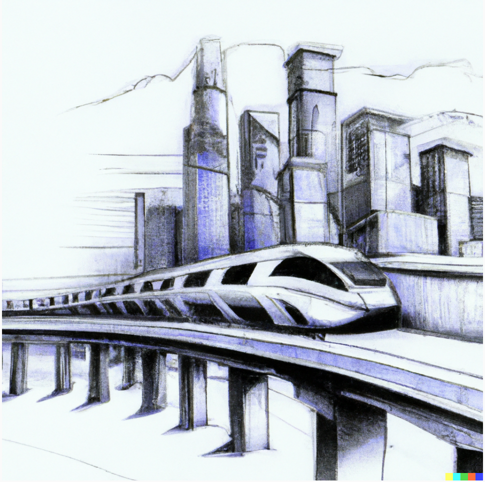
Joint work with Max Fathi and Yair Shenfeld

Transportation of measure

Given probability measures ν and μ, for now in $\mathbb{R}^d$, we seek a transport map T from ν to μ with "good properties".



Drawing of a train leaving a high-dimensional city (DALLL·E)

Transportation of measure

Transportation: T transports ν to μ if

$$X \sim \nu \quad \Rightarrow \quad T(X) \sim \mu,$$
$$\mu(A) = \nu(T^{-1}(A))$$

and in terms of densities

$$d\nu(x) = d\mu(T(x))|\det DT(x)|$$

Good properties: T should be L-Lipschitz:

$$|T(x) - T(y)| \leq L|x - y|,$$

and in terms of derivatives

$$|\nabla T(x)| \leq L.$$

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Transportation of functional inequalities

$*$ log-Sobolev: $\int f^2 \log(f) d\nu =: \mathrm{Ent}_\nu(f^2) \leq C_{LS}(\nu) \mathbb{E}_\nu \left[|\nabla f|^2 \right]$

Claim

Suppose ν satisfies a log-Sobolev* inequality with constant $C_{LS}(\nu)$.
Suppose there exist an L-Lipschitz map $T : \mathbb{R}^d \to \mathbb{R}^d$ which transports ν to μ. Then, μ satisfies a log-Sobolev inequality with constant $C_{LS}(\mu) \leq C_{LS}(\nu) L^2$

Proof.

$$\begin{aligned}\mathrm{Ent}_\mu(f^2) &= \mathrm{Ent}_\nu((f \circ T)^2) \leq C_{LS}(\nu) \mathbb{E}_\nu \left[|\nabla (f \circ T)|^2 \right] \\ &\leq C_{LS}(\nu) \mathbb{E}_\nu \left[|DT|^2 |\nabla f(T)|^2 \right] \leq C_{LS}(\nu) L^2 \mathbb{E}_\nu \left[|\nabla f(T)|^2 \right] \\ &= C_{LS}(\nu) L^2 \mathbb{E}_\mu \left[|\nabla f|^2 \right].\end{aligned}$$

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Optimal transport

Brenier 87': For reasonable μ, ν there exists an optimal transport map $\psi^{\mathrm{opt}} : \mathbb{R}^d \to \mathbb{R}^d$, satisfying:

$$\psi^{\mathrm{opt}} = \arg\min_{\psi_* \nu = \mu} \mathbb{E}_\nu \left[\|\psi(x) - x\|^2 \right].$$

Caffarelli 00': If $\nu = \gamma_d$ is the standard Gaussian and μ is more log-concave, ψ^{opt} is 1-Lipschitz.

(strong log-concavity: $-\nabla^2 \log \left(\frac{d\mu}{dx}(x) \right) \succeq \mathrm{Id}$.)

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Gaussian log-Sobolev inequality (Gross 75'): For γ_d the standard Gaussian and any test function f,

$$\mathrm{Ent}_{\gamma_d}(f^2) \leq \mathbb{E}_{\gamma_d}\left[\|\nabla f\|^2\right].$$

Theorem (Bakry-Emery 85')

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Further results

Caffarelli's original result was extended in several directions, mostly when $\nu = \gamma_d$, and

- μ is a structured perturbation. Colombo, Figali, Jhaveri (2017), Colombo, Fathi (2019), and Neeman (2022)
- μ is log-concave with bounded support. Kolesnikov (2011) and M., Shenfeld (2021)
- μ is a Gaussian mixture. M., Shenfeld (2021) and Klartag, Putterman (2021)
- μ is isotropic and log-concave.* M., Shenfeld (2021)

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Beyond Euclidean spaces

If ν and μ are measures on a Riemannian manifold (M, d) much less is known.

McCann 2001': For reasonable μ, ν there exists an optimal transport map $\psi^{\text{opt}} : M \to M$, satisfying:

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When should Lipschitz transport maps exist?

Question

For a given ν, for which target μ should we expect to have Lipschitz transport maps?

Rough intuition: the target measure μ should be more "concentrated" than the source measure ν.

- μ is more log-concave than ν.
- μ is supported on a smaller set than ν.
- μ is a mixture of ν.
- μ is a bounded perturbation of ν.
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What to expect?

Suppose that $\nu = \gamma_d$ and that $\frac{d\mu}{d\gamma_d} = e^{-W}$, with W L-Lipschitz.

Miclo's trick: μ satisfies a log-Sobolev inequality with constant $e^{\sqrt{d}L^2}$. The proof decomposes $W =$ bounded $+$ concave and then invokes Holley-Stroock.

Lower bound: If $W(x) = L|x|$ is is straightforward to show that μ satisfies a log-Sobolev inequality with constant e^{L^2}.

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Theorem (informal)

Let ν and μ be two measures on a Riemannian manifold (M, d). Assume that (M, d, ν) satisfies an appropriate curvature assumption and that μ is an L-log-Lipschitz perturbation of ν. Then, there exists a transport with Lipschitz constant $e^{e^{L^2}}$.

Moreover, if $M \in \{\mathbb{R}^d, \mathbb{S}^d\}$ then the Lipschitz constant can be improved to e^{L^2}.

Theorem (Improved Miclo's trick)

Let $\nu = \gamma_d$ and μ as above. Then, $C_{LS}(\mu) \leq e^{L^2}$.

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Our approach - Transportation along Langevin dynamics

Kim and E. Milman (2012) were the first to consider transportation along Langevin dynamics, building on the work of Otto, Villani (2000). In particular, they were the first to consider Lipschitz properties.

Rather than constructing the transport map at once as a solution to an optimization problem, the map is constructed infinitesimally along the Langevin dynamics.

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Transportation along Langevin dynamics

Let $(X_t)_{t\geq 0}$ be the Langevin process:

$$dX_t = \nabla \log\left(\frac{d\nu}{dx}\right)(X_t)dt + \sqrt{2}dB_t, \quad X_0 \sim \mu,$$

with $(B_t)_{t\geq 0}$ a Brownian motion.

$$P_t\eta(x) := E[\eta(X_t)|X_0 = x] \quad \text{Langevin semigroup.}$$

$$\rho_t := P_t\left(\frac{d\mu}{d\nu}\right)d\nu = \mathrm{Law}(X_t)$$

is a path of measures interpolating between $\rho_0 = \mu$ to $\rho_\infty = \nu$.

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The continuity equation

Recall

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The Langevin path (ρ_t) satisfies the continuity equation

$$\partial_t \rho_t + \nabla \cdot (-V_t \rho_t) = 0.$$

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S_t transports $\mu = \rho_0$ to ρ_t and $T_t := S_t^{-1}$ transports ρ_t to $\rho_0 = \mu$.

The transport map along Langevin dynamics is

$$T_{\text{LVN}} := \lim_{t \to \infty} T_t \quad \text{transporting } \nu = \rho_\infty \text{ to } \rho_0 = \mu.$$

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Lipschitz properties of T_{LVN}

Recall

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so

$$\partial_t \nabla S_t(x) = \nabla V_t(S_t(x)) \, \nabla S_t(x).$$

Lemma

The Lipschitz constant of T_{LVN} is at most $\exp\left(\int_0^\infty \sup_x \lambda_{\max}(-\nabla V_t)\, dt\right)$.

Hence, the **key point** is to bound $-\nabla V_t = \nabla^2 \log P_t\left(\frac{d\mu}{d\nu}(x)\right)$.

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Examples of upper bounds

Known bounds on $\nabla^2 \log P_t \left(\frac{d\mu}{d\nu}(x)\right)$:

- μ is more log-concave than $\nu =$ Gaussian [*Kim and E. Milman (2012)*]. The Ornstein-Uhlenbeck semigroup (P_t) preserves log-concavity.

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For $f = \frac{d\mu}{d\nu}$, integration by parts on Wiener space (Malliavin calculus) $\Rightarrow$

$$\nabla^2 P_t f(x) = \nabla^2 \mathbb{E}\left[f(X_t^x)\right] = \frac{1}{t\sqrt{2}} \mathbb{E}\left[\nabla f(X_t^x) \nabla X_t^x \int_0^t \langle \nabla X_s, dB_s \rangle\right] + \frac{1}{t}\int_0^t \mathbb{E}\left[\nabla P_{t-s} f(X_s^x) \nabla^2 X_s^x\right] ds.$$

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$$\nabla_u X_t^x := \lim_{\varepsilon \downarrow 0} \frac{X_t^{x+\varepsilon u} - X_t^x}{\varepsilon} \in \mathbb{R}^d,$$

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We need to upper bound: $\nabla f(X_t^x), \nabla X_t^x, \nabla^2 X_t^x$ and $\int_0^t \langle \nabla X_s, dB_s \rangle$.

Relative density $\nabla f(X_t^x)$: Use L-log-Lipschitz assumption.

First variation ∇X_s: Use κ-log-concavity.

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Recall

$$dX_t^x = \nabla \log \left(\frac{d\nu}{dx}\right)(X_t^x)dt + \sqrt{2}dB_t, \quad X_0^x = x.$$

Differentiate to get

$$\frac{d}{dt}\nabla X_t^x = \nabla^2 \log \left(\frac{d\nu}{dx}\right)(X_t^x)\nabla X_t^x, \quad \nabla X_0^x = \mathrm{Id}.$$

So, ∇X_t^x can be controlled since $-\nabla^2 \log\left(\frac{d\nu}{dx}\right) \geq \kappa \mathrm{Id}$.

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Theorem (Fathi, M., Shenfeld (Work in progress))

Let ν and μ be two measures on a Riemannian manifold (M, d).

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