Spectral gaps for measures on the cube via a generalized stochastic localization process

Dan Mikulincer

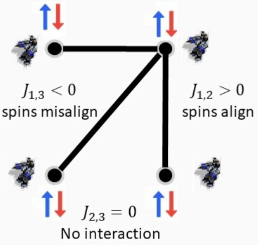
UW

Joint work with Arianna Piana

Ising Models

For a matrix J, consider $\mu(x) \propto \exp(\langle x, Jx\rangle)$, for $\{-1,1\}^n$.

Assigns state to particle system with pairwise interactions.

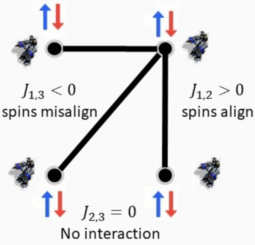


Question

How to sample from μ?

Ising Models

For a matrix J, consider $\mu(x) \propto \exp(\langle x, Jx\rangle)$, for $\{-1,1\}^n$.
Assigns state to particle system with pairwise interactions.

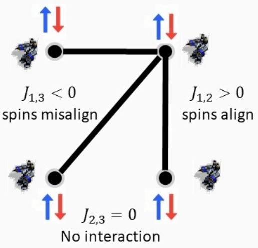


Question

How to sample from μ?

Ising Models

For a matrix J, consider $\mu(x) \propto \exp(\langle x, Jx \rangle)$, for $\{-1, 1\}^n$.
Assigns state to particle system with pairwise interactions.



Question

How to sample from μ?

Ising Models - Glauber Dynamics

Question

How to sample from μ?

Markov chain – construct $(X_t)_{t\geq 0}$ such that $\mathrm{Law}(X_t) \xrightarrow{t\to\infty} \mu$.

Glauber Dynamics – to go from X_t to X_{t+1}

1. Choose a random coordinate $i = 1, \ldots n$.
2. Update $X_{t,i}$ according to $\mu | X_t^{\hat{i}}$[1].

How efficient?

[1] $x^{\hat{i}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

Ising Models - Glauber Dynamics

Question

How to sample from μ?

Markov chain – construct $(X_t)_{t\geq 0}$ such that $\mathrm{Law}(X_t) \xrightarrow{t\to\infty} \mu$.

Glauber Dynamics – to go from X_t to X_{t+1}

1. Choose a random coordinate $i = 1, \ldots n$.
2. Update $X_{t,i}$ according to $\mu | X_t^{\hat{i}}$[1].

How efficient?

[1] $x^{\hat{i}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

Ising Models - Glauber Dynamics

Question

How to sample from μ?

Markov chain – construct $(X_t)_{t\geq 0}$ such that $\mathrm{Law}(X_t) \xrightarrow{t\to\infty} \mu$.

Glauber Dynamics – to go from X_t to X_{t+1}

1. Choose a random coordinate $i = 1, \ldots n$.
2. Update $X_{t,i}$ according to $\mu | X_t^{\hat{i}}$[1].

How efficient?

[1] $x^{\hat{i}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

Glauber Dynamics - Curie Weiss

Take $J = \frac{\beta}{n}\mathbb{1} \otimes \mathbb{1}$, so $\mu(x) \propto \exp\left(\frac{\beta}{n}\left(\sum x_i\right)^2\right)$.
Assume $X_0 = \mathbb{1}$ and consider the magnetization chain $S_t = \frac{1}{n}\langle X_t, \mathbb{1}\rangle$,

- $S_t \in [-1, 1]$
- $S_{t+1} - S_t \in \{-\frac{2}{n}, 0, \frac{2}{n}\}$.

$$p := p(\beta, m) = \mathbb{P}\left(S_{t+1} = S_t + 2 | S_t = m\right),$$

$$q := q(\beta, m) = \mathbb{P}\left(S_{t+1} = S_t - 2 | S_t = m\right).$$

$\frac{p}{q} \simeq \frac{(1-m)(1+\tanh(2\beta m))}{(1+m)(1-\tanh(2\beta m))}$ measures the positive drift.

Glauber Dynamics - Curie Weiss

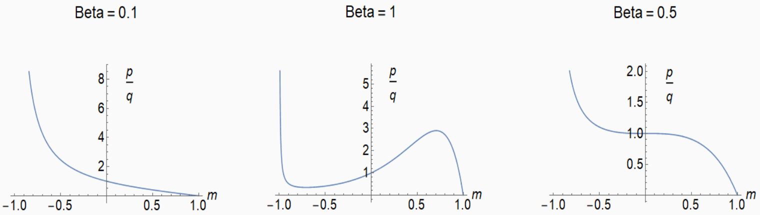
Take $J = \frac{\beta}{n} \mathbb{1} \otimes \mathbb{1}$, so $\mu(x) \propto \exp\left(\frac{\beta}{n}\left(\sum x_i\right)^2\right)$.
Assume $X_0 = \mathbb{1}$ and consider the magnetization chain
$S_t = \frac{1}{n}\langle X_t, \mathbb{1}\rangle$,

- $S_t \in [-1, 1]$
- $S_{t+1} - S_t \in \{-\frac{2}{n}, 0, \frac{2}{n}\}$.

$$p := p(\beta, m) = \mathbb{P}\left(S_{t+1} = S_t + 2 | S_t = m\right),$$
$$q := q(\beta, m) = \mathbb{P}\left(S_{t+1} = S_t - 2 | S_t = m\right).$$

$\frac{p}{q} \simeq \frac{(1-m)(1+\tanh(2\beta m))}{(1+m)(1-\tanh(2\beta m))}$ measures the positive drift.

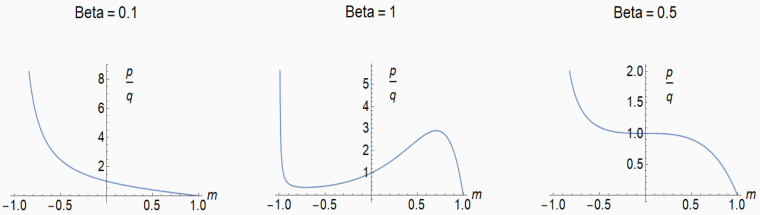
Glauber Dynamics - Curie Weiss



Theorem (Griffiths, Weng, Langer 66')

Glauber dynamics for the Curie Weiss model mixes rapidly if and only if $\beta \leq \frac{1}{2}$.

Glauber Dynamics - Curie Weiss



Theorem (Griffiths, Weng, Langer 66')

Glauber dynamics for the Curie Weiss model mixes rapidly if and only if $\beta \leq \frac{1}{2}$.

Stochastic Localization 101

One approach to handling general interaction matrices goes through Eldan's stochastic localization (SL)

- SL is a set of techniques, useful in the study of high-dimensional probability.
- Decomposes a measure into a mixture of simple measures in a tractable way.
- Simple: $\mu(x) \to e^{-Q(x)}\mu(x)$, with Q PSD quadratic.
- Tractable: amenable to stochastic analysis.

Stochastic Localization 101

One approach to handling general interaction matrices goes through Eldan's stochastic localization (SL)

- SL is a set of techniques, useful in the study of high-dimensional probability.
- Decomposes a measure into a mixture of simple measures in a tractable way.
- Simple: $\mu(x) \to e^{-Q(x)}\mu(x)$, with Q PSD quadratic.
- Tractable: amenable to stochastic analysis.

Stochastic Localization 101

One approach to handling general interaction matrices goes through Eldan's stochastic localization (SL)

- SL is a set of techniques, useful in the study of high-dimensional probability.
- Decomposes a measure into a mixture of simple measures in a tractable way.
- Simple: $\mu(x) \to e^{-Q(x)}\mu(x)$, with Q PSD quadratic.
- Tractable: amenable to stochastic analysis.

Stochastic Localization - Technical Details

Formally, define a measure-valued martingale via the SDE,

$$\partial F_t(x) = \langle x - \mathbf{a_t}, \mathbf{C_t} dB_t \rangle F_t(x), F_0(x) = 1.$$

- $\mu_t = F_t \mu$ is a martingale so $\mu = \mathbb{E}[\mu_t]$.
- C_t is a matrix process and $a_t := \int x F_t(x) \mu(x)$.

$$\partial \int \mu_t(x) = \partial \int F_t(x)\mu(x) = \left\langle \int x F_t(x)\mu(x) - a_t, C_t dB_t \right\rangle = 0.$$

- $\mu_t(x) = \exp\left(\int_0^t \langle x - a_s, C_s dB_s \rangle - \frac{1}{2} \int_0^t \| C_s(x - a_s) \|^2 ds \right) \mu(x)$.
- $\mu_t(x) \propto \exp\left(-\frac{1}{2} \langle x, \left(\int_0^t C_s^2 ds \right) x \rangle + \langle L_t, x \rangle \right) \mu(x)$.

Stochastic Localization - Technical Details

Formally, define a measure-valued martingale via the SDE,

$$\partial F_t(x) = \langle x - \mathbf{a_t}, \mathbf{C_t} dB_t \rangle F_t(x), F_0(x) = 1.$$

- $\mu_t = F_t \mu$ is a martingale so $\mu = \mathbb{E}[\mu_t]$.
- C_t is a matrix process and $a_t := \int x F_t(x) \mu(x)$.

$$\partial \int \mu_t(x) = \partial \int F_t(x)\mu(x) = \left\langle \int x F_t(x)\mu(x) - a_t, C_t dB_t \right\rangle = 0.$$

- $\mu_t(x) = \exp\left(\int_0^t \langle x - a_s, C_s dB_s \rangle - \frac{1}{2} \int_0^t \| C_s(x - a_s) \|^2 ds \right) \mu(x).$
- $\mu_t(x) \propto \exp\left(-\frac{1}{2} \langle x, \left(\int_0^t C_s^2 ds \right) x \rangle + \langle L_t, x \rangle \right) \mu(x).$

Stochastic Localization - Technical Details

Formally, define a measure-valued martingale via the SDE,

$$\partial F_t(x) = \langle x - \mathbf{a_t}, \mathbf{C_t} dB_t \rangle F_t(x), F_0(x) = 1.$$

- $\mu_t = F_t \mu$ is a martingale so $\mu = \mathbb{E}[\mu_t]$.
- C_t is a matrix process and $a_t := \int x F_t(x)\mu(x)$.

$$\partial \int \mu_t(x) = \partial \int F_t(x)\mu(x) = \left\langle \int x F_t(x)\mu(x) - a_t, C_t dB_t \right\rangle = 0.$$

- $\mu_t(x) = \exp\left(\int_0^t \langle x - a_s, C_s dB_s \rangle - \frac{1}{2} \int_0^t \| C_s(x - a_s) \|^2 ds \right) \mu(x).$
- $\mu_t(x) \propto \exp\left(-\frac{1}{2} \langle x, \left(\int_0^t C_s^2 ds \right) x \rangle + \langle L_t, x \rangle \right) \mu(x).$

Stochastic Localization - Technical Details

Formally, define a measure-valued martingale via the SDE,

$$\partial F_t(x) = \langle x - \mathbf{a_t}, \mathbf{C_t} dB_t \rangle F_t(x), F_0(x) = 1.$$

- $\mu_t = F_t \mu$ is a martingale so $\mu = \mathbb{E}[\mu_t]$.
- C_t is a matrix process and $a_t := \int x F_t(x)\mu(x)$.

$$\partial \int \mu_t(x) = \partial \int F_t(x)\mu(x) = \left\langle \int x F_t(x)\mu(x) - a_t, C_t dB_t \right\rangle = 0.$$

- $\mu_t(x) = \exp\left(\int\limits_0^t \langle x - a_s, C_s dB_s \rangle - \frac{1}{2} \int\limits_0^t \| C_s(x - a_s) \|^2 ds \right) \mu(x).$
- $\mu_t(x) \propto \exp\left(-\frac{1}{2} \langle x, \left(\int\limits_0^t C_s^2 ds \right) x \rangle + \langle L_t, x \rangle \right) \mu(x).$

Stochastic Localization - Technical Details

Formally, define a measure-valued martingale via the SDE,

$$\partial F_t(x) = \langle x - \mathbf{a_t}, \mathbf{C_t} dB_t \rangle F_t(x), F_0(x) = 1.$$

- $\mu_t = F_t \mu$ is a martingale so $\mu = \mathbb{E}[\mu_t]$.
- C_t is a matrix process and $a_t := \int x F_t(x) \mu(x)$.

$$\partial \int \mu_t(x) = \partial \int F_t(x)\mu(x) = \left\langle \int x F_t(x)\mu(x) - a_t, C_t dB_t \right\rangle = 0.$$

- $\mu_t(x) = \exp\left(\int_0^t \langle x - a_s, C_s dB_s \rangle - \frac{1}{2} \int_0^t \| C_s(x - a_s) \|^2 ds \right) \mu(x).$
- $\mu_t(x) \propto \exp\left(-\frac{1}{2} \langle x, \left(\int_0^t C_s^2 ds \right) x \rangle + \langle L_t, x \rangle \right) \mu(x).$

Stochastic Localization - Ising Model

Consider $\mu(x) = \exp\left(\langle x, Jx\rangle\right), x \in \{-1,1\}^n$ and its Glauber dynamics.

Want: control the spectral gap, $\mathrm{Var}_\mu(\varphi) \leq \mathbf{C}(\mu)\mathcal{E}_\mu(\varphi)$.

(Possible) Solution: choose C_t in SL to satisfy:

- $\mathrm{Var}_{\mu_t}(\varphi)$ is a martingale.
- For $J_t = J - \frac{1}{2}\int_0^t C_s^2 ds$, J_t is strictly decreasing when $\mathrm{rank}(J_t) \geq 2$.

If $\tau := \min\{t | \mathrm{rank}(J_t) = 1\}$, then

$$\mathrm{Var}_\mu(\varphi) = \mathbb{E}\left[\mathrm{Var}_{\mu_\tau}(\varphi)\right] \leq C\mathbb{E}\left[\mathcal{E}_{\mu_\tau}(\varphi)\right] \leq C\mathcal{E}_\mu(\varphi).$$

with C the maximal Poincaré constant ranging over measures

$$\exp\left(\langle x, Rx\rangle + \langle \ell, x\rangle\right), R \leq J, \mathrm{rank}(R) = 1.$$

Stochastic Localization - Ising Model

Consider $\mu(x) = \exp(\langle x, Jx \rangle), x \in \{-1, 1\}^n$ and its Glauber dynamics.

Want: control the spectral gap, $\mathrm{Var}_\mu(\varphi) \leq \mathbf{C}(\mu)\mathcal{E}_\mu(\varphi)$.

(Possible) Solution: choose C_t in SL to satisfy:

- $\mathrm{Var}_{\mu_t}(\varphi)$ is a martingale.
- For $J_t = J - \frac{1}{2}\int_0^t C_s^2 ds$, J_t is strictly decreasing when $\mathrm{rank}(J_t) \geq 2$.

If $\tau := \min\{t | \mathrm{rank}(J_t) = 1\}$, then

$$\mathrm{Var}_\mu(\varphi) = \mathbb{E}\left[\mathrm{Var}_{\mu_\tau}(\varphi)\right] \leq C\mathbb{E}\left[\mathcal{E}_{\mu_\tau}(\varphi)\right] \leq C\mathcal{E}_\mu(\varphi).$$

with C the maximal Poincaré constant ranging over measures

$$\exp(\langle x, Rx \rangle + \langle \ell, x \rangle), R \leq J, \mathrm{rank}(R) = 1.$$

Stochastic Localization - Ising Model

Consider $\mu(x) = \exp(\langle x, Jx \rangle), x \in \{-1, 1\}^n$ and its Glauber dynamics.

Want: control the spectral gap, $\mathrm{Var}_\mu(\varphi) \leq \mathbf{C}(\mu)\mathcal{E}_\mu(\varphi)$.

(Possible) Solution: choose C_t in SL to satisfy:

- $\mathrm{Var}_{\mu_t}(\varphi)$ is a martingale.
- For $J_t = J - \frac{1}{2}\int_0^t C_s^2 ds$, J_t is strictly decreasing when $\mathrm{rank}(J_t) \geq 2$.

If $\tau := \min\{t | \mathrm{rank}(J_t) = 1\}$, then

$$\mathrm{Var}_\mu(\varphi) = \mathbb{E}\left[\mathrm{Var}_{\mu_\tau}(\varphi)\right] \leq C\mathbb{E}\left[\mathcal{E}_{\mu_\tau}(\varphi)\right] \leq C\mathcal{E}_\mu(\varphi).$$

with C the maximal Poincaré constant ranging over measures

$$\exp(\langle x, Rx \rangle + \langle \ell, x \rangle), R \leq J, \mathrm{rank}(R) = 1.$$

Stochastic Localization - Ising Model

Consider $\mu(x) = \exp\left(\langle x, Jx\rangle\right), x \in \{-1, 1\}^n$ and its Glauber dynamics.

Want: control the spectral gap, $\mathrm{Var}_\mu(\varphi) \leq \mathbf{C}(\mu)\mathcal{E}_\mu(\varphi)$.

(Possible) Solution: choose C_t in SL to satisfy:

- $\mathrm{Var}_{\mu_t}(\varphi)$ is a martingale.
- For $J_t = J - \frac{1}{2}\int_0^t C_s^2 ds$, J_t is strictly decreasing when $\mathrm{rank}(J_t) \geq 2$.

If $\tau := \min\{t | \mathrm{rank}(J_t) = 1\}$, then

$$\mathrm{Var}_\mu(\varphi) = \mathbb{E}\left[\mathrm{Var}_{\mu_\tau}(\varphi)\right] \leq C\mathbb{E}\left[\mathcal{E}_{\mu_\tau}(\varphi)\right] \leq C\mathcal{E}_\mu(\varphi).$$

with C the maximal Poincaré constant ranging over measures

$$\exp\left(\langle x, Rx\rangle + \langle \ell, x\rangle\right), R \leq J, \mathrm{rank}(R) = 1.$$

Stochastic Localization - Ising Model

If $\nu(x) \propto \exp\left(\langle v, x\rangle^2 + \langle \ell, x\rangle\right)$ then $C(\nu) \leq \frac{1}{1-2\|v\|^2}$.

Theorem (Eldan, Koehler, Zeitouni 20')

If J is PSD with $\|J\|_{\mathrm{op}} < \frac{1}{2}$ and $\mu(x) \propto \exp\left(\langle x, Jx\rangle\right)$,

$$C(\mu) \leq \frac{1}{1 - 2\|J\|_{\mathrm{op}}}.$$

1. Essentially sharp for the Curie-Weiss model $J = \frac{\beta}{n}\mathbb{1} \otimes \mathbb{1}$.
2. Meaningful bounds for the Sherrington-Kirkpatrick model.
3. **Crucially relies on J being quadratic.**

Stochastic Localization - Ising Model

If $\nu(x) \propto \exp\left(\langle v, x\rangle^2 + \langle \ell, x\rangle\right)$ then $C(\nu) \leq \frac{1}{1-2\|v\|^2}$.

Theorem (Eldan, Koehler, Zeitouni 20')

If J is PSD with $\|J\|_{\mathrm{op}} < \frac{1}{2}$ and $\mu(x) \propto \exp\left(\langle x, Jx\rangle\right)$,

$$C(\mu) \leq \frac{1}{1 - 2\|J\|_{\mathrm{op}}}.$$

1. Essentially sharp for the Curie-Weiss model $J = \frac{\beta}{n}\mathbb{1} \otimes \mathbb{1}$.
2. Meaningful bounds for the Sherrington-Kirkpatrick model.
3. **Crucially relies on J being quadratic.**

Stochastic Localization - Ising Model

If $\nu(x) \propto \exp\left(\langle v, x\rangle^2 + \langle \ell, x\rangle\right)$ then $C(\nu) \leq \frac{1}{1-2\|v\|^2}$.

Theorem (Eldan, Koehler, Zeitouni 20')

If J is PSD with $\|J\|_{\mathrm{op}} < \frac{1}{2}$ and $\mu(x) \propto \exp\left(\langle x, Jx\rangle\right)$,

$$C(\mu) \leq \frac{1}{1 - 2\|J\|_{\mathrm{op}}}.$$

1. Essentially sharp for the Curie-Weiss model $J = \frac{\beta}{n}\mathbb{1} \otimes \mathbb{1}$.
2. Meaningful bounds for the Sherrington-Kirkpatrick model.
3. **Crucially relies on J being quadratic.**

Stochastic Localization - Ising Model

If $\nu(x) \propto \exp\left(\langle v, x\rangle^2 + \langle \ell, x\rangle\right)$ then $C(\nu) \leq \frac{1}{1-2\|v\|^2}$.

Theorem (Eldan, Koehler, Zeitouni 20')

If J is PSD with $\|J\|_{\mathrm{op}} < \frac{1}{2}$ and $\mu(x) \propto \exp\left(\langle x, Jx\rangle\right)$,

$$C(\mu) \leq \frac{1}{1-2\|J\|_{\mathrm{op}}}.$$

1. Essentially sharp for the Curie-Weiss model $J = \frac{\beta}{n}\mathbb{1} \otimes \mathbb{1}$.
2. Meaningful bounds for the Sherrington-Kirkpatrick model.
3. **Crucially relies on J being quadratic.**

Stochastic Localization - Tensor Models?

Crucially relies on J being quadratic. Why?

1. Waiting for $J_t = J - \int_0^t C_s^2 ds$ to collapse.
2. Came from a 'Gaussian tilt',

$$\mu_t(x) \simeq \exp\left(-\frac{1}{2}\langle x, \int_0^t C_s^2 ds x\rangle\right)\mu(x) = \exp\left(\langle x, J - \frac{1}{2}\int_0^t C_s^2 ds x\rangle\right).$$

3. Induced by the quadratic variation of $C_t dB_t$.

Question

Can a similar technique work for interactions of higher degrees?

Can we induce non-Gaussian tilts?

Stochastic Localization - Tensor Models?

Crucially relies on J being quadratic. Why?

1. Waiting for $J_t = J - \int_0^t C_s^2 ds$ to collapse.
2. Came from a 'Gaussian tilt',

$$\mu_t(x) \simeq \exp\left(-\frac{1}{2}\langle x, \int_0^t C_s^2 ds x\rangle\right)\mu(x) = \exp\left(\langle x, J - \frac{1}{2}\int_0^t C_s^2 ds x\rangle\right).$$

3. Induced by the quadratic variation of $C_t dB_t$.

Question

Can a similar technique work for interactions of higher degrees?

Can we induce non-Gaussian tilts?

Tensor Models - Possible Approaches

Let $T \in (\mathbb{R}^n)^{\otimes 4}$, equivalently $T : \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$.
Pedestrian view: T is an $n^2 \times n^2$ matrix.

For $\mu(x) = \exp(T(x))^2$, define the SL process,

$$\partial F_t(x) = \langle x \otimes x - a_t, M_t dW_t \rangle F_t(x), F_0(x) = 1.$$

1. a_t - Barycenter, W_t - Dyson Brownian motion, M_t - 4 tensor.
2. Set $\mu_t = F_t \mu$, a probability measure, because of a_t.
3. $\mu_t(x) = \exp\left(\int_0^t \langle x \otimes x - a_s, M_s dW_s\rangle - \frac{1}{2}\int_0^t \|M_s(x \otimes x - a_s)\|^2 ds\right)\mu(x)$.

[2] $T(x) := T(x,x,x,x) = \langle T(xx^T), xx^T\rangle_{HS}$

Tensor Models - Possible Approaches

Let $T \in (\mathbb{R}^n)^{\otimes 4}$, equivalently $T : \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$.
Pedestrian view: T is an $n^2 \times n^2$ matrix.

For $\mu(x) = \exp(T(x))$[2], define the SL process,

$$\partial F_t(x) = \langle x \otimes x - a_t, M_t dW_t \rangle F_t(x), F_0(x) = 1.$$

1. a_t - Barycenter, W_t - Dyson Brownian motion, M_t - 4 tensor.
2. Set $\mu_t = F_t \mu$, a probability measure, because of a_t.
3. $\mu_t(x) = \exp\left(\int_0^t \langle x \otimes x - a_s, M_s dW_s \rangle - \frac{1}{2} \int_0^t \|M_s(x \otimes x - a_s)\|^2 ds\right) \mu(x)$.

[2] $T(x) := T(x,x,x,x) = \langle T(xx^T), xx^T \rangle_{HS}$

Tensor Models - Possible Approaches

$$\mu_t(x) = \exp\left(\int_0^t \langle x \otimes x - a_s, M_s dW_s \rangle - \frac{1}{2}\int_0^t \|M_s(x \otimes x - a_s)\|^2 ds\right)\mu(x)$$

Or, equivalently

$$\mu_t(x) \propto \exp\left(T(x) - \frac{1}{2}\int_0^t M_s^2(x) ds + \langle L_t, x \otimes x \rangle\right)$$

Problem 1: $\langle L_t, x \otimes x \rangle$ is not linear in x. Spectral gap can be bad.

Problem 2: We treat T as a matrix, so $T(x) - \frac{1}{2}\int_0^t M_s^2(x) ds$ collapses to $D \otimes D$, rather than $v^{\otimes 4}$.

Tensor Models - Possible Approaches

$$\mu_t(x) = \exp\left(\int_0^t \langle x \otimes x - a_s, M_s dW_s \rangle - \frac{1}{2}\int_0^t \|M_s(x \otimes x - a_s)\|^2 ds\right)\mu(x)$$

Or, equivalently

$$\mu_t(x) \propto \exp\left(T(x) - \frac{1}{2}\int_0^t M_s^2(x) ds + \langle L_t, x \otimes x \rangle\right)$$

Problem 1: $\langle L_t, x \otimes x \rangle$ is not linear in x. Spectral gap can be bad.

Problem 2: We treat T as a matrix, so $T(x) - \frac{1}{2}\int_0^t M_s^2(x) ds$ collapses to $D \otimes D$, rather than $v^{\otimes 4}$.

Tensor Models - Possible Approaches

$$\mu_t(x) = \exp\left(\int_0^t \langle x \otimes x - a_s, M_s dW_s\rangle - \frac{1}{2}\int_0^t \|M_s(x \otimes x - a_s)\|^2 ds\right)\mu(x)$$

Or, equivalently

$$\mu_t(x) \propto \exp\left(T(x) - \frac{1}{2}\int_0^t M_s^2(x) ds + \langle L_t, x \otimes x\rangle\right)$$

Problem 1: $\langle L_t, x \otimes x\rangle$ is not linear in x. Spectral gap can be bad.

Problem 2: We treat T as a matrix, so $T(x) - \frac{1}{2}\int_0^t M_s^2(x) ds$ collapses to $D \otimes D$, rather than $v^{\otimes 4}$.

About Problem 1

Problem 1: $\langle L_t, x \otimes x \rangle$ is not linear in x. Need to eliminate

$$L_t = \int_0^t M_s dW_s + \int_0^t a_s ds.$$

- Remove a_t. Add a constraint to M_t to fix $\int 1 \mu_t(x)$.
- Introduce a new drift v_t to ensure that $\int_0^t M_s dW_s + \int_0^t v_s ds$ is arbitrarily small.

About Problem 1

Problem 1: $\langle L_t, x \otimes x \rangle$ is not linear in x. Need to eliminate

$$L_t = \int_0^t M_s dW_s + \int_0^t a_s ds.$$

- Remove a_t. Add a constraint to M_t to fix $\int 1 \mu_t(x)$.
- Introduce a new drift v_t to ensure that $\int_0^t M_s dW_s + \int_0^t v_s ds$ is arbitrarily small.

About Problem 1

Problem 1: $\langle L_t, x \otimes x \rangle$ is not linear in x. Need to eliminate

$$L_t = \int_0^t M_s dW_s + \int_0^t a_s ds.$$

- Remove a_t. Add a constraint to M_t to fix $\int 1\mu_t(x)$.
- Introduce a new drift v_t to ensure that $\int_0^t M_s dW_s + \int_0^t v_s ds$ is arbitrarily small.

About Problem 1

Lemma

For any 'reasonable' adapted matrix-process M_t and $\delta > 0$, there exists an adapted drift v_t, such that

$$\left\| \int_0^t M_s dW_s + \int_0^t v_s ds \right\| < \delta,$$

almost surely, for t arbitrarily large.

Define now the SL process

$$\partial F_t(x) = \langle x \otimes x - v_t, M_t dW_t \rangle F_t(x), F_0(x) = 1.$$

M_t encodes constraints: mass preservation, variance martingale.

About Problem 1

Lemma

For any 'reasonable' adapted matrix-process M_t and $\delta > 0$, there exists an adapted drift v_t, such that

$$\left\| \int_0^t M_s dW_s + \int_0^t v_s ds \right\| < \delta,$$

almost surely, for t arbitrarily large.

Define now the SL process

$$\partial F_t(x) = \langle x \otimes x - v_t, M_t dW_t \rangle F_t(x), F_0(x) = 1.$$

M_t encodes constraints: mass preservation, variance martingale.

About Problem 2

$$\partial F_t(x) = \langle x \otimes x - v_t, M_t dW_t \rangle F_t(x), F_0(x) = 1.$$

Define $T_t = T - \frac{1}{2}\int_0^t M_s^2 ds$ and $\tau := \min\{t | \mathrm{rank}(T_t) = 2\}$. We have the expression,

$$\mu_\tau \propto \exp\left(\langle x \otimes x, D_1 \otimes D_1\rangle + \langle x \otimes x, D_2 \otimes D_2\rangle\right)$$
$$= \exp\left(\langle x, D_1 x\rangle^2 + \langle x, D_2 x\rangle^2\right).$$

with $n \times n$ matrices. $\|D_1\|_{\mathrm{op}}, \|D_2\|_{\mathrm{op}} \le \|T\|_{\mathrm{inj}}$.

No immediate results for measures $\exp\left(\langle x, D_1 x\rangle^2\right)$.

About Problem 2

$$\partial F_t(x) = \langle x \otimes x - v_t, M_t dW_t \rangle F_t(x), F_0(x) = 1.$$

Define $T_t = T - \frac{1}{2}\int_0^t M_s^2 ds$ and $\tau := \min\{t | \mathrm{rank}(T_t) = 2\}$. We have the expression,

$$\begin{aligned}\mu_\tau &\propto \exp\left(\langle x \otimes x, D_1 \otimes D_1 \rangle + \langle x \otimes x, D_2 \otimes D_2 \rangle\right) \\ &= \exp\left(\langle x, D_1 x\rangle^2 + \langle x, D_2 x\rangle^2\right).\end{aligned}$$

with $n \times n$ matrices. $\|D_1\|_{\mathrm{op}}, \|D_2\|_{\mathrm{op}} \leq \|T\|_{\mathrm{inj}}$.

No immediate results for measures $\exp\left(\langle x, D_1 x\rangle^2\right)$.

About Problem 2

$$\partial F_t(x) = \langle x \otimes x - v_t, M_t dW_t \rangle F_t(x), F_0(x) = 1.$$

Define $T_t = T - \frac{1}{2}\int\limits_0^t M_s^2 ds$ and $\tau := \min\{t | \mathrm{rank}(T_t) = 2\}$. We have the expression,

$$\begin{aligned}\mu_\tau &\propto \exp\left(\langle x \otimes x, D_1 \otimes D_1 \rangle + \langle x \otimes x, D_2 \otimes D_2 \rangle\right) \\ &= \exp\left(\langle x, D_1 x\rangle^2 + \langle x, D_2 x\rangle^2\right).\end{aligned}$$

with $n \times n$ matrices. $\|D_1\|_{\mathrm{op}}, \|D_2\|_{\mathrm{op}} \leq \|T\|_{\mathrm{inj}}$.

No immediate results for measures $\exp\left(\langle x, D_1 x\rangle^2\right)$.

About Problem 2

Iterate!!

$$\exp\left(\langle x, D_1 x\rangle^2 + \langle x, D_2 x\rangle^2\right) \implies$$

$$\exp\left(\left(\langle v_1, x\rangle^2 + \langle v_2, x\rangle^2\right)\langle x, D_1 x\rangle + \langle x, D_2 x\rangle^2\right) \implies$$

$$\exp\left(\left(\langle v_1, x\rangle^2 + \langle v_2, x\rangle^2\right)\left(\langle u_1, x\rangle^2 + \langle u_2, x\rangle^2\right) + \langle x, D_2 x\rangle^2\right) \implies$$

$$\exp\left(\Sigma_{i,j=1}^{2}\langle v_i, x\rangle^2\langle u_i, x\rangle^2 + \Sigma_{i,j=3}^{4}\langle v_i, x\rangle^2\langle u_i, x\rangle^2\right)$$

About Problem 2

Iterate!!

$$\exp\left(\langle x, D_1 x\rangle^2 + \langle x, D_2 x\rangle^2\right) \Longrightarrow$$

$$\exp\left(\left(\langle v_1, x\rangle^2 + \langle v_2, x\rangle^2\right)\langle x, D_1 x\rangle + \langle x, D_2 x\rangle^2\right) \Longrightarrow$$

$$\exp\left(\left(\langle v_1, x\rangle^2 + \langle v_2, x\rangle^2\right)\left(\langle u_1, x\rangle^2 + \langle u_2, x\rangle^2\right) + \langle x, D_2 x\rangle^2\right) \Longrightarrow$$

$$\exp\left(\Sigma_{i,j=1}^2 \langle v_i, x\rangle^2 \langle u_i, x\rangle^2 + \Sigma_{i,j=3}^4 \langle v_i, x\rangle^2 \langle u_i, x\rangle^2\right)$$

Iterate!!

$$\exp\left(\langle x, D_1 x\rangle^2 + \langle x, D_2 x\rangle^2\right) \Longrightarrow$$

$$\exp\left(\left(\langle v_1, x\rangle^2 + \langle v_2, x\rangle^2\right)\langle x, D_1 x\rangle + \langle x, D_2 x\rangle^2\right) \Longrightarrow$$

$$\exp\left(\left(\langle v_1, x\rangle^2 + \langle v_2, x\rangle^2\right)\left(\langle u_1, x\rangle^2 + \langle u_2, x\rangle^2\right) + \langle x, D_2 x\rangle^2\right) \Longrightarrow$$

$$\exp\left(\Sigma_{i,j=1}^2 \langle v_i, x\rangle^2 \langle u_i, x\rangle^2 + \Sigma_{i,j=3}^4 \langle v_i, x\rangle^2 \langle u_i, x\rangle^2\right)$$

Iterate!!

$$\exp\left(\langle x, D_1 x\rangle^2 + \langle x, D_2 x\rangle^2\right) \Longrightarrow$$

$$\exp\left(\left(\langle v_1, x\rangle^2 + \langle v_2, x\rangle^2\right)\langle x, D_1 x\rangle + \langle x, D_2 x\rangle^2\right) \Longrightarrow$$

$$\exp\left(\left(\langle v_1, x\rangle^2 + \langle v_2, x\rangle^2\right)\left(\langle u_1, x\rangle^2 + \langle u_2, x\rangle^2\right) + \langle x, D_2 x\rangle^2\right) \Longrightarrow$$

$$\exp\left(\Sigma_{i,j=1}^{2}\langle v_i, x\rangle^2\langle u_i, x\rangle^2 + \Sigma_{i,j=3}^{4}\langle v_i, x\rangle^2\langle u_i, x\rangle^2\right)$$

Iterate!!

$$\exp\left(\langle x, D_1x\rangle^2 + \langle x, D_2x\rangle^2\right) \Longrightarrow$$

$$\exp\left(\left(\langle v_1, x\rangle^2 + \langle v_2, x\rangle^2\right)\langle x, D_1x\rangle + \langle x, D_2x\rangle^2\right) \Longrightarrow$$

$$\exp\left(\left(\langle v_1, x\rangle^2 + \langle v_2, x\rangle^2\right)\left(\langle u_1, x\rangle^2 + \langle u_2, x\rangle^2\right) + \langle x, D_2x\rangle^2\right) \Longrightarrow$$

$$\exp\left(\Sigma_{i,j=1}^2\langle v_i, x\rangle^2\langle u_i, x\rangle^2 + \Sigma_{i,j=3}^4\langle v_i, x\rangle^2\langle u_i, x\rangle^2\right)$$

Spectral Gaps of Tensor Models

$\exp\left(\sum_{i,j=1}^{2}\langle v_i, x\rangle^2\langle u_i, x\rangle^2 + \sum_{i,j=3}^{4}\langle v_i, x\rangle^2\langle u_i, x\rangle^2\right)$

Enough to bound spectral gap of rank 8 tensors.

Theorem

If T is PSD 4-tensor with $\|T\|_{\mathrm{inj}} < \frac{1}{8\cdot 12n}$ and $\mu(x) \propto \exp(T(x))$,

$$C(\mu) \lesssim \frac{1}{1 - 8\cdot 12n\|T\|_{\mathrm{inj}}}.$$

1. Not tight, by about a factor of 8.
2. Also applies to spin glass models, lose another constant factor.
3. Can be extended to higher order degrees or mixed spins.

Spectral Gaps of Tensor Models

$\exp\left(\sum_{i,j=1}^{2}\langle v_i, x\rangle^2\langle u_i, x\rangle^2 + \sum_{i,j=3}^{4}\langle v_i, x\rangle^2\langle u_i, x\rangle^2\right)$

Enough to bound spectral gap of rank 8 tensors.

Theorem

If T is PSD 4-tensor with $\|T\|_{\mathrm{inj}} < \frac{1}{8\cdot 12n}$ and $\mu(x) \propto \exp(T(x))$,

$$C(\mu) \lesssim \frac{1}{1 - 8\cdot 12n\|T\|_{\mathrm{inj}}}.$$

1. Not tight, by about a factor of 8.
2. Also applies to spin glass models, lose another constant factor.
3. Can be extended to higher order degrees or mixed spins.

Spectral Gaps of Tensor Models

$\exp\left(\sum_{i,j=1}^{2}\langle v_i, x\rangle^2\langle u_i, x\rangle^2 + \sum_{i,j=3}^{4}\langle v_i, x\rangle^2\langle u_i, x\rangle^2\right)$

Enough to bound spectral gap of rank 8 tensors.

Theorem

If T is PSD 4-tensor with $\|T\|_{\mathrm{inj}} < \frac{1}{8\cdot 12n}$ and $\mu(x) \propto \exp(T(x))$,

$$C(\mu) \lesssim \frac{1}{1 - 8\cdot 12n\|T\|_{\mathrm{inj}}}.$$

1. Not tight, by about a factor of 8.
2. Also applies to spin glass models, lose another constant factor.
3. Can be extended to higher order degrees or mixed spins.

Thank You