18.303 Problem Set 7

Due Thursday, 3 May 2018

Problem 1. Solve the following linear first-order PDE:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (x+y)u \quad \text{with} \quad u = 1 \text{ on the line segment } x = 1,\ 1 < y < 2.$$

Problem 2. For each integer n, let

$$g_n(x) = \begin{cases} n - n^2|x|, & |x| < 1/n, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch a graph of $g_n(x)$.

(b) Show that $\lim_{n\to\infty} g_n(x) = \delta_0(x)$. Justify your answer using either the definition of the Dirac's delta as a limit of ordinary functions, or using the formal definition of a distribution f as linear functional $L_f : C_0^\infty(\mathbb{R}) \to \mathbb{R}$ in the space $C_0^\infty(\mathbb{R})$ of infinitely differentiable functions that vanish outside a sufficiently large interval $[-M, M]$. In the latter case you may want to use the identity

$$L_{g_n}[u] = \int_{-\infty}^{\infty} g_n(x)u(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} g_n(x)[u(x) - u(0)]\,\mathrm{d}x + u(0)$$

which is valid for all $u \in C_0^\infty(\mathbb{R})$, and then take the limit as $n \to \infty$.

(c) Evaluate $f_n(x) = \int_{-\infty}^{x} g_n(y)\,\mathrm{d}y$ and sketch a graph. Does the sequence f_n converge to the step function $\sigma_0(x)$ as $n \to \infty$?

(d) Find the derivative $h_n(x) = g_n'(x)$.

(e) Does the sequence h_n converge to $\delta_0'(x)$ as $n \to \infty$. Justify your answer using either the definition of δ_0' as a limit of ordinary functions, or using the formal definition of a distribution. In the latter case you may want to use the definition of the distributional derivative; that is, f' is the distributional derivative of f if $L_{f'}[u] = -L_f[u']$ for all $u \in C_0^\infty(\mathbb{R})$, where $L_{f'}, L_f : C_0^\infty(\mathbb{R}) \to \mathbb{R}$ are linear functionals. These linear functionals admit integral representations $L_f[u] = \int_{-\infty}^{\infty} u(x)f(x)\,\mathrm{d}x$ and $L_{f'}[u] = \int_{-\infty}^{\infty} u(x)f'(x)\,\mathrm{d}x$ when f and f' are ordinary functions.

Problem 3. For n a positive integer, set

$$f_n(x) = \begin{cases} \frac{1}{2}n, & |x - \xi| < \frac{1}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the solution $u_n(x)$ to the boundary value problem $-u'' = f_n(x)$, $u(0) = u(1) = 0$, assuming $0 < \xi - \frac{1}{n} < \xi + \frac{1}{n} < 1$.

(b) Prove that $u_n(x)$ converges to the Green's function $g_\xi(x) = \begin{cases} x(1-\xi), & 0 < x < \xi < 1, \\ \xi(1-x), & 0 < \xi < x < 1 \end{cases}$ as $n \to \infty$. Why should this be the case?

(c) Reconfirm the result in part (b) by graphing $u_5(x)$, $u_{15}(x)$ and $u_{25}(x)$ along with $g_\xi(x)$ when $\xi = 0.3$.