18.303 Problem Set 4

Due Thursday, 15 March 2018

Problem 1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Consider the initial-boundary value problem for the nonhomogeneous heat equation in Ω:

$$\begin{aligned} u_t - \Delta u &= F(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega, \quad t > 0, \\ u(\boldsymbol{x}, t) &= 0, \quad \boldsymbol{x} \in \partial\Omega, \quad t \geq 0, \\ u(\boldsymbol{x}, 0) &= 0, \quad \boldsymbol{x} \in \Omega. \end{aligned} \tag{1}$$

Use "separation in time" to derive the following formula for the solution of (1):

$$u(\boldsymbol{x}, t) = \sum_{n=1}^{\infty} \left[\int_0^t f_n(\tau)\, \mathrm{e}^{-\lambda_n (t-\tau)}\, \mathrm{d}\tau \right] v_n(\boldsymbol{x}),$$

where λ_1, λ_2, ... and $v_1(\boldsymbol{x})$, $v_2(\boldsymbol{x})$, ..., are respectively the eigenvalues of the eigenfunctions of the operator $-\Delta : U \to U$ where $U = \{u \in L^2(\Omega) : u = 0 \text{ on } \partial\Omega\}$, and

$$f_n(\tau) = \frac{\displaystyle\int_\Omega F(\boldsymbol{x}, \tau) \overline{v_n}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}}{\displaystyle\int_\Omega |v_n(\boldsymbol{x})|^2\, \mathrm{d}\boldsymbol{x}}.$$

Problem 2. Suppose a ring-shaped membrane bounded by $1 < r < 2$ (using polar coordinates) has its edges held fixed. The vibrations of this membrane can be modeled by the wave equation

$$u_{tt} = c^2 \Delta u,$$

where u is the displacement, over $1 < r < 2$ with Dirichlet boundary conditions $u = 0$ at $r = 1$ and $r = 2$. If the initial conditions for this membrane are circularly-symmetric, then the solution will remain circularly-symmetric for all time, and the θ-dependence can be neglected. Let's suppose that the ring has some initial displacement that is circularly-symmetric, but has no initial velocity. In other words, if $u = u(r, t)$,

$$u(r, 0) = f(r), \qquad u_t(r, 0) = 0.$$

Solve for $u(r, t)$ by means of separation of variables and Bessel functions using the following steps:

1. Separate variables to obtain a Sturm-Liouville eigenvalue problem in the variable r. Is this eigenvalue problem a regular Sturm-Liouville eigenvalue problem?

2. You can assume that negative eigenvalues will always lead to the trivial solution. However, does the zero eigenvalue correspond to the trivial solution?

3. Using the asymptotic form of the Bessel functions for large arguments, derive an approximate transcendental equation for the eigenvalues in terms of trigonometric functions. Use this approximate transcendental equation and a root-finding algorithm

to approximate the first (smallest) 10 eigenvalues. You may want to used the Matlab commands `besselj` and `bessely` that produce the Bessel functions of first and second kind, respectively, to check the accuracy of your approximations.

4. Assuming that all of the eigenvalues are known, derive the series expansion for the solution.