18.303 Problem Set 3

Due Thursday, 8 March 2018

Problem 1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Construct a set of homogeneous boundary conditions on $\partial\Omega$ that, using the standard $L^2(\Omega)$ inner product, make the biharmonic operator $\Delta^2 = \Delta \circ \Delta$:
(a) self-adjoint
(b) positive definite
(c) positive semi-definite but not positive definite.

Problem 2. Find the adjoint of the gradient operator ∇ with respect to the L^2 inner product between scalar fields and the weighted inner product between vector fields defined as

$$\langle \mathbf{v}, \tilde{\mathbf{v}} \rangle = \int_\Omega \mathbf{v}^T(\boldsymbol{x}) C(\boldsymbol{x}) \widetilde{\mathbf{v}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}, \quad (\Omega \subset \mathbb{R}^2)$$

for $\mathbf{v}(\boldsymbol{x}) = [v_1(\boldsymbol{x}), v_2(\boldsymbol{x})]^T$ and $\widetilde{\mathbf{v}} = [\tilde{v}_1(\boldsymbol{x}), \tilde{v}_2(\boldsymbol{x})]^T$, where the 2×2 matrix $C(\boldsymbol{x}) = \begin{bmatrix} \alpha(\boldsymbol{x}) & \beta(\boldsymbol{x}) \\ \beta(\boldsymbol{x}) & \gamma(\boldsymbol{x}) \end{bmatrix}$ is symmetric and positive definite at all points $\boldsymbol{x} \in \Omega$. What kind of boundary conditions do you need to impose? Write down the corresponding boundary value problem for the equilibrium equation $\nabla^* \circ \nabla u = f$.

Problem 3. Consider the heat equation with no sources but with non-constant thermal properties:

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) \quad \text{in} \quad \Omega \subset \mathbb{R}^3$$

where c, ρ and K_0 are functions of $\boldsymbol{x} = (x, y, z)$. Assume that $u|_{\partial\Omega} = 0$.
(a) Show that the time variable can be separated by assuming that

$$u(\boldsymbol{x}, t) = \phi(\boldsymbol{x}) h(t)$$

and show that $\phi(\boldsymbol{x})$ satisfies the eigenvalue problem

$$\nabla \cdot (p \nabla \phi) + \lambda \sigma \phi = 0$$

with $\phi|_{\partial\Omega} = 0$. What are $\sigma(\boldsymbol{x})$ and $p(\boldsymbol{x})$?
(b) Prove that eigenfunctions belonging to different eigenvalues are orthogonal but in a weighted inner product.
(b) Prove that all the eigenvalues are real.