18.303 Problem Set 2

Due Thursday, 1 March 2018

Problem 1:

1. Consider the linear operators $L : U \to V$ and $M : V \to W$ on the inner product spaces $U$, $V$ and $W$. Show that:
   
   (a) $L = (L^*)^*$
   
   (b) $(L^*)^{-1} = (L^{-1})^*$
   
   (c) $(M \circ L)^* = L^* \circ M^*$

2. Let $c(x) \in C^0([a,b])$ be a continuous function. Prove that the linear multiplication operator $S[u](x) = c(x)u(x)$ is self-adjoint with respect to the real $L^2$ inner product. What sort of boundary conditions need to be imposed?

3. Prove that the complex differential operator $L[u] = iu'$ is self-adjoint with respect to the complex $L^2$ inner product (i.e., $\langle u, v \rangle = \int_\pi^{-\pi} u(x)\overline{v(x)} \, dx$) on the space of continuously differentiable complex-valued $2\pi$-periodic functions: $u(x + 2\pi) = u(x)$ for all $x$.

Problem 2: Let $D[u] = u'$, $D : U \to V$, be the derivative operator acting on the vector space $U = \{u(x) \in C^2[0,1] \mid u(0) = 0, \ u(1) = 0\}$.

1. Given the weighted inner product $\langle u, \tilde{u} \rangle = \int_0^1 u(x)\tilde{u}(x) e^x \, dx$ on both spaces $U$ and $V$, determine the corresponding adjoint operator $D^*$.

2. Let $S = D^* \circ D : U \to U$. Show that $D$ is self-adjoint.

3. Write down and solve the boundary value problem $S[u] = 2e^x$.

Problem 4: Let $\beta$ be a real constant. Consider the differential operator $S : U \to U$, $S[u] = -u''$, where $U = \{u(x) \in C^2[0,1] \mid u(0) = 0, \ u'(1) + \beta u(1) = 0\}$ with the $L^2$ inner product (i.e. $\langle u, v \rangle = \int_0^1 u(x)v(x) \, dx$).

1. Prove that $S$ is self-adjoint.

2. Find the transcendental equation for the eigenvalues of $S$ and use it to show that $S$ has infinitely many distinct real eigenvalues.

3. Prove that $S$ is positive definite (i.e., $\langle Su, u \rangle > 0$ for all $u \neq 0$) if and only if $\beta > -1$. (Hint: Assume that any function $u \in U$ has a convergent expansion in terms of the eigenfunctions of $S$).

4. Let $\beta = 1$. We now attempt to numerically approximate the smallest eigenvalue of $S$.

(a) Use the Matlab command $fzero$ (or a root finding method of your choice) to approximate the smallest eigenvalue of $S$ with at least 10 digits of accuracy.
(b) Use finite differences, with grid points \( x_j = jh, \ h = 1/N, \ j = 1, \ldots, N \), to discretize the eigenvalue problem \(-u'' = \lambda u, \ u \in U\). To do so, use centered differences to approximate the second order derivative \( u'' \), and backward differences to approximate the term \( u'(1) \) in the boundary condition, to obtain the a discrete eigenvalue problem \( Au = \lambda u \) with \( A \in \mathbb{R}^{N \times N} \), where \((u)_j \approx u(x_j)\). Denote by \( \lambda_D \) the smallest eigenvalue of \( A \) and by \( \lambda_C \) the smallest eigenvalue of \( S \). How does the error in the approximation \( \lambda_C \approx \lambda_D \) tend to zero as \( h \to 0 \)? To answer the question, assume that the error tends to zero as \( O(h^\alpha) \) and estimate \( \alpha > 0 \). To compute the error, assume that the approximate value obtained in the previous part is the exact value \( \lambda_C \).