Problem 1:

1. Consider the linear operators \( L : U \to V \) and \( M : V \to W \) on the inner product spaces \( U, V \) and \( W \). Show that:

   (a) \( L = (L^*)^* \)

   (b) \( (L^*)^{-1} = (L^{-1})^* \)

   (c) \( (M \circ L)^* = L^* \circ M^* \)

2. Let \( c(x) \in C^0([a,b]) \) be a continuous function. Prove that the linear multiplication operator \( S[u](x) = c(x)u(x) \) is self-adjoint with respect to the real \( L^2 \) inner product. What sort of boundary conditions need to be imposed?

3. Prove that the complex differential operator \( L[u] = iu' \) is self-adjoint with respect to the complex \( L^2 \) inner product (i.e., \( \langle u, v \rangle = \int_{-\pi}^{\pi} u(x)\overline{v(x)} \, dx \)) on the space of continuously differentiable complex-valued \( 2\pi \)-periodic functions: \( u(x + 2\pi) = u(x) \) for all \( x \).

Problem 2: Let \( D[u] = u' \), \( D : U \to V \), be the derivative operator acting on the vector space \( U = \{ u(x) \in C^2[0,1] \mid u(0) = 0, u(1) = 0 \} \).

1. Given the weighted inner product \( \langle u, \tilde{u} \rangle = \int_0^1 u(x)\overline{\tilde{u}(x)} e^x \, dx \) on both spaces \( U \) and \( V \), determine the corresponding adjoint operator \( D^* \).

2. Let \( S = D^* \circ D : U \to U \). Show that \( S \) is self-adjoint.

3. Write down and solve the boundary value problem \( S[u] = 2 e^x \).

Problem 3: Let \( \beta \) be a real constant. Consider the differential operator \( S : U \to U \), \( S[u] = -u'' \), where \( U = \{ u(x) \in C^2[0,1] \mid u(0) = 0, u'(1) + \beta u(1) = 0 \} \) with the \( L^2 \) inner product (i.e. \( \langle u, v \rangle = \int_0^1 u(x)\overline{v(x)} \, dx \)).

1. Prove that \( S \) is self-adjoint.

2. Find the transcendental equation for the eigenvalues of \( S \) and use it to show that \( S \) has infinitely many distinct real eigenvalues.

3. Prove that \( S \) is positive definite (i.e., \( \langle Su, u \rangle > 0 \) for all \( u \neq 0 \)) if and only if \( \beta > -1 \). (Hint: Assume that any function \( u \in U \) has a convergent expansion in terms of the eigenfunctions of \( S \)).

4. Let \( \beta = 1 \). We now attempt to numerically approximate the smallest eigenvalue of \( S \).

   (a) Use the Matlab command \texttt{fzero} (or a root finding method of your choice) to approximate the smallest eigenvalue of \( S \) with at least 10 digits of accuracy.
(b) Use finite differences, with grid points \( x_j = jh, \ h = 1/N, \ j = 1, \ldots, N, \) to discretize the eigenvalue problem \(-u'' = \lambda u, \ u \in U.\) To do so, use centered differences to approximate the second order derivative \( u'' \), and backward differences to approximate the term \( u'(1) \) in the boundary condition, to obtain the a discrete eigenvalue problem \( Au = \lambda u \) with \( A \in \mathbb{R}^{N \times N}, \) where \( (u)_j \approx u(x_j). \) Denote by \( \lambda_D \) the smallest eigenvalue of \( A \) and by \( \lambda_C \) the smallest eigenvalue of \( S. \) How does the error in the approximation \( \lambda_C \approx \lambda_D \) tend to zero as \( h \to 0? \) To answer the question, assume that the error tends to zero as \( O(h^\alpha) \) and estimate \( \alpha > 0. \) To compute the error, assume that the approximate value obtained in the previous part is the exact value \( \lambda_C. \)