18.303 Problem Set 2

Due Thursday, 1 March 2018

Problem 1:

1. Consider the linear operators $L : U \to V$ and $M : V \to W$ on the inner product spaces U, V and W. Show that:

   (a) $L = (L^*)^*$

   (b) $(L^*)^{-1} = (L^{-1})^*$

   (c) $(M \circ L)^* = L^* \circ M^*$

2. Let $c(x) \in C^0([a, b])$ be a continuous function. Prove that the linear multiplication operator $S[u](x) = c(x)u(x)$ is self-adjoint with respect to the real L^2 inner product. What sort of boundary conditions need to be imposed?

3. Prove that the complex differential operator $L[u] = iu'$ is self-adjoint with respect to the complex L^2 inner product (i.e., $\langle u, v\rangle = \int_{-\pi}^{\pi} u(x)\overline{v(x)}\,\mathrm{d}x$) on the space of continuously differentiable complex-valued 2π-periodic functions: $u(x + 2\pi) = u(x)$ for all x.

Problem 2: Let $D[u] = u'$, $D : U \to V$, be the derivative operator acting on the vector space $U = \{u(x) \in C^2[0, 1]|\ u(0) = 0,\ u(1) = 0\}$.

1. Given the weighted inner product $\langle u, \tilde{u}\rangle = \int_0^1 u(x)\tilde{u}(x)\,\mathrm{e}^x\,\mathrm{d}x$ on both spaces U and V, determine the corresponding adjoint operator D^*.

2. Let $S = D^* \circ D : U \to U$. Show that S is self-adjoint.

3. Write down and solve the boundary value problem $S[u] = 2\,\mathrm{e}^x$.

Problem 3: Let β be a real constant. Consider the differential operator $S : U \to U$, $S[u] = -u''$, where $U = \{u(x) \in C^2[0, 1]|\ u(0) = 0,\ u'(1) + \beta u(1) = 0\}$ with the L^2 inner product (i.e. $\langle u, v\rangle = \int_0^1 u(x)\overline{v(x)}\,\mathrm{d}x$).

1. Prove that S is self-adjoint.

2. Find the transcendental equation for the eigenvalues of S and use it to show that S has infinitely many distinct real eigenvalues.

3. Prove that S is positive definite (i.e., $\langle Su, u\rangle > 0$ for all $u \neq 0$) if and only if $\beta > -1$. (Hint: Assume that any function $u \in U$ has a convergent expansion in terms of the eigenfunctions of S).

4. Let $\beta = 1$. We now attempt to numerically approximate the smallest eigenvalue of S.

   (a) Use the Matlab command `fzero` (or a root finding method of your choice) to approximate the smallest eigenvalue of S with at least 10 digits of accuracy.

(b) Use finite differences, with grid points $x_j = jh$, $h = 1/N$, $j = 1, \ldots, N$, to discretize the eigenvalue problem $-u'' = \lambda u$, $u \in U$. To do so, use centered differences to approximate the second order derivative u'', and backward differences to approximate the term $u'(1)$ in the boundary condition, to obtain the a discrete eigenvalue problem $A\mathbf{u} = \lambda \mathbf{u}$ with $A \in \mathbb{R}^{N \times N}$, where $(\mathbf{u})_j \approx u(x_j)$. Denote by λ_D the smallest eigenvalue of A and by λ_C the smallest eigenvalue of S. How does the error in the approximation $\lambda_C \approx \lambda_D$ tend to zero as $h \to 0$? To answer the question, assume that the error tends to zero as $O(h^\alpha)$ and estimate $\alpha > 0$. To compute the error, assume that the approximate value obtained in the previous part is the exact value λ_C.