18.303 Problem Set 1

Due Thursday, 22 February 2018

Problem 1: Linear algebra warm up.

1. Show that if a matrix A is real-symmetric and invertible, then A^{-1} is real-symmetric too.

2. Consider the eigenvalue problem $B^{-1}Ax = \lambda x$ where A and B are both real-symmetric matrices and B is positive-definite, i.e. $(Bx, x) = (Bx)^T x > 0$ for $x \neq 0$. Consider a modified "dot product" $(x, y)_B = x^T By$ of vectors x and y.

   (a) Show that $(\cdot, \cdot)_B$ is indeed a dot (or inner) product.

   (b) Show that $C = B^{-1}A$ is symmetric with respect to this new dot product.

Problem 2: Quasi-periodic boundary conditions.

In class, we considered the 1d Poisson equation $\hat{A}[u](x) = f(x)$ where $\hat{A} = -\frac{d^2}{dx^2}$ for the vector space of functions $u(x)$ on $x \in [0, L]$ that satisfy "Dirichlet" boundary conditions $u(0) = u(L) = 0$, and solved it in terms of the eigenfunctions of $\hat{A}$ (giving a Fourier sine series). Here, we will consider a small variation on this: Suppose that we change the boundary conditions to the quasi-periodic boundary conditions $u(0) = e^{i\phi}\, u(L)$ and $u'(0) = e^{i\phi}\, u'(L)$ for some real number ϕ.

1. What are the eigenfunctions and eigenvalues of $\hat{A}$ now?

2. Under what conditions (if any) on $f(x)$ and ϕ would a solution exist? (You can assume that f has a convergent Fourier series.)

3. Consider the linear space of functions $u(x)$ on $x \in (0, L)$ that satisfy quasi-periodic boundary conditions, with the inner product $\int_0^L u(x)\overline{v}(x)\, dx$. Show that $\hat{A}$ defined in that space is self adjoint-adjoint. That is, show that $(Au, v) = (u, Av)$ for all u and v in that space, or equivalently, that $\hat{A} = \hat{A}^*$. For what values (if any) of ϕ is it positive definite?

Problem 3: Finite-difference approximations.

Note: For computational (Matlab-based) homework problems in 18.303, turn in with your solutions a printout of any commands used and their results. You can also email your notebook (.mlx) file to Andrew (rzeznik@mit.edu), our TA.

Suppose that we want to analyze the operator

$$\hat{A}u = -c\frac{d^2}{dx^2}$$

where $c(x) > 0$ is a real-values positive function. Now, we want to construct a finite-difference approximation for $\hat{A}$ with $u(x)$ on $[0, L]$ satisfying Dirichlet boundary conditions $u(0) = u(L) = 0$, similar to class, approximating $u(jh) \approx u_j$ for N equally spaced points $j = 1, 2, ..., N$, $u_0 = u_{N+1} = 0$, and $h = L/(N + 1)$.

1. Write down a finite-difference approximation, using center differences as in class, that corresponds to approximating $\hat{A}u$ by $A\mathbf{u}$ where $\mathbf{u}$ is the column vector of the N points u_j and A is a matrix of the form $A = CD^TD$. That is, give the matrix C, where D is the same as the 1st-derivative matrix from lecture.

2. Explain why you expect the matrix A to have real, positive eigenvalues, even though $A \neq A^T$.

3. In Matlab, construct the matrix A from part (1) for $N = 100$, $L = 1$ and $c(x) = \mathrm{e}^{3x}$. Then, get the eigenvalues and eigenvectors of A by `[U,D] = eig(A)` where the vector `lam=diag(D)` contains the eigenvalues of A and the columns of U contain the corresponding eigenvectors (note that the eigenvalues are positive because A is positive-definite).

   (a) Plot the eigenvectors for the smallest-magnitude four eigenvalues.

   (b) Verify that the eigenfunctions corresponding two to the two smallest eigenvalues are orthogonal for the correct inner product with `dot(U(:,1),X*U(:,2))` in Matlab, where you replace X by an appropriate matrix (hint: see problem 1.2).

4. For $c(x) = 1$, we saw in class that the eigenfunctions are $\sin(n\pi x/L)$. How do these compare to the eigenvectors you plotted in the previous part? Try changing $c(x)$ to some other function (note: still needs to be real and > 0), and see how different you can make the eigenfunctions from $\sin(n\pi x/L)$. Is there some feature that always remains similar, no matter how much you change c?