18.303 Midterm Solution

Problem 1: SLP with mixed boundary conditions

Consider the following (regular) Sturm-Liouville eigenvalue problem consisting in finding scalars λ and functions $v : [0, b] \to \mathbb{R}$ $(b > 0)$, $v \neq 0$, such that

$$-v''(x) = \lambda v(x), \quad x \in (0, b) \quad \text{with boundary conditions} \quad v(0) = 0 \text{ and } v'(b) = 0. \tag{1}$$

(a) **(5 points)** Define an inner product space in which; a) eigenfunctions v_1 and v_2 corresponding to distinct eigenvalues λ_1 and λ_2 $(\lambda_1 \neq \lambda_2)$ are orthogonal; b) and the eigenvalues λ are real and positive. Justify your answers.

Solution. We consider the space $U = \{u \in L^2([0, b]) : u(0) = 0,\ u'(b) = 0\}$ with the standard inner product: $\langle u, v\rangle = \int_0^b u(x)\overline{v}(x)\,\mathrm{d}x$. We first show that the operator (with the given boundary conditions) is self-adjoint. Integrating by parts we have

$$\begin{aligned}\langle Lv, w\rangle = -\langle v'', w\rangle = -(v'\bar{w})\big|_0^b + \langle v', w'\rangle &= -(v'\bar{w})\big|_0^b + (v\bar{w}')\big|_0^b - \langle v, w''\rangle \\ &= -(v'\bar{w})\big|_0^b + (v\bar{w}')\big|_0^b + \langle v, Lw\rangle\end{aligned} \tag{2}$$

for all $v, w \in U$. The boundary conditions yield $(v'\bar{w})\big|_0^b = (v\bar{w}')\big|_0^b = 0$. Therefore, from (2) it follows that L is self-adjoint.

To show that L is positive definite, in turn, we set $w = v$ in (2). Doing so we get $\langle Lv, v\rangle = \langle v', v'\rangle \geq 0$ for all $v \in U$ and if $v \neq 0$ it easy to show that $\langle Lv, v\rangle > 0$. In fact, $\langle Lv, v\rangle = 0$ iff $v' = 0$ in $(0, b)$, or equivalently $v(x) = c$. From the boundary condition $v(0) = 0$, we get that $c = 0$, and thus $v = 0$. Therefore we conclude that $\langle Lv, v\rangle > 0$ for all $v \neq 0$.

Parts a) and b), finally, follow directly from the fact that $L : U \to U$ is self-adjoint and positive definite.

(b) **(10 points)** Compute explicitly the eigenfunctions v and the eigenvalues λ that solve (1).

Solution. Since $\lambda > 0$ we have that the general solution of the ODE is given by $v(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$. Imposing the boundary condition $v(0) = 0$ we obtain $a = 0$. From the boundary conditions at $x = b$ we get $v'(b) = b\sqrt{\lambda}\cos(\sqrt{\lambda}b) = 0$, which a transcendental equation for λ. The solutions to this equation are

$$\boxed{\lambda_n = \left((n + \frac{1}{2})\frac{\pi}{b}\right)^2 \quad \text{for} \quad n = 0, 1, 2, ...}$$

The associated eigenfunctions are then

$$\boxed{v_n(x) = \sin\left((n + \frac{1}{2})\frac{\pi}{b}x\right).}$$

Problem 2: BVP with mixed boundary conditions in 1D

Consider the (steady-state) boundary value problem

$$-u''(x) = f(x), \quad x \in (0, b), \qquad u(0) = \alpha \quad \text{and} \quad u'(b) = \beta, \tag{3}$$

where f is a "nice" (say, continuous) function on $(0, b)$.

(a) **(5 points)** Show that the solution u of (3) can be expressed as the sum $u = u_1 + u_2$, where

$$-u_1''(x) = f(x), \quad x \in (0, b), \qquad u_1(0) = 0, \quad u_1'(b) = 0, \tag{4}$$

$$\text{and} \quad -u_2''(x) = 0, \quad x \in (0, b), \qquad u_2(0) = \alpha, \quad u_2'(b) = \beta. \tag{5}$$

Solution. The result follows from the identities: $u(0) = (u_1 + u_2)(0) = u_1(0) + u_2(0) = \alpha$, $u'(b) = (u_1 + u_2)'(b) = u_1'(b) + u_2'(b) = \beta$ and $u'' = (u_1 + u_2)'' = u_1'' + u_2'' = -f$.

(b) **(15 points)** Find the solution u of the boundary value problem (3). In order to do so, solve (4) for u_1 first using the eigenfunctions obtained in Problem 1(b). Make sure to provide an explicit formula for the expansion coefficients in terms of f, the eigenfunctions and the eigenvalues. Then solve (5) for u_2 using the fact that the general solution of the homogeneous ODE $y''(x) = 0$ is simply $y(x) = c_1 x + c_2$, where c_1 and c_2 are arbitrary constants.

Solution. The solution of (3) is given by $u_1(x) = \sum_{n=0}^{\infty} \frac{f_n}{\lambda_n} v_n(x)$ where $f_n = \langle f, v_n \rangle / \langle v_n, v_n \rangle$. To find u_2 we consider the ODE $-u_2'' = 0$ whose solution is given by $u_2(x) = c_1 + c_2 x$. Imposing the boundary condition at $x = 0$ we get $u_2(0) = \alpha = c_1$ and imposing the boundary condition at $x = b$ we get $u_2'(b) = c_2 = \beta$. Therefore $u_2(x) = \alpha + \beta x$ and, consequently,

$$\boxed{u(x) = \alpha + \beta x + \sum_{n=0}^{\infty} \frac{\langle f, v_n \rangle}{\lambda_n \langle v_n, v_n \rangle} v_n(x).}$$

(c) **(5 points)** In light of the Fredholm alternative, what can you say about the conditions (if any) that f has to satisfy in order for (3) to have a solution? Is the solution you found in part (b) the unique solution of (3)?

Solution. Since the associated differential operator is positive definite (Problem 1(a)), we have that the kernel (or null space) of the operator is zero. Therefore, the Fredholm alternative does not pose any constrain on f for (3) to have a solution and, furthermore, the solution found in (b) is the unique solution of the problem.

(d) **(15 points)** Derive a second-order accurate finite difference scheme for the numerical solution of (3) using the grid points $x_j = jh$, $j = 1, \ldots, N$, with $h = b/N$ ($N > 2$). Make sure to specify the resulting linear system including the $N \times N$ matrix, the $N \times 1$ unknown vector, and the $N \times 1$ right-hand-side vector.

Solution. Let $\mathbf{u} = (u_1, \ldots, u_N)^T \in \mathbb{R}^N$ with $u_j \approx u(x_j)$ and set $f_j = f(x_j)$. Here we use the following second-order finite difference approximations:

$$\boxed{-u''(x_j) = f_j \approx \frac{-u_{j-1} + 2u_j - u_{j+1}}{h^2} \quad \text{and} \quad u'(x_N) = u'(b) = \beta \approx \frac{u_{N+1} - u_{N-1}}{2h}.}$$

An additional equation is obtained by assuming that $-u'' = f$ holds true at $x_N = b$. This equation becomes $f_N = \frac{-u_{N-1} + 2u_N - u_{N+1}}{h^2}$ after using the finite difference discretization. It follows from here that $u_{N+1} = -u_{N-1} + 2u_N - h^2 f_N$.

We have thus obtained the following set of equations:

- for $j = 1$: $\quad 2u_1 - u_2 = h^2 f_1 + \alpha$,

- for $1 < j < N$: $\quad -u_{j-1} + 2u_j - u_{j+1} = h^2 f_j$,
- for $j = N$: $\quad 2u_N - 2u_{N-1} = h^2 f_N + 2h\beta$,

which can be expressed as the linear system $A\mathbf{u} = \mathbf{b}$, where

$$\boxed{A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 \\ \vdots & & \ddots & \ddots & & \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} h^2 f_1 + \alpha \\ h^2 f_2 \\ h^2 f_3 \\ \vdots \\ h^2 f_{N-1} \\ h^2 f_N + 2h\beta \end{bmatrix}.}$$

Problem 3: IBVP with mixed boundary conditions in 2D

Consider a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega = \Gamma$ is given by the union $\Gamma = \Gamma_N \cup \Gamma_D$ with $\Gamma_N \cap \Gamma_D = \emptyset$ (see Figure 1a). Consider also the function space $U = \{u(\boldsymbol{x}) \in L^2(\Omega) : u = 0 \text{ on } \Gamma_D,\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N\}$ with the standard L^2 inner product $\langle u, v\rangle_U = \int_\Omega u(\boldsymbol{x})\overline{v(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}$.

(a) **(10 points)** Show that the Laplace operator $L = -\Delta : U \to U$ is self-adjoint and positive definite.

Solution. Let $u, v \in U$. Integrating by parts (using the divergence theorem) we obtain

$$\langle Lu, v\rangle = -\langle \Delta u, v\rangle = -\int_\Gamma \frac{\partial u}{\partial n}\bar{v}\,\mathrm{d}s + \int_\Omega \nabla u \cdot \nabla \bar{v}\,\mathrm{d}x$$

$$= \int_\Gamma \left(u\frac{\partial \bar{v}}{\partial n} - \frac{\partial u}{\partial n}\bar{v}\right)\mathrm{d}s - \int_\Omega u\Delta\bar{v}\,\mathrm{d}x = \int_\Gamma \left(u\frac{\partial \bar{v}}{\partial n} - \frac{\partial u}{\partial n}\bar{v}\right)\mathrm{d}s + \langle u, Lv\rangle.$$

Splitting the boundary integral above and using the boundary conditions we get:

$$\boxed{\int_\Gamma \left(u\frac{\partial \bar{v}}{\partial n} - \frac{\partial u}{\partial n}\bar{v}\right)\mathrm{d}s = \underbrace{\int_{\Gamma_D} \left(u\frac{\partial \bar{v}}{\partial n} - \frac{\partial u}{\partial n}\bar{v}\right)\mathrm{d}s}_{I_D} + \underbrace{\int_{\Gamma_N} \left(u\frac{\partial \bar{v}}{\partial n} - \frac{\partial u}{\partial n}\bar{v}\right)\mathrm{d}s}_{I_N} = 0.}$$

This is so because $v = u = 0$ on Γ_D (which implies that $I_D = 0$), and $\frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} = 0$ on Γ_N (which implies that $I_N = 0$). Since $\langle Lu, v\rangle = \langle u, Lv\rangle$ for all $u, v \in U$, we conclude that L self-adjoint.

On the other hand, letting $v = u$ in the identities above we get

$$\boxed{\langle Lu, u\rangle = -\langle \Delta u, u\rangle = \int_\Omega |\nabla u|^2\,\mathrm{d}x \geq 0,}$$

from where it follows that L is positive semi-definite. In order to show that L is positive definite, we note that $\langle Lu, u\rangle = 0$ implies $\nabla u = 0$ in Ω. Integrating we get $u(x, y) = c$ and since $u = 0$ on Γ_D it follows that $u(x, y) = c = 0$. Therefore L is positive definite.

(b) **(20 points)** Using separation of variables (in time) derive the solution for the following initial boundary value problems:

$$\begin{cases} \dfrac{\partial h(\boldsymbol{x},t)}{\partial t} &=& \Delta h(\boldsymbol{x},t), & \boldsymbol{x}\in\Omega,\ t>0, \\ h(\boldsymbol{x},0) &=& h_0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega, \\ h(\boldsymbol{x},t) &=& 0, & \boldsymbol{x}\in\Gamma_D,\ t>0, \\ \dfrac{\partial h(\boldsymbol{x},t)}{\partial n} &=& 0, & \boldsymbol{x}\in\Gamma_N,\ t>0, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial^2 w(\boldsymbol{x},t)}{\partial t^2} &=& \Delta w(\boldsymbol{x},t), & \boldsymbol{x}\in\Omega,\ t>0, \\ w(\boldsymbol{x},0) &=& w_0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega, \\ \dfrac{\partial w}{\partial t}(\boldsymbol{x},0) &=& v_0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega, \\ w(\boldsymbol{x},t) &=& 0, & \boldsymbol{x}\in\Gamma_D,\ t>0, \\ \dfrac{\partial v(\boldsymbol{x},t)}{\partial n} &=& 0, & \boldsymbol{x}\in\Gamma_N,\ t>0. \end{cases}$$

What can you say about the limits $\lim_{t\to\infty} h(\boldsymbol{x},t)$ and $\lim_{t\to\infty} w(\boldsymbol{x},t)$?

Solution. L is self-adjoint and positive definite so the space U has an orthonormal basis of eigenfunctions $\{v_n\}_{n=1}^{\infty}$ ($\|v_n\|^2 = \langle v_n, v_n\rangle = 1$) which satisfy $-\Delta v_n = \lambda_n v_n$ in Ω, $v_n = 0$ on Γ_D, and $\frac{\partial v_n}{\partial n} = 0$ on Γ_N, with positive eigenvalues $\lambda_n > 0$. Therefore, using separation of variables we obtain

$$\boxed{h(\boldsymbol{x},t) = \sum_{n=1}^{\infty} c_n\,\mathrm{e}^{-\lambda_n t}\,v_n(\boldsymbol{x}),}$$

$$\boxed{w(\boldsymbol{x},t) = \sum_{n=1}^{\infty} \Big(a_n\cos(\sqrt{\lambda_n}t) + b_n\sin(\sqrt{\lambda_n}t)\Big)\,v_n(\boldsymbol{x}).}$$

Imposing the initial conditions and using the orthogonality of the eigenfunctions we get

$$h(\boldsymbol{x},0) = h_0(\boldsymbol{x}) = \sum_{n=1}^{\infty} c_n v_n(\boldsymbol{x}) \implies \boxed{c_n = \langle h_0, v_n\rangle = \int_\Omega h_0(\boldsymbol{x})\bar{v}_n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},}$$

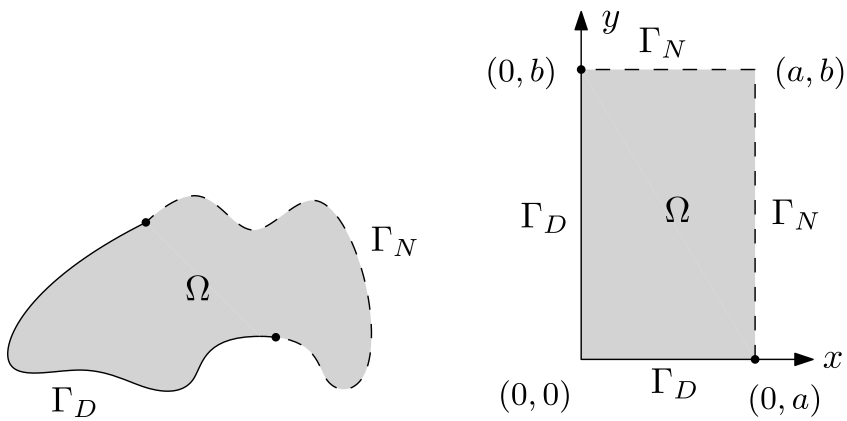
for the heat equation, and

$$w(\boldsymbol{x},0) = w_0(\boldsymbol{x}) = \sum_{n=1}^{\infty} a_n v_n(\boldsymbol{x}) \implies \boxed{a_n = \langle w_0, v_n\rangle = \int_\Omega w_0(\boldsymbol{x})\bar{v}_n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},}$$

$$\frac{\partial w}{\partial t}(\boldsymbol{x},0) = v_0(\boldsymbol{x}) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} b_n v_n(\boldsymbol{x}) \implies \boxed{b_n = \frac{\langle v_0, v_n\rangle}{\sqrt{\lambda_n}} = \frac{1}{\sqrt{\lambda_n}}\int_\Omega w_0(\boldsymbol{x})\bar{v}_n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},}$$

for the wave equation.

Since $\lambda_n > 0$ (because L is positive definite), we conclude from the formula above that $h(\boldsymbol{x},t)\to 0$ as $t\to\infty$. For the wave equation, in turn, no limit solution exists as $t\to\infty$. We can only note that each one of the modes oscillates as functions of t with frequencies $\sqrt{\lambda_n}$.



(a) General 2D domain. (b) Rectangle $(0,a)\times(0,b)$.

Figure 1

(c) **(15 points)** Suppose that Ω is the rectangle $(0,a)\times(0,b)$ whose boundary is given by the union of $\Gamma_D = ((0,a)\times\{0\})\cup(\{0\}\times(0,b))$ and $\Gamma_N = ((0,a)\times\{b\})\cup(\{a\}\times(0,b))$ (see Figure 1b). Using separation of variables (in space) find the eigenfunctions and eigenvalues of the operator L. Is your result consistent with the fact that L is self-adjoint and positive definite? Justify your answer.

Solution. Assuming the eigenfunctions have the form $v(\boldsymbol{x}) = X(x)Y(y)$, we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda \implies X'' + \mu X = 0 \quad \text{and} \quad Y'' + \gamma Y = 0 \quad (\mu + \gamma = \lambda).$$

with boundary conditions

$$X(0) = X'(a) = 0 \quad \text{and} \quad Y(0) = Y'(b) = 0.$$

From Problem 1(b), we readily conclude that $\mu = ((n+1/2)\pi/a)^2$, $X_n(x) = \sin(\sqrt{\mu_n}x)$, $n = 0,1,\ldots$ and $\gamma = ((m+1/2)\pi/b)^2$, $Y_m(x) = \sin(\sqrt{\gamma_m}y)$, $m = 0,1,\ldots$. Therefore, the eigenfunctions and eigenvalues of L are given by

$$\boxed{v_{n,m}(x,y) = \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{a}\right)\sin\left(\left(m+\frac{1}{2}\right)\frac{\pi y}{b}\right) \quad \text{and} \quad \lambda_{n,m} = ((n+\frac{1}{2})\frac{\pi}{a})^2 + ((m+\frac{1}{2})\frac{\pi}{b})^2,}$$

for $n, m = 0,1,\ldots$ respectively. Everything is consistent with what we found in part (a). In fact, the eigenvalues are positive ($\lambda_{n,m} > 0$) and the eigenfunctions are orthogonal as they should be given the fact that L is self-adjoint and positive definite.