# Properties of symmetric matrices

## 18.303: Linear Partial Differential Equations: Analysis and Numerics

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Let $A \in \mathbb{R}^{N\times N}$ be a symmetric matrix, i.e., $(Ax, y) = (x, Ay)$ for all $x, y \in \mathbb{R}^N$. The following properties hold true:

- Eigenvectors of $A$ corresponding to different eigenvalues are orthogonal.

  Let $\lambda$ and $\mu$, $\lambda \neq \mu$, be eigenvalues of $A$ corresponding to eigenvectors $x$ and $y$, respectively. Then $(Ax, y) = \lambda(x, y)$ and, on the other hand, $(Ax, y) = (x, Ay) = \mu(x, y)$. Subtracting these two identities we obtain $(\lambda - \mu)(x, y) = 0$. Since $\lambda \neq \mu$ we conclude that $(x, y) = 0$.

- If $(Ax, x) \leq 0$ (resp. $(Ax, x) \geq 0$) for all $x \neq 0$, then $A$ has eigenvalues $\lambda \leq 0$ (resp. $\lambda \geq 0$).

  Let $x$ be an eigenvector of $A$ with eigenvalue $\lambda$. Then $(Ax, x) = \lambda(x, x) = \lambda\|x\|^2 \leq 0$ which in view of the fact that $\|x\| > 0$, implies that $\lambda \leq 0$.

- If $(Ax, x) < 0$ (resp. $(Ax, x) > 0$) for all $x \neq 0$, then $A$ has eigenvalues $\lambda < 0$ (resp. $\lambda > 0$). In particular, $A$ is invertible.

  The same argument used above shows that $\lambda < 0$ is this case. Since all the eigenvalues are strictly negative, none of them is zero. Therefore, $A$ is invertible.

- $A$ is diagonalizable.

  Since $A$ is symmetric, it is possible to select an orthonormal basis $\{x_j\}_{j=1}^N$ of $\mathbb{R}^N$ given by eigenvectors or $A$. Letting $V = [x_1, \ldots, x_N]$, we have from the fact that $Ax_j = \lambda_j x_j$, that $AV = VD$ where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ and where the eigenvalues are repeated according to their multiplicities. Therefore $A = VDV^T$. Note that we have used the fact that $VV^T = V^TV = I$.

- If $(Ax, x) < 0$ (resp. $(Ax, x) > 0$) for all $x \neq 0$, then $A$ admits a factorization of the form $A = -D^TD$ (resp. $A = D^TD$) for some full-rank matrix $D$.

  Since $A$ is negative definite ($(Ax, x) < 0$), it has negative eigenvalues. The matrix of eigenvalues can thus be written as $D = -\Lambda^2$ with $\Lambda = \operatorname{diag}(\sqrt{|\lambda_1|}, \cdots, \sqrt{|\lambda_N|})$. From the identity $A = -V\Lambda^2V^T = -(V\Lambda)(\Lambda V^T) = -D^TD$ we finally recognize the factor $D = \Lambda V^T$. The fact that $D$ is full rank follows from both $V$ and $\Lambda$ being non-singular matrices.