Let us prove theorems in chapter 11.5. It is worth to learn the proofs of those theorems.

**Theorem 1** (Theorem 11.5A). Suppose that a sequence $x_n$ with $x_n \neq a$ has the limit $a$, and $\lim_{x \to a} f(x) = L$. Then, $\lim_{n \to \infty} f(x_n) = L$.

**Remind that $f(x)$ is not necessarily defined at $a$ in the theorem.**

*Proof.* Given $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon$ holds for $x \in (a - \delta, a + \delta) \setminus \{a\}$. Next, there exists a natural number $N$ such that $|x_n - a| < \delta$ for $n \geq N$. Therefore, $|f(x_n) - L| < \epsilon$ holds for $n \geq N$, namely $\lim_{n \to \infty} f(x_n) = L$. \(\square\)

**Example 2.** Let $\lim_{x \to a} x = a$ and $f(x)$ be continuous at $a$. Then, $\lim_{x \to a} f(x) = f(a)$.

*Proof.* Since $f(x)$ is continuous at $a$, we have $\lim_{x \to a} f(x) = f(a)$. So, the above theorem implies the desired result. \(\square\)

**Example 3.** Let $f(x)$ be a continuous function defined on $\mathbb{R}$, and let $f(x) \leq 0$ hold for all $x \in \mathbb{Q}$. Then, $f(x) \leq 0$ holds for all $x \in \mathbb{R}$.

*Proof.* Given a real number $x$ and a natural number $n$, we choose a rational number $r_n \in (a - \frac{1}{n}, a + \frac{1}{n})$. Then, we have $\lim_{n \to \infty} r_n = a$. Since $f(x)$ is continuous, we have $\lim_{n \to \infty} f(r_n) = f(a)$ by the example above.

On the other hand, $f(r_n) \leq 0$ for all $n$, because $r_n \in \mathbb{Q}$. Thus, the limit location theorem for sequences gives $\lim_{x \to a} f(x) = \lim_{x \to a} f(r_n) \leq 0$. \(\square\)

**Theorem 4** (Theorem 11.5B). Let $f(x)$ be defined for $x \in (a - \delta_0, a + \delta_0) \setminus \{a\}$. Suppose that for any sequence $\{x_n\}_{n \geq 0}$ with $x_n(a - \delta_0, a + \delta_0) \setminus \{a\}$ and $\lim_{x \to a} x_n = a$, we have $\lim_{x \to a} f(x_n) = L$. Then, $\lim_{x \to a} f(x) = L$ holds.

*Proof.* Assume that $f(x)$ does not converge to $L$ as $x \to a$, namely diverges or converges to another number. Then, by definition of the limit, there exists $\epsilon > 0$ such that given any $\delta \in (0, \delta_0)$, $|f(x) - L| \geq \epsilon$ holds for some number $x \in (a - \delta, a + \delta) \setminus \{a\}$. Hence, for all natural number $n$ with $\frac{1}{n} < \delta_0$, there exists a number $x_n \in (a - \frac{1}{n}, a + \frac{1}{n}) \setminus \{a\}$ such that $|f(x_n) - L| \geq \epsilon$. However, we have $\lim_{n \to \infty} f(x_n) = L$ because $\lim_{n \to \infty} x_n = a$. They are contradict. \(\square\)

**Example 5.** Let $f(x)$ be defined for $x \approx a$. Suppose that for any sequence $\{x_n\}_{n \geq 0}$ with $\lim_{x \to a} x_n = a$, we have $\lim_{x \to a} f(x_n) = f(a)$. Then, $f(x)$ is continuous at $a$.

*Proof.* The Theorem 11.5B and definition of limit yield the desired result. \(\square\)
**Example 6.** Let \( f(x) \) be defined for \( x \in (a, a + \delta_0) \). Suppose that for any sequence \( \{x_n\}_{n \geq 0} \) with \( x_n \in (a, a + \delta_0) \) and \( \lim x_n = a \), we have \( \lim_{x \to a^+} f(x) = L \). Then, \( \lim_{x \to a^+} f(x) = L \) holds.

**Proof.** Assume that \( f(x) \) does not converge to \( L \) as \( x \to a^+ \). Then, there exist some \( \varepsilon > 0 \) such that for each natural number \( n \) with \( \frac{1}{n} < \delta_0 \) we can choose a number \( x_n \in (a, a + \frac{1}{n}) \) satisfying \( |f(x_n) - L| \geq \varepsilon \). However, we have \( \lim f(x_n) = L \) because \( \lim x_n = a \). They are contradict. \( \square \)

**Example 7.** We define \( f(x) = \int_x^1 \frac{\sin(1/t)}{t} \, dt \) for \( x \in (0, 1) \). Then, \( f(x) \) is right-continuous at 0.

**Proof.** For \( 0 < x \leq y < 1 \), we have

\[
(1) \quad |f(x) - f(y)| = \left| \int_x^y \frac{\sin(1/t)}{t} \, dt \right| \leq \int_x^y \left| \frac{\sin(1/t)}{t} \right| \, dt \leq \int_x^y \frac{1}{t} \, dt = |y - x|.
\]

Suppose that a sequence \( \{y_n\} \) satisfies \( y_n \in (0, 1) \) and \( \lim y_n = 0 \). Then, given \( \varepsilon > 0 \), there exists a large number \( N \) such that \( |y_n| < \varepsilon/2 \) holds for \( n \geq N \). Therefore, \( |y_n - y_m| \leq |y_n| + |y_m| < \varepsilon \) holds for all \( n, m \geq N \). Hence, combining with (1) yields

\[
|f(y_n) - f(y_m)| \leq |y_n - y_m| < \varepsilon,
\]

for \( n, m \geq N \), namely \( \{f(y_n)\} \) is a Cauchy sequence. We denote by \( L \) the limit of \( \{f(y_n)\} \).

Given \( \varepsilon \in (0, 1) \), we have \( |f(y_n) - L| < \varepsilon/2 \) for \( n \gg 1 \). Since \( |y_n| < \varepsilon/2 \) for \( n \gg 1 \), there exist some term \( y_N \) of the sequence \( \{y_n\} \) such that \( y_N \in (0, \varepsilon/2) \) and \( |f(y_N) - L| < \varepsilon/2 \). Then, for any \( x \in (0, \varepsilon/2) \) the following holds

\[
|f(x) - L| \leq |f(x) - f(y_N)| + |f(y_N) - L| < |x - y_L| + \frac{\varepsilon}{2} \leq \varepsilon.
\]

Therefore, \( \lim_{x \to 0^+} f(x) = L \). \( \square \)

**Exercise 8.** Prove Example 7 by using Example 6 as follows:

1. For any sequence \( \{x_n\} \) with \( \lim x_n = 0 \), \( \{f(x_n)\} \) is a Cauchy sequence and thus has the limit, as the proof above.
2. Given two sequences \( \{x_n\} \) and \( \{y_n\} \) with \( \lim x_n = \lim y_n = 0 \), the limits of \( \{f(x_n)\} \) and \( \{f(y_n)\} \) are the same.
3. Applying the result of Example 6 proves Example 7.