Theorem 1. A non-negative $f(x)$ is integrable on $[a,b]$. Then, $f^2(x)$ is integrable on $[a,b]$.

Proof. Since $f$ is integrable, it is bounded. Hence, we have $0 \leq f(x) \leq M$ for some constant $M$.

In addition, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|U_f(P) - L_f(P)| < \frac{\epsilon}{2M}$$

holds if $|P| < \delta$. Namely,

$$\frac{\epsilon}{2M} > U_f(P) - L_f(P) = \sum_{i=1}^{n} (x_i - x_{i-1})(M_i - m_i).$$

where $P = \{x_i\}_{i=0}^{n}$, $M_i = \sup_{[x_{i-1},x_i]} f$, and $m_i = \inf_{[x_{i-1},x_i]} f$.

Now, $f \geq 0$ implies

$$M_i^2 = \sup_{[x_{i-1},x_i]} f^2, \quad m_i^2 = \inf_{[x_{i-1},x_i]} f^2.$$ 

Moreover, $0 \leq f(x) \leq M$ yields $0 \leq m_i, M_i \leq M$. Hence,

$$|U_{f^2}(P) - L_{f^2}(P)| = \sum_{i=1}^{n} (x_i - x_{i-1})(M_i^2 - m_i^2)$$

$$= \sum_{i=1}^{n} (x_i - x_{i-1})(M_i - m_i)(M_i + m_i)$$

$$\leq \sum_{i=1}^{n} (x_i - x_{i-1})(M_i - m_i)(2M)$$

$$= 2M|U_f(P) - L_f(P)| < \epsilon.$$ 

Thus, $f^2$ is integrable on $[a,b]$. □