**Problem 1**

(a). If \( \sum a_n \) is absolutely convergent, then it is convergent, so \( a_n \to 0 \) as \( n \to \infty \). Thus, there is \( N \) such that for \( n > N \), we have \( |a_n| < 1 \). It follows that for such \( n \), we have \( |a_n^2| = |a_n|^2 \leq |a_n| \).

By tail convergence, we know that \( \sum_{n>N} |a_n| \) converges, so by the comparison theorem for positive series, we find that \( \sum_{n>N} a_n^2 = \sum_{n>N} a_n^2 \) converges.

Again, by tail convergence, this implies that \( \sum_n a_n^2 \) converges.

(b). We consider \( a_n = \frac{(-1)^n}{\sqrt{n}} \). By Cauchy’s test for alternating series, this converges. However, \( a_n^2 \) is the harmonic series, which is known to diverge.

**Problem 2**

For each \( n \), set

\[
    a_n^+ = \frac{|a_n| + a_n}{2}, \quad a_n^- = \frac{|a_n| - a_n}{2}.
\]

Then \( a_n = a_n^+ - a_n^- \) for all \( n \), and \( a_n^+, a_n^- \geq 0 \).

Suppose that \( a_n \) has finitely many positive terms. Then \( a_n^+ = 0 \) for all but finitely many \( n \), so the series \( \{a_n^+\} \) converges. It follows that \( a_n^- = a_n^+ - a_n \).
converges, hence so does \(|a_n| = a_n^+ + a_n^-\), a contradiction to conditional convergence.

Suppose that \(a_n\) has finitely many negative terms. Then \(a_n^- = 0\) for all but finitely many \(n\), so the series \(\{a_n^-\}\) converges. It follows that \(a_n^+ = a_n - a_n^-\) converges, hence so does \(|a_n| = a_n^+ + a_n^-\), a contradiction to conditional convergence.

In either case, we see that if \(a_n\) converges conditionally (i.e., \(|a_n|\) does not converge), then \(a_n\) either has infinitely many positive or infinitely many negative terms.

\textbf{Problem 3}

\textbf{(b).} Setting \(a_n = \frac{n^2}{2^n}\), we have

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^2/2^{n+1}}{n^2/2^n} \right| = \frac{1}{2} \left( \frac{n+1}{n} \right)^2
\]

Thus

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{1}{2} \left( \frac{n+1}{n} \right)^2 = \frac{1}{2} < 1.
\]

So by the ratio test, this series converges.

\textbf{(d).} Setting \(a_n = \frac{(n!)^2}{(2n)!}\), we have

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)!^2/(2n+2)!}{(n!)^2/(2n)!} \right| = \frac{(n+1)!^2/(2n+2)!}{(2n+2)!/(2n)!} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n^2 + 2n + 1}{4n^2 + 6n + 2} = \frac{1 + 2/n + 1/n^2}{4 + 6/n + 2/n^2}
\]
Thus
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{1 + 2/n + 1/n^2}{4 + 6/n + 2/n^2} = \frac{1}{4} < 1. \]

So by the ratio test, this series converges.

(j). By the integral test, we may compare this with  \[ \int_2^\infty \frac{dx}{x(ln x)^p}. \]

For \( p \neq 1 \), the corresponding indefinite integral is \( \frac{(ln x)^{1-p}}{1-p} \). We thus have
\[ \int_2^\infty \frac{dx}{x(ln x)^p} = \left[ \frac{(ln x)^{1-p}}{1-p} \right]_2^\infty. \]

As \( \lim_{x \to \infty} \ln x = \infty \), this converges only when \( p > 1 \).

Finally, if \( p = 1 \), the corresponding indefinite integral is \( \ln \ln x \). We thus have
\[ \int_2^\infty \frac{dx}{x \ln x} = [\ln \ln x]_2^\infty \]

As \( \lim_{x \to \infty} \ln x = \infty \), we also have \( \lim_{x \to \infty} \ln \ln x = \infty \), so the integral diverges.

In summary, we have convergence only when \( p > 1 \).

**Problem 4**

Let’s suppose that the limit \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \) exists. Then if we apply the ratio test to \( \sum a_n x^n \), we are considering the limit
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = |x| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|. \]

This is less than one iff
\[ |x| < \left( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \right)^{-1}, \]

so
\[ R = \left( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \right)^{-1} = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|. \]
(a). Setting $a_n = \frac{1}{2^n \sqrt{n}}$, we have

$$\left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{1/(2^n \sqrt{n})}{1/(2^{n+1} \sqrt{n+1})} \right| = \frac{2^{n+1} \sqrt{n+1}}{2^n \sqrt{n}} = 2 \sqrt{1 + \frac{1}{n}}$$

Thus

$$\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} 2 \sqrt{1 + \frac{1}{n}} = 2.$$

Therefore, the ratio of convergence is 2.

(f). The $n$th root of the $n$th term is $\frac{x}{\ln n}$. For all $x$, we have $\lim_{n \to \infty} \frac{x}{\ln n} = 0$, so by the $n$th root test, we see that the $n$th root of the $n$th term approaches 0. It follows that the series converges for all $x$, i.e., the radius of convergence is $\infty$.

Problem 5

Let $f(x) = \frac{x}{1 + x}$. We have $f(1) = \frac{1}{2}$, so we need to show that $\lim_{x \to 1} f(x) = \frac{1}{2}$.

Given $\epsilon > 0$, let $\delta = \min(1, \epsilon)$.

Then if $|x - 1| < \delta$, we have

$$\left| f(x) - \frac{1}{2} \right| = \left| \frac{x}{1 + x} - \frac{1}{2} \right| = \left| \frac{2x}{2 + 2x} - \frac{1 + x}{2 + 2x} \right| = \left| \frac{x - 1}{2 + 2x} \right| = \frac{|x - 1|}{2 + 2x}$$
As $|x - 1| < \delta \leq 1$, we have $x > 0$, so $|2 + 2x| > 2$, so

$$
|f(x) - \frac{1}{2}| = \frac{|x - 1|}{|2 + 2x|} \leq |x - 1| < \delta \leq \epsilon.
$$

As $\epsilon > 0$ was arbitrary, we are done.

**Problem 6**

We have $\sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x)$, so

$$1 - \cos x = \frac{\sin^2 x}{1 + \cos x},$$

unless $\cos x = -1$. But $\cos 0 = 1$, so $\cos x \neq -1$ for $x$ in a neighborhood of 0.

We therefore have

$$
\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)} = \left( \lim_{x \to 0} \frac{\sin x}{x} \right) \left( \lim_{x \to 0} \frac{\sin x}{1 + \cos x} \right) = (1) \left( \frac{0}{1+1} \right) = 0.
$$

**Problem 7**

As the function contains $\sqrt{x}$, we are only considering $x \geq 0$. This will be assumed implicitly in all that follows.

For all $x \neq 0$, we have $|\cos(1/x)| \leq 1$. It follows that for $x \neq 0$, we have $|f(x)| = \sqrt{x} |\cos 1/x| \leq \sqrt{x}$. Thus $0 \leq |f(x)| \leq \sqrt{x}$, so by the squeeze theorem for limits, $0 \leq \lim_{x \to 0} |f(x)| \leq \lim_{x \to 0} \sqrt{x} = 0$. It follows that $\lim_{x \to 0} |f(x)| = 0$, hence also $\lim_{x \to 0} f(x) = 0 = f(0)$, so $f$ is continuous at 0.

**Problem 8**

(a). Suppose there were $x_0$ such that $f(x_0) \neq 0$. Let $\epsilon = |f(x_0)|/2 > 0$. Then there is $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for $x \in (x_0 - \delta, x_0 + \delta)$. 

In particular, for such $x$, we have $f(x) > |f(x_0)|/2 > 0$. But we can find a rational number $r$ in $(x_0 - \delta, x_0 + \delta)$, so $f(r) = 0$, a contradiction.

(b). Again, suppose there were $x_0$ such that $f(x_0) > g(x_0)$. Let $\epsilon = |f(x_0) - g(x_0)|/2 > 0$. Then there is $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ and $|g(x) - g(x_0)| < \epsilon$ for $x \in (x_0 - \delta, x_0 + \delta)$. In particular, for such $x$, we have $f(x) > f(x_0) - \epsilon = \frac{f(x_0) + g(x_0)}{2}$, and $g(x) < g(x) + \epsilon = \frac{f(x_0) + g(x_0)}{2}$.

But we can find a rational number $r$ in $(x_0 - \delta, x_0 + \delta)$, so $f(r) \leq g(r)$, a contradiction.

As a counterexample, take $f(x) = 0$ and $g(x) = (x - \sqrt{2})^2$. Then $f(x) < g(x)$ for all rational $x$, but $f(\sqrt{2}) = g(\sqrt{2}) = 0$.

**Problem 9**

Let us take $x_n = \frac{n\pi}{2}$. Assume that $\lim_{x \to \infty} \sin x$ exists.

Applying Theorem 11.5A for $a = \infty$ (if one were worried about the theorem applying with $a = \infty$, one could also apply it to $\sin(1/x)$ with $a = 0$), we find that since $\lim_{n \to \infty} x_n = \infty$, the limit $\lim_{n \to \infty} \sin x_n$ also exists. Call this limit $L$.

Taking the subsequence $x_{2n}$, we have

$$\lim_{n \to \infty} \sin x_{2n} = \lim_{n \to \infty} \sin (2\pi n) = \lim_{n \to \infty} 0 = 0,$$

so $L = 0$. Taking the subsequence $x_{4n+1}$, we have

$$\lim_{n \to \infty} \sin x_{4n+1} = \lim_{n \to \infty} \sin \left(2\pi n + \frac{\pi}{2}\right) = \lim_{n \to \infty} 1 = 1,$$

so $L = 1$.

This is a contradiction, so $\lim_{x \to \infty} \sin x$ does not exist.

**Problem 10**

If $f$ is multiplicatively periodic with constant $c$, we note that $f(x) = f(cc^{-1}x) = f(c^{-1}x)$, so $f$ is multiplicatively periodic with constant $c^{-1}$. If $c > 1$, then $c^{-1} < 1$, so we may assume that $f$ is multiplicatively periodic for a constant less than one. In other words, without lack of generality, we may assume $c < 1$. 

Applying the relation \(f(x) = f(cx)\) iteratively, we find that \(f(x) = f(c^n x)\) for any \(x\).

Consider the sequence \(\{c^n x\}\). This sequence has limit 0 as \(n \to \infty\). Thus \(\lim_{n \to \infty} f(c^n x) = f(0)\) by continuity of \(f\). But \(f(c^n x) = f(x)\), so this limit is also \(\lim_{n \to \infty} f(c^n x) = \lim_{n \to \infty} f(x) = f(x)\). Thus \(f(x) = f(0)\) for all \(x\), so the function is constant.