Problem 1

Let $a_n = (-1)^n$. First, we note that for all $n$, we either have $a_n = 1$ or $a_n = -1$. In either case, we have $-1 \leq a_n \leq 1$. Therefore, $\{a_n\}$ is bounded below by $-1$ and above by $1$, i.e., it is bounded.

For the next part, we present two possible solutions:

Solution 1. Next, let us suppose that the sequence has a limit, call it $L$. Then for some $N > 0$, we have $|a_n - L| < 1/2$ whenever $n > N$.

But for any such $n$, we can always find an even $n$ such that $n > N$. Then for such an $n$, we have $a_n = 1$, so $|L - 1| < 1/2$, hence $L > 1/2$. Similarly, we can always find odd $n$ such that $n > N$, so $a_n = -1$ for such an $n$, and so $|(-1) - L| < 1/2$, so $L < -1/2$. But this contradicts $L > 1/2$, so a limit cannot exist.

Solution 2. We consider the subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$. By the Subsequence Theorem, if the sequence $\{a_n\}$ has a limit, then any subsequence has the same limit. But the first subsequence has limit $1$, and the second has limit $-1$, a contradiction, so the original sequence cannot have a limit.

Comments.

- Some people’s explanations were overly complicated
- The key is to choose epsilon less than 1 (or realize that any such epsilon works).
- A lot of people bounded it by $\pm 2$ or even $\pm 10$. That might be good for intuition, but you only need $\pm 1$.
- Some people said “choose some $n$ even” or “choose some $n$ odd” and that the limit is then 1 (or $-1$). But a limit isn’t about a single
value ("some" $n$), rather it’s about what happens as $n$ (say, even) gets larger and larger.

- An important point people weren’t making explicit: you need to note that there are *arbitrarily large* even and odd $n$.

### Problem 2

Letting $a_n = \frac{n - 1}{3^n}$, we have

\[
a_{n+1} - a_n = \frac{n}{3^{n+1}} - \frac{n - 1}{3^n}
\]

\[
= \frac{n}{3^{n+1}} - \frac{3n - 3}{3^{n+1}}
\]

\[
= \frac{n - (3n - 3)}{3^{n+1}}
\]

\[
= \frac{3n - 3}{3^{n+1}}
\]

For $n \geq 2$ (in fact any $n > 3/2$), we have $-2n + 3 < 0$, and, noting that $3^{n+1} > 0$, this implies that $a_{n+1} - a_n < 0$. But this implies that $a_{n+1} < a_n$, which, by definition, says that the sequence is decreasing.

### Comments.

- Some people are starting with the conclusion and then getting to an inequality that’s true. If you do that, you need to explain that the steps are reversible!

### Problem 3

To show that the sequence is bounded above, note that for $n \geq 1$, we have $a_n = -a_{n-1}^2$, which is $\leq 0$. Noting that $a_0 < 0$, we have $a_n \leq 0$ for all $n \geq 0$, so the sequence is bounded above by 0.

To show that the sequence is increasing, we need to know that $a_n \geq -1$ for all $n$. Such a property for $a_n$ depends on the same property for $a_{n-1}$, so we need to use induction (notice how, in the previous paragraph, $-a_{n-1}$ is $\geq 0$ regardless of the value of $a_{n-1}$, so we do not need induction).
For \( n = 0 \), we have \( a_0 \geq -1 \). For some \( n \geq 0 \), suppose \( a_n \geq -1 \). We also know that \( a_n < 0 \) (by the first paragraph), so \( a_n^2 = (-a_n)^2 \) is between 0 and 1, meaning that \( a_{n+1} = -a_n^2 \) is \( \geq -1 \), as desired. By induction, we have \( a_n \geq -1 \) for all \( n \).

We have now shown that \(-1 \leq a_n < 0\) for all \( n \geq 0 \). It follows that \(-a_n\) is positive, so we may multiply both sides of the inequality \(-1 \leq a_n\) to get the inequality \( a_n \leq -a_n^2\). But this says that \( a_n \leq a_{n+1} \), which implies that the sequence is increasing.

Comments.

- It’s important to understand where induction is needed and where it isn’t needed. A lot of people used induction on the wrong part.
- As an alternative proof, one can show \( a_n = -a_0^{2^n} \) for all \( n \) and proceed directly (i.e., without even using induction). (Though technically, proving that formula involves induction, albeit a very intuitive example of induction.)

Problem 4

We first note that \( a_n > 0 \) for all \( n \), as the numerator and denominator are clearly positive. In particular, this implies that the sequence is bounded below.

Next, we note that

\[
\frac{a_{n+1}}{a_n} = \frac{\frac{2^{2n+2}(n+1)!^2}{(2n+3)!}}{\frac{2^{2n}(n)!^2}{(2n+1)!}} = \frac{\frac{2^{2n+2}(n+1)!^2}{2^{2n}(n)!^2}}{\frac{(2n+1)!}{(2n+3)!}} = \frac{4(n + 1)^2}{(2n + 3)(2n + 2)} = \frac{2n + 2}{2n + 3} < 1.
\]
As everything is positive, this implies that \( a_{n+1} < a_n \), i.e., the sequence is decreasing. But a bounded below decreasing sequence always has a limit by the Completeness Theorem, so it has a limit.

Comments.

- One can also show that the limit is 0 by noting that \( a_0 = 1 \), so

\[
a_n = \prod_{k=1}^{n} \frac{2k}{2k + 1} = \prod_{k=1}^{n} \frac{1}{1 + \frac{1}{2k}} < \frac{1}{\sum_{k=1}^{n} \frac{1}{2k}},
\]

but the bottom diverges to \( \infty \), so the limit approaches 0.

Problem 5

Solution 1. We let \( H_n = \sum_{k=1}^{n} \frac{1}{k} \) denote the Harmonic series. We note that

\[
a_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n-2} = \frac{1}{2} H_{n-1}.
\]

But we know that the Harmonic series grows arbitrarily large, hence so does \( a_n \).

(More precisely, if \( a_n < B \) for some \( B \) and all \( n \), the above would imply that \( H_{n-1} < 2B \) for all \( n \), contradicting the fact that \( H_n \) is unbounded.)

Solution 2. We note that \( a_n \) is an upper Riemann sum for the integral

\[
\int_{1}^{2n+1} \frac{1}{2x - 1} \, dx.
\]

The antiderivative of the integrand is \( \frac{\log(2x - 1)}{2} \), so the integral evaluates to

\[
\frac{\log(2(n + 1) - 1)}{2} - \frac{\log(2 - 1)}{2} = \frac{1}{2} \log(2n + 1).
\]

But the log function is unbounded so this approaches \( +\infty \), hence so does \( a_n \).

Comments.

- Be careful, \( a_n \) is the sum of \( n \) terms, so you need to compare it to an integral from 1 to \( n + 1 \), not from 1 to \( n \)

Problem 6

There is a counterexample. We choose any sequences \( a_n \) and \( b_n \) such that each is increasing and always negative. For example, let \( a_n = b_n = -\frac{1}{n} \). Then \( |a_n| \) is decreasing, hence so is \( a_n^2 \), and the same is true for \( b_n^2 \), so
$a_n^2 + b_n^2$ is also decreasing. (In that example, we have $a_n^2 + b_n^2 = \frac{2}{n^2}$, which is decreasing.)

Comments.

• Technically, you can find something where $a_n^2 + b_n^2$ increases for sufficiently large $n$, but fails in general due to behavior for small values of $n$. But it’s more interesting to note that there’s a counterexample even if you write “increasing for sufficiently large $n$” instead of “increasing.”
• Note that the word ‘counterexample’ has the word ‘example’ in it, and although I didn’t take off points for this, it’s good to include an example.

Problem 7

We first prove that for all $n$, we have $0 \leq a_n < \frac{2}{\sqrt{3}}$. We do this by induction. For $n = 0$, this is automatically true. Now assume that $0 \leq a_n < \frac{2}{\sqrt{3}}$ for some $n$.

Then $a_{n+1}$ is a positive square root, so it is clearly $\geq 0$. We also know $a_n^2 < 4/3$, so $a_{n+1}^2 = 1 + a_n^2/4 < 1 + (4/3)/4 = 1 + 1/3 = 4/3$, which implies that $a_{n+1} < \frac{2}{\sqrt{3}}$.

Comments.

• When proving this by induction, it’s important to be clear about *what statement* you’re trying to prove by induction. This was confusing in a lot of the problem sets.
• As an alternative proof, one can actually find a closed form for $a_n$.

More specifically, one has $a_n^2 = \frac{a_0^2}{4^n} + \sum_{k=1}^{n-1} \left( \frac{1}{4} \right)^{k-1} \frac{a_0^2}{4^n} + \frac{4 - \left( \frac{1}{4} \right)^{n-1}}{3}$, so the limit of $a_n$ is $\frac{2}{\sqrt{3}}$. 
(b). We have

\[
\left| \frac{2n - 1}{n + 2} - 2 \right| = \left| \frac{-5}{n + 2} \right|
= \frac{5}{n + 2}
< \frac{5}{n}
\]

For any \( \epsilon > 0 \), this is less than \( \epsilon \) for \( n > \frac{5}{\epsilon} \). This implies that \( \lim_{n \to \infty} \frac{2n - 1}{n + 2} = 2 \).

(c). As \( n > 0 \), we have

\[
\left| \frac{n}{n^2 + 3n + 1} \right| = \frac{n}{n^2 + 3n + 1}
< \frac{n}{n^2}
= \frac{1}{n}
\]

For any \( \epsilon > 0 \), this is less than \( \epsilon \) for \( n > \frac{1}{\epsilon} \), so \( \lim_{n \to \infty} \frac{n}{n^2 + 3n + 1} = 0 \).

(e). We note that \( (\sqrt{n^2 + 2} - n)(\sqrt{n^2 + 2} + n) = n^2 + 2 - n^2 = 2 \), so

\[ \sqrt{n^2 + 2} - n = \frac{2}{\sqrt{n^2 + 2} + n} \]. For \( n > 0 \), we have \( \sqrt{n^2 + 2} \) is well defined and \( \geq 0 \), so

\[
\left| \sqrt{n^2 + 2} - n \right| = \left| \frac{2}{\sqrt{n^2 + 2} + n} \right|
= \frac{2}{\sqrt{n^2 + 2} + n}
< \frac{2}{n}
\]
For any $\epsilon > 0$, this is less than $\epsilon$ for $n > \frac{2}{\epsilon}$. This implies that $\lim_{n \to \infty} \sqrt{n^2 + 2} - n = 0$.

**Problem 9**

**(a).** We first note that

$$a_n - a_{n-1} = \left(\frac{1}{n+1} + \cdots + \frac{1}{2n}\right) - \left(\frac{1}{n} + \cdots + \frac{1}{2n-2}\right)$$

$$= \frac{1}{2n} + \frac{1}{2n-1} - \frac{1}{n}$$

$$= \frac{1}{2n-1} - \frac{1}{2n}$$

$$= \frac{2n}{2n(2n-1)} - \frac{2n-1}{2n(2n-1)}$$

$$= \frac{1}{2n(2n-1)} > 0$$

In particular, we find that $\{a_n\}$ is increasing for $n \geq 1$.

Furthermore, we note that

$$a_n = \sum_{k=n+1}^{2n} \frac{1}{k} < \sum_{k=n+1}^{2n} \frac{1}{n+1} = \frac{2n}{n+1} = \frac{n}{n+1} < 1,$$

so $a_n$ is bounded above. By the Completeness Theorem, it follows that $a_n$ has a limit.

**(b).** In the $K$-$\epsilon$ principle, $K$ must be a constant. But this proposed “proof” is taking $K = n$, which is not constant.

**Problem 10**

We will actually do both cases at once. Let $M$ be a positive integer greater than $2|r|$. We note that for $n \geq M$, we have $\frac{|r|}{n} < \frac{1}{2}$. Therefore, for $n > M$, we have
\[ |a_n| = \frac{|r|^n}{\prod_{k=1}^{n} k} = |a_M| \frac{|r|^{n-M}}{\prod_{k=M+1}^{n} k} = |a_M| \prod_{k=M+1}^{n} \frac{|r|}{k} < |a_M| \prod_{k=M+1}^{n} \frac{1}{2} = |a_M| \left( \frac{1}{2} \right)^{n-M} \]

As \( \lim_{n \to \infty} \left( \frac{1}{2} \right)^{n-M} = 0 \), we know that for any \( \epsilon > 0 \), that \( |a_n| = |a_M| \left( \frac{1}{2} \right)^{n-M} < |a_M| \epsilon \) for \( n >> 1 \). By the \( K-\epsilon \) principle, it follows that \( \lim_{n \to \infty} a_n = 0 \).