18.100A Practice problems for Chapter 1-11

The midterm exam will take place on March 22th Thursday 9:35-10:50.

As an open book exam, during the exam you can see
1. the textbook: *Introduction to Real Analysis* by A. Mattuck,
2. notes, copies, and scratch papers (at most 500 sheets of paper).

However, the following are NOT allowed to use
1. electronic devices.
2. the other books except the textbook.

When you write the proofs of problems, you can cite Theorems, Properties, and examples with proofs in the textbook Chapter 1-11. Moreover, a sheet of facts will be given and you can cite them.

However, you can not use exercises and problems in the textbook as well as problem sets, practice problems, and their solutions. If you have copies of the solutions and want to use them, please rewrite the proofs.

The problems in this file will be continuously updated until March 16th without announcements. The solutions will be provided in a separate file.

The problems with stars are challenging.

**Problem 1.** Determine whether the following statements are true or false. If true then prove it, and if false then provide a counterexample.

1. Suppose $a_n > M$ for $n \gg 1$ and $\lim a_n = L$. Then, $L > M$.
2. Suppose $\lim a_n^2 = L$. Then, $\lim a_n = \sqrt{L}$.
3. Suppose $\{a_n b_n\}$ and $\{a_n\}$ converge. Then, $b_n$ also converges.
4. Suppose $\{a_n\}$ and $\{b_n\}$ with $b_n \neq 0$ are bounded. Then, $a_n/b_n$ is also bounded.
5. Suppose a non-empty set $S$ has its supremum. Then, the set $S^2 = \{s^2 : s \in S\}$ has its supremum and $\sup(S^2) = \sup S^2$.
6. A sequence of open intervals $I_n = (a_n, b_n)$ satisfies $I_{n+1} \subseteq I_n$ and $\lim |b_n - a_n| = 0$. Then, there exists a number $L$ such that $\lim a_n = \lim b_n = L$ and $L \in I_n$.
7. If $\lim a_n = M$, then $\lim |a_n| = |M|$.
8. A sequence $\{n^2 a_n\}$ converges. Then, the series $\sum a_n$ converges.
9. Let $a_n$ and $b_n$ be Cauchy sequences. Then, $a_n b_n$ is also a Cauchy sequence.
10. Let $a_n > 0$ be a Cauchy sequence. Then, $1/a_n$ is also a Cauchy sequence.
Problem 2. Determine whether the following sequences are convergent or divergent. If convergent, find the limit and explain why it is the limit. If divergent, explain why the sequence is not convergent.

(1) \( a_n = \frac{(-1)^n n}{2n+1} \), \hspace{1cm} (2) \( a_n = \frac{n^3}{3^n} \), \hspace{1cm} (3) \( a_n = \frac{2^n + 1}{3^n + n^3} \), \hspace{1cm} (4^*) \( a_n = \frac{n!}{n^n} \)

(5) \( a_{n+1} = \left( \frac{a_n}{2} \right)^2 \), \( a_0 < 4 \), \hspace{1cm} (6) \( a_{n+1} = \left( \frac{a_n}{2} \right)^2 \), \( a_0 > 4 \).

Fact needed for (4*): \( \lim (1 + \frac{1}{n})^n = e \approx 2.71828... > 1 \).

Problem 3. Let \( a_{n+1} = \frac{2}{1+a_n} \) and \( a_0 > 1 \).

(1) Show that the subsequence of even terms \( a_{2n} \) is decreasing and bounded below, and the subsequence of odd terms \( a_{2n-1} \) is increasing and bounded above.

(2) Show the convergence of \( a_n \), and find the limit.

Problem 4. Let \( S, T \) be non-empty sets bounded above. Suppose \( s, t > 0 \) holds for all \( s \in S \) and \( t \in T \). Then, we have \( (\sup S)(\sup T) = \sup ST \), where \( ST = \{ st : s \in S, t \in T \} \).