4 18.152 Problem set 4 solutions

PROBLEM 4.1.

People took two different approaches to this problem: either use Fourier series or extend the problem to all of \( \mathbb{R} \). Both approaches, done correctly, amount to the same thing. We will use the second approach here. The key thing to notice is that the boundary conditions \( u(0,t) = u_x(1,t) = 0 \) dictates how to extend the initial data to all of \( \mathbb{R} \) (or in the Fourier series approach, it tells us we’re looking for a superposition of solutions to the wave equation of the form \( \cos(\pi(2k+1)\tau/2) \sin(\pi(2k+1)x/2) \), because these are the separation-of-variables solutions that satisfy the boundary conditions).

Solution 1. In order for the conditions \( u(0,t) = 0 \) and \( u_x(1,t) = 0 \) to be met for all \( t \geq 0 \), we must have \( g(0) = g'(1) = 0 \). Using this, we extend \( g \) continuously to an odd periodic function on \( \mathbb{R} \) with the additional property that \( g(1 + x) = g(1 - x) \) for all \( x \): for \( x \in [1, 2] \), set \( g(x) = g(2 - x) \) so that \( g \) is even across the line \( x = 1 \). Next, for \( x \in [-2, 0] \) set \( g(x) = -g(-x) \) so that \( g \) is an odd function on \( [-2, 2] \). Finally, extend \( g \) to \( \mathbb{R} \) so that it is periodic with period 4: require that \( g(x + 4) = g(x) \) for all \( x \in \mathbb{R} \). Notice the periodic extension of \( g \) has the properties that (1) \( g(-x) = -g(x) \) for all \( x \in \mathbb{R} \) (it is an odd function) and (2) \( g(1 + x) = g(1 - x) \) for all \( x \in \mathbb{R} \) (it is even across the line \( x = 1 \)).

Now we search for a solution \( u \) to the wave equation on \( \mathbb{R} \) satisfying \( u(x,0) = g(x) \) and \( u_x(0,0) = 0 \) and \( u(0,t) = u_x(1,t) = 0 \) for \( t \geq 0 \). The solution to the wave equation on \( \mathbb{R} \) with the initial conditions \( u(x,0) = g(x) \) and \( u_x(x,0) = 0 \) is

\[
u(x,t) = \frac{1}{2} g(x + t) + \frac{1}{2} g(x - t).
\]

We will see that, because of the way we extended \( g \) to all of \( \mathbb{R} \), the boundary conditions are met as well. Indeed, \( 2u(0,t) = g(t) + g(-t) = 0 \) because \( g \) was extended to be an odd function. On the other hand,

\[
2u_x(1,t) = g'(1 + t) + g'(1 - t) = 0
\]

using the fact that \( g(1 + x) - g(1 - x) = 0 \) for all \( x \in \mathbb{R} \) (just differentiate this equation).

Solution 2. Here I’ll run through the separation of variables proof and explain why it ends up giving the same solution. Consider a (nonzero) separation-of-variables solution \( v(x,t) = a(t)b(x) \) to the wave equation satisfying the boundary conditions \( v(0,t) = \partial_t v(1,t) = 0 \) for \( t \geq 0 \). We find that \( a''/a = b''/b \) where \( v \) is nonzero, so that both sides are constant and \( b \) solves an eigenvalue problem \( b'' = \lambda b \) and \( b(0) = b'(1) = 0 \). The solutions to this have the form \( b(x) = \sin(\pi(2k + 1)x/2) \). For \( a(t) \) we then get

\[
a(t) = \alpha \cos(\pi(2k + 1)t/2) + \beta \sin(\pi(2k + 1)t/2)
\]

for constants \( \alpha, \beta \). But since we are searching for \( u \) with \( u_x(0,0) = 0 \), we will need \( a'(0) = 0 \). This means that \( \beta = 0 \) and our solution should be a linear combination of the functions \( \cos(\pi(2k + 1)t/2) \sin(\pi(2k + 1)x/2) \):

\[
u(x,t) = \sum_{k=0}^{\infty} c_k \cos(\pi(2k + 1)t/2) \sin(\pi(2k + 1)x/2).
\]

To find the coefficients \( c_k \), we use the initial condition \( u(x,0) = g(x) \). This tells us that

\[
c_k = 2 \int_{0}^{1} \sin(\pi(2k + 1)x/2)g(x) \, dx,
\]

that is, \( c_k \) are the Fourier coefficients of \( g \).
How does this relate to Solution 1? I will show how the two solutions are in fact the same. The Fourier series

\[ g(x) = \sum_{k \geq 0} c_k \sin(\pi(2k + 1)x/2) \]

is precisely the continuous and odd periodic extension of \( g \) that was found in Solution 1 (notice the Fourier series is defined for all \( x \in \mathbb{R} \)). Moreover, because of the trigonometric identity

\[ \cos(\pi(2k + 1)t/2) \sin(\pi(2k + 1)x/2) = \frac{1}{2} \sin(\pi(2k + 1)(x + t)/2) + \frac{1}{2} \sin(\pi(2k + 1)(x - t)/2), \]

we see that

\[ u(x, t) = \sum_{k=0}^{\infty} c_k \cos(\pi(2k + 1)t/2) \sin(\pi(2k + 1)x/2) \]

\[ = \frac{1}{2} \sum_{k=0}^{\infty} c_k \sin(\pi(2k + 1)(x + t)/2) + \frac{1}{2} \sum_{k=0}^{\infty} c_k \sin(\pi(2k + 1)(x - t)/2) \]

\[ = \frac{1}{2} g(x + t) + \frac{1}{2} g(x - t), \]

where we’re using the continuous periodic extension of \( g \) provided by the Fourier series. So this solution is the same as the one from Solution 1. ■

**Problem 4.2.**

**Solution.** The problem is linear in the function \( u \), so it is enough to show that when \( f, g, h \) vanish identically the solution \( u \) vanishes identically as well. We will show that

\[ \frac{d}{dt} \int_{\Omega} |\nabla u(x, t)|^2 + u_t(x, t)^2 \, dx = 0. \]  

(1)

This is sufficient because (1) implies that the integral is constant, and if \( g \) and \( h \) vanish identically on \( \Omega \) then the integral on the left side of (1) vanishes at the initial time. Consequently, \( u_t^2 \) is identically zero and \( u \) is constant (equal to zero).

To prove (1), observe that

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u(x, t)|^2 + u_t(x, t)^2 \, dx = \int_{\Omega} \nabla u \cdot \nabla u_t + u_t u_{tt} \]

\[ = \int_{\Omega} \nabla u \cdot \nabla u_t + u_t \Delta u. \]

\[ = \int_{\Omega} \text{div} (u_t \nabla u). \]

The divergence theorem then gives

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u(x, t)|^2 + u_t(x, t)^2 \, dx = \int_{\partial \Omega} u_t \nabla u \cdot v, \]

where \( v \) is the outer unit normal for \( \partial \Omega \). But since \( u(x, t) = 0 \) for any \( x \in \partial \Omega \) and any \( t \), it follows that \( u_t(x, t) = 0 \) for \( x \in \partial \Omega \). Thus (1) is proved. ■

**Problem 4.3.**
Solution. The solution to this problem is \( u(x, t) = (x + t)^2 \). The graph of \( u(x, 10) \) therefore looks like a parabola centered at \( x = -10 \).

Here is a way to find this solution (you can also compute directly with d’Alembert’s formula). Search for a solution to the wave equation on \( \mathbb{R} \) having the form \( A(x + t) + B(x - t) \) for some real numbers \( A \) and \( B \) satisfying \( A + B = 1 \). Any such solution \( u \) satisfies the initial condition \( u(x, 0) = x^2 \). If we impose the initial condition \( u_t(x, 0) = 2x \), we find that \( A = 1 \) and \( B = 0 \), so that \( u(x, t) = (x + t)^2 \) as claimed.

\[ \Box \]

**Problem 4.4.**

Everyone used the correct approach on this problem, but there were many computation errors. Many of these errors could have been avoided by double checking at the end that your solution satisfies the equation and the initial values.

Solution. The solution is

\[ u(x, t) = \frac{1}{3} g(x + 2t) + \frac{2}{3} g(x - t) + \frac{1}{3} \int_{x-t}^{x+2t} h(y) \, dy. \]

Using the hint, we see the general solution to this problem has the form \( u(x, t) = F(x + 2t) + G(x - t) \) for functions \( F \) and \( G \). We search for a solution of the form

\[ u(x, t) = Ag(x + 2t) + Bg(x - t) + CH(x + 2t) + DH(x - t), \]

where \( H' = h \) (so \( H \) is the integral of \( h \)) and \( A, B, C, \) and \( D \) are chosen so that the initial conditions are met.

From \( u(x, 0) = g(x) \) we get \( A + B = 1 \) and \( C + D = 0 \). And then from \( u_t(x, 0) = h(x) \) we get \( 2A - B = 0 \) and \( 2C - D = 1 \). Solving for \( A, B, C, D \) gives \( (A, B, C, D) = (1/3, 2/3, 1/3, -1/3) \), and then we represent \( H(x + 2t) - H(x - t) \) as an integral to get the formula from the beginning.

\[ \Box \]