Problem 3.1.

Solution. Part (1). Everyone did this correctly. We can easily obtain the conclusion by applying the weak maximum principle to the function \( u^2 \), which is subharmonic because \( \Delta u^2 = 2|\nabla u|^2 \geq 0 \). By the maximum principle, \( \max_{\Omega} u^2 \leq \max_{\partial \Omega} g^2 \), which is equivalent to the conclusion from the problem statement.

You could also apply the maximum principle to \( u \) and \(-u\).

Part (2). The main issue I saw with this problem was not manipulating the inequalities clearly. The basic idea with this sort of problem (and we've seen several in this vein) is to take the Laplacian and then use Cauchy–Schwarz and AM-GM inequalities to bound the possibly negative terms below. You just need to be clever in how you apply the inequalities and group the terms. Here is the argument.

We will show that

\[
|\nabla u(x_0)| \leq \frac{4}{r} \max_{\partial \Omega} |g|.
\]

Let \( \eta = r^2 - |x - x_0|^2 \) and \( v = \eta^2|\nabla u|^2 + Ku^2 \) for a constant \( K \) to be determined. Since \( u \) is harmonic, you can compute that \( \Delta u^2 = 2|\nabla u|^2 \) and \( \Delta |\nabla u|^2 = 2|\nabla^2 u|^2 \). Therefore,

\[
\Delta v = (\Delta \eta^2)|\nabla u|^2 + 2\eta^2|\nabla^2 u|^2 + 8\eta \nabla \eta \cdot \nabla^2 u \nabla u + 2K|\nabla u|^2 \\
= (2\eta \Delta \eta + 2|\nabla \eta|^2 + 2K)|\nabla u|^2 + 2\eta^2|\nabla^2 u|^2 + 8\eta \nabla \eta \cdot \nabla^2 u \nabla u.
\]

Now we use Cauchy–Schwarz and \( 2ab \leq a^2 + b^2 \) to get

\[
8\eta \nabla \eta \cdot \nabla^2 u \nabla u \geq -2(2^{1/2}\eta|\nabla u|)(2^{1/2}|\nabla \eta||\nabla u|) \geq -2\eta^2|\nabla^2 u|^2 - 8|\nabla \eta|^2|\nabla u|^2.
\]

Inserting this inequality into the expression for \( \Delta v \) gives

\[
\Delta v \geq (2\eta \Delta \eta - 6|\nabla \eta|^2 + 2K)|\nabla u|^2.
\]

Since \( 2\eta \Delta \eta - 6|\nabla \eta|^2 = -16|x - x_0|^2 - 8r^2 \geq -24r^2 \) on \( B_r(x_0) \), it follows that \( \Delta v \geq 0 \) if \( K = 12r^2 \).

Now we apply the maximum principle in \( B_r \) to \( v = \eta^2|\nabla u|^2 + 12r^2 u^2 \). We get

\[
r^4|\nabla u(x_0)|^2 \leq v(x_0) \leq \max_{\partial B_r(x_0)} v = \max_{\partial B_r(x_0)} 12r^2 u^2 \leq 12r^2 \max_{\partial \Omega} g^2,
\]

where for the last inequality we used the conclusion of part (1).

Part (3). To make the formulas simpler, we will assume \( x_0 = 0 \). The Poisson representation formula says that

\[
u(x) = \frac{r^2 - |x|^2}{2\pi r} \int_{\partial B_r(0)} \frac{u(y)}{|x - y|^2} \, dy.
\]

Differentiating in \( x \) gives

\[
\nabla u(x) = -\frac{x}{\pi r} \int_{\partial B_r(0)} \frac{u(y)}{|x - y|^2} \, dy - \frac{r^2 - |x|^2}{\pi r} \int_{\partial B_r(0)} \frac{u(y)}{|x - y|^2} \frac{x - y}{|x - y|^2} \, dy.
\]
Taking \( x = 0 \) then gives
\[
\|\nabla u(0)\| = \left| \frac{r^2}{\pi r} \int_{\partial B_r(0)} \frac{u(y) \ y}{|y|^3} \ dy \right| \\
\leq \frac{2}{r} \left( \frac{1}{2r} \int_{\partial B_r(0)} |u| \right) \\
\leq \frac{2}{r} \max_\Omega |u| \\
\leq \frac{2}{r} \max_\partial \Omega |g|.
\]

**Problem 3.2.**

**Solution.** The approach described in the hint is an important technique that goes by the name “freezing coefficients.”

Following the hint, define the linear differential operator \( L \) by
\[
Lv = v_{xx} + \frac{v_{yy}}{1 + x^2 + y^2} + u_{xx}v_x.
\]

Then \( Lu = 0 \) on \( \Omega \) and \( L \) satisfies the conditions required for the weak maximum principle from lecture to apply, namely, there exists \( \lambda > 0 \) such that if \( \xi \in \mathbb{R}^2 \) and \((x, y) \in \Omega \) then
\[
\xi_1^2 + \frac{1}{1 + x^2 + y^3} \xi_2^2 \geq \lambda |\xi|^2.
\]

Since \( Lu = 0 \), both \( u \) and \( -u \) are subsolutions, and applying the weak maximum principle to each of them gives \( |u| \leq \max_{\partial \Omega} |g| \).

We could alternatively prove the maximum principle directly in this case: consider \( v = u + \epsilon y^2/2 \). Then \( v_{xx}(1 + v_x) + v_{yy}(1 + x^2 + y^2)^{-1} = \epsilon(1 + x^2 + y^2)^{-1} > 0 \) on \( \Omega \), and we can check as usual that \( v \) does not attain a maximum in the interior of \( \Omega \) and so must attain its maximum on the boundary. We then make \( \epsilon \to 0 \) and conclude as usual. ■

**Problem 3.3.**

**Solution.** I will first give a heuristic solution and then will make it rigorous. The Green’s function for Laplace’s equation on \( \Omega \) can be thought of as the solution to the following family of boundary value problems: for each \( x \in \Omega \), the function \( G \) satisfies
\[
G(x, y) = 0 \quad y \in \partial \Omega, \\
\Delta_y G(x, y) = \delta(x - y), \quad y \in \Omega.
\]

Here \( \delta \) is the Dirac delta or point mass—it is not really a function but is characterized by the property that \( \int f(z)\delta(z) \ dz = f(0) \) for every function \( f \). Using this it’s easy to come up with a formal proof that we will then make precise.
As in the hint, define \( u(z) = G(x, z) \) and \( v(z) = G(y, z) \). Then \( u = v = 0 \) for \( z \in \partial \Omega \) and \( \Delta u(z) = \delta(x - z) \) and \( \Delta v(z) = \delta(y - z) \). We need to show that \( u(y) = v(x) \). To do this (formally), notice that

\[
v(x) = \int_\Omega v(z) \delta(x - z) \, dz = \int_\Omega v(z) \Delta u(z) \, dz = \int_\Omega (\Delta v(z)) u(z) \, dz = \int_\Omega \delta(y - z) u(z) \, dz = u(y).
\]

For the third equality we are just integrating by parts twice using the divergence theorem and the fact that \( u(z) = v(z) = 0 \) for \( z \in \partial \Omega \). So this is the idea behind the proof.

We now make the proof rigorous as suggested in the hint. Suppose \( x, y \in \Omega \) and \( x \neq y \). We can choose \( \epsilon \) so small that \( B_\epsilon(x) \) and \( B_\epsilon(y) \) are contained in \( \Omega \) and do not intersect. Now notice that \( \Delta u = \Delta v = 0 \) on the domain \( \Omega_1 := \Omega \setminus (B_\epsilon(x) \cup B_\epsilon(y)) \), and so applying the divergence theorem on this domain gives

\[
0 = \int_{\Omega_1} (v(z) \Delta u(z) - u(z) \Delta v(z)) \, dz = \int_{\partial \Omega_1} (v \nabla u - u \nabla v) \cdot \nu - \int_{\partial B_\epsilon(x)} (v \nabla u - u \nabla v) \cdot \nu,
\]

where \( \nu \) is the outer unit normal and we’ve used the fact that \( v \nabla u - u \nabla v = 0 \) on \( \partial \Omega \) to pass from the second line to the third. I will show that the last expression converges to \( v(x) - u(y) \) as \( \epsilon \to 0 \). I’ll treat the first integral but the second one is similar. First, the divergence theorem gives

\[
\int_{\partial B_\epsilon(x)} (v \nabla u - u \nabla v) \cdot \nu = \int_{\partial B_\epsilon(x)} v \nabla u \cdot \nu \, dz - \int_{\partial B_\epsilon(x)} (u \Delta v + \nabla u \cdot \nabla v) \, dz = \int_{\partial B_\epsilon(x)} v \nabla u \cdot \nu \, dz - \int_{\partial B_\epsilon(x)} \nabla u \cdot \nabla v
\]

using the fact that \( v \) is harmonic in \( B_\epsilon(x) \) to pass to the second line. Now \( |\nabla u(z) \cdot \nabla v(z)| \leq C |z - x|^{-n+1} \) on \( B_\epsilon(x) \) for some constant \( C \) and since \( |z - x|^{-n+1} \) is integrable over \( B_\epsilon(x) \) the integral over \( B_\epsilon(x) \) converges to zero with \( \epsilon \). To deal with the remaining integral, remember first that \( u(z) = \Phi(x - z) - \varphi(x, z) \) in \( B_\epsilon(x) \). Since \( \varphi \) is smooth in \( z \) near \( x \) its contribution is negligible (it is \( O(\epsilon) \)). On the other hand, \( \nabla \Phi(x - z) = -\omega_n (x - z)^{-n} (z - x) = -\omega_n \epsilon^{-n+1} \nu \) for \( z \in \partial B_\epsilon(x) \), where \( \nu = |z - x|^{-1} (z - x) \) is the outer unit normal to \( \partial B_\epsilon(x) \) at the point \( z \). Thus

\[
\int_{\partial B_\epsilon(x)} v \nabla u \cdot \nu \, dz = -\frac{1}{\omega_n \epsilon^{n-1}} \int_{\partial B_\epsilon(x)} v(z) \nu(z) \cdot \nu(z) \, dz + O(\epsilon) = -\frac{1}{\omega_n \epsilon^{n-1}} \int_{\partial B_\epsilon(x)} v(z) \, dz + O(\epsilon)
\]

Finally, \( \omega_n \epsilon^{n-1} \) is just the surface area of \( \partial B_\epsilon(x) \) and because \( v \) is continuous at \( z = x \) the integral, which is the average of \( v \) over \( \partial B_\epsilon(x) \), converges to \( v(x) \) as \( \epsilon \to 0 \). In sum,

\[
\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(x)} (v \nabla u - u \nabla v) \cdot \nu = -v(x).
\]
The same argument shows that \( \lim \int_{\partial B_r(y)} (v \nabla u - u \nabla v) \cdot \nu = u(y) \), and we conclude that \( u(y) = v(x) \). This completes the proof. \hfill \blacksquare

**Problem 3.4.**

**Solution.** Suppose \( u : \mathbb{R}^n \to \mathbb{R} \) is a nonnegative harmonic function. By the Harnack inequality (Theorem 4 in the March 7 lecture notes) we have

\[
\frac{R^{n-2}(R - r)}{(R + r)^{n-1}} u(0) \leq u(x) \leq \frac{R^{n-2}(R + |x|)}{(R - |x|)^{n-1}} u(0)
\]

for any \( R > 0 \) such that \( x \in B_R(0) \). Making \( R \to \infty \) then shows that \( u(0) \leq u(x) \leq u(0) \). Since \( x \) was arbitrary, this means that \( u \) is constant. \hfill \blacksquare

**Problem 3.5.**

**Solution.** We will first show that \( \Delta u = f \) in \( \Omega \), and we will then show that \( \lim_{x \to x_0} u(x) = g(x_0) \) for \( x_0 \in \partial \Omega \). Recall that

\[
u(x) = -\int_{\Omega} f(y) G(x, y) \, dy - \int_{\partial \Omega} g(y) \nu \cdot \nabla_y G(x, y) \, dy.
\]

**Proof that \( \Delta u = f \) in \( \Omega \).** As the hint suggests, we first notice that the integral over the boundary is harmonic: since \( G \) is smooth away from \( x = y \), we can simply compute

\[
\Delta \int_{\partial \Omega} g(y) \nu \cdot \nabla_y G(x, y) \, dy = \int_{\partial \Omega} g(y) \Delta_u \nu \cdot \nabla_y G(x, y) \, dy
\]

\[
= \int_{\partial \Omega} g(y) \nu \cdot \nabla_y \Delta_u G(x, y) \, dy
\]

\[
= 0,
\]

since \( \Delta_u G(x, y) = \Delta_u G(y, x) = 0 \) by Problem 3 and the properties of the Green’s function.

Now we turn to the integral over \( \Omega \). Begin by writing \( G(x, y) = \Phi(x - y) = \varphi(x, y) \) as in the hint, where \( \Phi \) is the fundamental solution for the Laplace operator (see the lecture notes of March 7) and by Problem 3 we have \( \varphi(x, y) = \varphi(y, x) \) and \( \Delta_u \varphi(x, y) = 0 \) for \( x \in \Omega \). It follows that \( \Delta \int_{\Omega} f(y) \varphi(x, y) \, dy = 0 \), and so we need only show that

\[
\Delta \int_{\Omega} f(y) \Phi(x - y) \, dy = -f(x)
\]

for \( x \in \Omega \). We will do this by reducing the problem to Theorem 1 from the March 7 lecture notes: that the Newtonian potential \( v(x) = \int_{\mathbb{R}^n} h(y) \Phi(x - y) \, dy \) of a smooth compactly supported function \( h \) satisfies the equation \( \Delta v = -h \).

We need to localize the problem. Fix \( x_0 \in \Omega \) and choose a small \( \varepsilon > 0 \) so that \( B_\varepsilon(x_0) \subset \Omega \), and let \( \eta \) be a smooth function that satisfies \( 0 \leq \eta \leq 1 \), \( \eta = 0 \) off of \( B_\varepsilon(x_0) \), and \( \eta = 1 \) on \( B_{\varepsilon/2}(x_0) \). Write \( f = \eta f + (1 - \eta) \int_{\Omega \setminus B_{\varepsilon/2}(x_0)} (1 - \eta) f(y) \Phi(x - y) \, dy \).

Since \( \Phi(x - y) \) is smooth and harmonic in \( x \) for \( (x, y) \in B_{\varepsilon/2}(x_0) \times (\Omega \setminus B_{\varepsilon/2}(x_0)) \) we can differentiate under the sign of integration to conclude that \( \int_{\Omega \setminus B_{\varepsilon/2}(x_0)} (1 - \eta(y)) f(y) \Phi(x - y) \, dy \) is harmonic for \( x \in B_{\varepsilon/2}(x_0) \). Therefore,
we are reduced to proving that \( v = \int_{\Omega} \eta(y)f(y)\Phi(x-y)\,dy \) satisfies \( \Delta v(x_0) = - f(x_0) \). But since \( \eta f = 0 \) off of \( B_\varepsilon(x_0) \), the function \( v \) is just the Newtonian potential of \( \eta f \), that is, \( v = \int_{\mathbb{R}^n} \eta(y)f(y)\Phi(x-y)\,dy \) (where we just extend \( \eta f \) smoothly by zero outside of \( \Omega \)), and the theorem from the lecture notes tells us then that \( \Delta v = -\eta f \). In particular, \( \Delta v(x_0) = -\eta f(x_0) = - f(x_0) \).

**Proof that** \( \lim_{x \to y_0} u(x) = g(y_0) \) for \( y_0 \in \partial \Omega \). If \( y_0 \in \partial \Omega \), we start by showing that

\[
\lim_{x \to y_0} \int_{\Omega} f(y)G(x, y)\,dy = 0.
\]

This is because \( G(x, y) = G(y, x) \to 0 \) as \( x \to y_0 \in \partial \Omega \). To prove it, divide the region \( \Omega \) of integration into a small ball centered at \( y_0 \) and the outside of this ball. For \( y \) outside the ball \( B_\varepsilon(y_0) \), the function \( G(x, y) \) is smooth for \( x \) sufficiently near \( y_0 \) and converges to zero pointwise, so \( f(y)G(x, y) \) is smooth and converges to zero uniformly on \( \Omega \setminus B_\varepsilon(y_0) \) as \( x \to y_0 \). On the other hand, the integral over \( B_\varepsilon(y_0) \cap \Omega \) tends to zero as \( \varepsilon \to 0 \) because \( f(y) \) is bounded and \( \int_{B_\varepsilon(y_0)} |G(x, y)|\,dy \leq C \varepsilon^2 \log \varepsilon \), which converges to zero as \( \varepsilon \). So it is enough to show that

\[
- \lim_{x \to y_0} \int_{\partial \Omega} g(y)G_v(x, y)\,dy = g(y_0),
\]

and we turn to this now.

Consider \( y_0 \) in the boundary of \( \Omega \). We will show two things: (1) \(- \int_{\partial \Omega} G_v(x, y)\,dy = 1 \) for all \( x \in \Omega \), and (2) if \( y_0 \in \partial \Omega \) and \( \varepsilon > 0 \), then \( \max_{y \in \partial \Omega \setminus B_\varepsilon(y_0)} |G_v(x, y)| \to 0 \) as \( x \to y_0 \). Suppose (1) and (2) are proved. Then we can write

\[
g(y_0) + \int_{\partial \Omega} g(y)G(x, y)\,dy = \int_{\partial \Omega} (g(y) - g(y_0))G_v(x, y)\,dy
\]

\[
= \int_{\partial \Omega \cap B_\varepsilon(y_0)} + \int_{\partial \Omega \setminus B_\varepsilon(y_0)} (g(y) - g(y_0))G_v(x, y)\,dy.
\]

Now

\[
\left| \int_{\partial \Omega \setminus B_\varepsilon(y_0)} (g(y) - g(y_0))G_v(x, y)\,dy \right| \leq 2\max_{\partial \Omega} |g(y)|\max_{y \in \partial \Omega \setminus B_\varepsilon(y_0)} |G_v(x, y)| \to 0
\]

as \( x \to y_0 \), by (2). On the other hand, since \( G > 0 \) and \( G(x, y) = 0 \) for \( y \in \partial \Omega \), we have \( G_v(x, y) < 0 \) for all \( x \in \Omega \) and \( y \in \partial \Omega \). This means that

\[
\left| \int_{\partial \Omega \cap B_\varepsilon(y_0)} (g(y) - g(y_0))G_v(x, y)\,dy \right| \leq \max_{y \in B_\varepsilon(y_0)} |g(y) - g(y_0)| \int_{\partial \Omega \cap B_\varepsilon(y_0)} (-G_v(x, y))\,dy
\]

\[
\leq \max_{y \in B_\varepsilon(y_0)} |g(y) - g(y_0)| \int_{\partial \Omega} (-G_v(x, y))\,dy
\]

\[
= \max_{y \in B_\varepsilon(y_0)} |g(y) - g(y_0)|.
\]

Since \( g \) is continuous, the last expression tends to zero as \( \varepsilon \to 0 \). So it is enough to show (1) and (2).

First we show (1) using the divergence theorem: note that \( G(x, y) \) is smooth away from \( x = y \), so if we fix \( x \) and choose a small \( \delta > 0 \), we can apply the divergence theorem to the region \( \Omega \setminus B_\delta(x) \) to obtain

\[
\int_{\partial \Omega} G_v(x, y)\,dy - \int_{\partial B_\delta(x)} G_v(x, y)\,dy = \int_{\Omega \setminus B_\delta(x)} \Delta G(x, y)\,dy = 0,
\]

using the fact that \( G \) is harmonic in \( y \) on \( \Omega \setminus B_\delta(x) \). So the two integrals on the left side are equal. We can directly compute the integral over \( \partial B_\delta(x) \). Indeed, remember that \( G(x, y) = \Phi(x - y) - q(x, y) \) where \( \varphi \) is
smooth and harmonic on \( B(x) \). So applying the divergence theorem again to \( \varphi \) shows that the integral of \( G_x \) over \( \partial B_B(x) \) is equal to the integral of \( \Phi_x \) over the same. We can translate the domain so that \( x = 0 \) and compute that \( \Phi_x(0 - y) = \frac{x}{|x|} \cdot \nabla \Phi(y) = (\omega_n |y|^{n-1})^{-1} = (\omega_n \delta^{n-1})^{-1} \) for \( y \in \partial B_B(0) \), and this is precisely the reciprocal of the surface area of \( \partial B_B(0) \). So the integral is equal to one and (1) is proved.

We now prove (2) using the fact that \( G(y_0, y) = 0 \) for all \( y \in \Omega \) if \( y_0 \in \partial \Omega \) (this follows from Problem 3). Since \( G(y_0, y) = 0 \) for all \( y \in \Omega \), we can write

\[
G(x, y) = \int_0^1 (x - y_0) \cdot \nabla_x G(y_0 + t(x - y_0), y) \, dt.
\]

Thus for \( y \in \partial \Omega \) away from corners (which form a set with zero length) we get

\[
G_x(x, y) = v(y) \cdot \nabla_y G(x, y) = v(y) \cdot \int_0^1 (x - y_0) \cdot \nabla_x \nabla_y G(y_0 + t(x - y_0), y) \, dt.
\]

Now notice that \( G \) is smooth on \( B_x(y_0) \times (\Omega \setminus B_x(y_0)) \), so that the derivative \( \nabla_x \nabla_y G \) is bounded uniformly there. Therefore the right expression is bounded by \( C|x - y_0| \) for \( y \in \partial \Omega \setminus B_x(y_0) \) and for \( x \in B_x(y_0) \), and so it converges to zero uniformly on that set. This completes the proof of (2).