2 18.152 Problem set 2 solutions

PROBLEM 2.1.

There were several different correct approaches to this problem and a couple that did not work. Some of you argued from first principles and others invoked the weak maximum principle to get started—either way is fine. Everyone who did this problem correctly used the following approach: introduce an auxiliary function \( w_\varepsilon(x,t) = u(x,t) + \varepsilon v(x,t) \), where \( v \) is a strong subsolution to the heat equation \( (v_t < v_{xx}) \), prove the strong maximum principle for \( w_\varepsilon \) (it cannot attain a max except at the initial time), and then make \( \varepsilon \to 0 \) in order to derive the weak maximum principle for \( u \).

A few people seemed to want to argue that \( u \) cannot attain a maximum except at the initial time \( t = 0 \). This is false: consider a constant function, for instance \( u(x,t) \equiv 0 \), which solves the heat equation and attains its maximum everywhere on \( Q_T \). The strong maximum principle says that this constant function is the only example, but you weren’t asked to prove this.

One more remark: the argument here is simplified slightly by the assumption that \( u \) solves the heat equation on all of \( \overline{Q}_T \). In particular, we’re assuming that \( u \) is \( C^2 \) on the boundary \( \partial Q_T \). If \( u \) solves the heat equation only on the interior \( Q_T \), then the proof can be modified by “coming in” a little bit from the boundary: replace \( Q_T \) with a region \( Q_{T-\varepsilon} \) that is compactly contained in \( Q_T \). One of you actually did this in their proof.

Solution. Suppose \( \varepsilon > 0 \), and let \( w(x,t) = u(x,t) - \varepsilon t \). Then

\[
  w_t = u_t - \varepsilon < u_t = u_{xx} = w_{xx},
\]

so \( w \) is a strong subsolution to the heat equation. Moreover, \( w \) satisfies the same Neumann conditions as \( u \) because \( w_x = u_x \). We will argue that \( w \) cannot attain a maximum at any point \( (x,t) \) with \( t > 0 \).

So suppose that \( t > 0 \) and \( 0 \leq x \leq L \). If \( w_\varepsilon(x,t) < 0 \), then \( w(x,t) < w(x,t) \) for small enough \( \delta \), meaning that \( w \) cannot attain a maximum at \( (x,t) \). We may assume then that \( w_\varepsilon(x,t) \geq 0 \). Because \( w \) is a strict subsolution to the heat equation, the assumption \( w_\varepsilon(x,t) \geq 0 \) implies that \( w_{xx}(x,t) > 0 \). From the second derivative test in single-variable calculus, it follows immediately that \( w \) cannot attain a maximum at \( (x,t) \) if \( 0 < x < L \). But it also follows from the Neumann boundary conditions that \( w \) cannot attain a maximum at a point \( (0,t) \) or \( (L,t) \) with \( t > 0 \) by assumption, \( w_x(0,t) = u_x(0,t) \geq 0 \), and since \( w_{xx}(0,t) > 0 \) by the preceding argument it follows that

\[
  w(\delta,t) \approx w(0,t) + w_x(0,t)\delta + w_{xx}(0,t)\frac{\delta^2}{2} > w(0,t)
\]

for small enough \( \delta > 0 \). (The correction in the approximation is order \( \delta^3 \).) A similar argument using the Neumann condition on the boundary \( (L,t) \) shows that \( w \) cannot attain a maximum at such a point if \( t > 0 \).

We have shown that \( w \) cannot attain a maximum at a point \( (x,t) \) with \( t > 0 \). Since \( \overline{Q}_T \) is a compact region and \( w \) is continuous, it must attain its maximum somewhere on \( \overline{Q}_T \). Thus it attains its maximum at the initial time \( t = 0 \). In other words,

\[
  \max_{\overline{Q}_T} w = \max_{0 \leq x \leq L} u(x,0) = \max_{0 \leq x \leq L} g(x). \tag{1}
\]

Finally, notice that \( u(x,t) = \varepsilon t + w(x,t) \leq \varepsilon T + w(x,t) \) for \( (x,t) \in \overline{Q}_T \), so that \( \text{(1)} \) implies

\[
  \max_{\overline{Q}_T} u \leq \varepsilon T + \max_{0 \leq x \leq L} g(x).
\]

Since \( \varepsilon > 0 \) is arbitrary, it follows that \( u \leq \max g \) and we’re done bounding \( u \) from above.
Finally, as many of you noticed, to deal with the minimum of $u$ in the second case we simply replace $u$ with $-u$ in the conclusion of the first case, giving

$$-\min_{\overline{Q}_T} u = \max(-u) \leq \max_{0 \leq x \leq L} (-g(x)) = -\min_{0 \leq x \leq L} g(x).$$

\[\square\]

**PROBLEM 2.2.**

**Solution.** (1) Everyone got the idea, but several of you missed the lower bound on $u$. Notice that the problem asks you to bound $\max |u|$ and not just $\max u$!

Let $A = \max_{\overline{Q}_T} |f|$ and define $v(x, s) = u(x, s) - As$ as in the hint. Then $v_x = u_x$ so that $v$ satisfies the same boundary conditions as $u$, and $v$ is a subsolution to the heat equation because

$$v_x = u_x - A = u_{xx} + f - A \leq u_{xx}.$$

Notice that the maximum principle from Problem 1 carries through without modification for subsolutions to the heat equation (several of you actually mentioned this in your solutions to Problem 1). Since $v$ is a subsolution to the heat equation that satisfies $v_x(0, t) = v_x(L, t) = 0$ for all $t$, we can apply the conclusion of Problem 1 to $v$ in order to obtain

$$v(x, t) \leq \max_{0 \leq x \leq L} v(x, 0) = \max_{0 \leq x \leq L} g(x) \leq \max_{0 \leq x \leq L} |g(x)|$$

for any $(x, t) \in \overline{Q}_T$. Substituting the definition $v(x, t) = u(x, t) - At$ into the left side and rearranging gives $u(x, t) \leq \max |g| + t \max |f|$.

Next, we establish a lower bound on $u$. The idea is exactly the same: introduce $w(x, s) = -u(x, s) - As$. Then $w$ is also a subsolution to the heat equation, since

$$w_x = -u_x - A = -u_{xx} - f - A \leq -u_{xx} = w_{xx}.$$

Moreover, $w$ satisfies the boundary conditions $w_x(0, t) = w_x(L, t) = 0$, the same as $u$. Arguing exactly as before, we get

$$-u(x, t) = At = w(x, t) \leq \max_{0 \leq x \leq L} w(x, 0) = \max_{0 \leq x \leq L} (-g(x)) \leq \max_{0 \leq x \leq L} |g(x)|.$$

Combining with the upper bound gives $|u(x, t)| \leq \max |g| + t \max |f|$, which is what we set out to show.

(2) We need to show that $w_{xx} - w_t \geq 0$, that’s what it means to be a subsolution of the heat equation. Let’s start by collecting the derivatives of $w$:

$$w_t = \frac{1}{(t + 1)^2} u_x^2 + \frac{2t}{t + 1} u_x u_{xt} + uu_t - A$$

$$= \frac{1}{(t + 1)^2} u_x^2 + \frac{2t}{t + 1} u_x (u_{xxx} + f_x) + uu_{xx} + uf - A,$$

and

$$w_{xx} = \frac{2t}{t + 1} (u_{xx}^2 + u_x u_{xxx}) + uu_{xx} + u_x^2.$$

Thus

$$w_{xx} - w_t = \frac{2t}{t + 1} u_x^2 + \left(1 - \frac{1}{(1 + t)^2}\right) u_x^2 - \frac{2t}{t + 1} u_x f_x - uf + A.$$
On the other hand, in part (2) that So, as mentioned, a simple computation shows that take the second approach but both are similar.

w

nonnegative. Therefore

\[ w_{xx} - w_i \geq \left( 1 - \frac{1}{(1+t)^2} \right) u_x^2 - \frac{2t}{t+1} u_x f_x + f_x^2. \]

The trick is now just to complete the square for the right two terms:

\[ -\frac{2t}{t+1} u_x f_x + f_x^2 = \left( \frac{t}{t+1} u_x - f_x \right)^2 - \frac{t^2}{(t+1)^2} u_x^2. \]

Inserting this into the lower bound for \( w_{xx} - w_i \) gives

\[ w_{xx} - w_i \geq \left( 1 - \frac{1}{(1+t)^2} \right) u_x^2 - \frac{2t}{t+1} u_x f_x + f_x^2 \]

\[ = \frac{2t}{(1+t)^2} u_x^2 + \left( \frac{t}{t+1} u_x - f_x \right)^2 \]

\[ \geq 0. \]

So \( w \) is a subsolution of the heat equation, and we’re done.

(3) There were a variety of computational errors in solutions to this problem. I want to mention two sanity checks that would have resolved a lot of these. First, the inequality that you get in the end should be homogeneous. That is, if you replace \( u \) by \( cu \) where \( c \) is a constant (so that \( g \) and \( f \) are also replaced by \( cg \) and \( cf \)), then the same power of \( c \) should factor out of both sides of the inequality. If you do end up with an inhomogeneous inequality, try replacing \( u \) with \( cu \) and making \( c \to 0 \) and \( c \to \infty \) and see what happens. The second sanity check is that you cannot have a uniform bound on \( |u_x| \) as \( t \to 0 \); there should be something making the upper bound infinite as \( t \to 0 \) (it turns out to be a factor of \( 1/t \) on one of the terms). To see why, consider the case when \( f \) is zero and \( g \) is not differentiable, that is, consider a solution to the heat equation for which the initial data is not differentiable. Such a solution must have the property that \( u_x \) becomes infinite somewhere as \( t \to 0 \).

Aside from the minor errors people did a good job with this. There are two ways to solve it: (1) use the weak maximum principle and notice that \( w \leq u^2/2 \) on the parabolic boundary, or (2) notice that \( w_x(0,t) = w_x(L,t) = 0 \) and apply the result from Problem 1 to conclude that \( w \) is bounded by \( \max_{0 \leq x \leq L} w(x,0) \). I will take the second approach but both are similar.

So, as mentioned, a simple computation shows that \( w_x(0,t) = w_x(L,t) = 0 \) for \( t > 0 \), and since we showed in part (2) that \( w \) is a subsolution to the heat equation, Problem 1 gives

\[ w(x,t) \leq \max_{0 \leq x \leq L} w(x,0) = \frac{1}{2} \max_{0 \leq x \leq L} g(x)^2. \]

Next, notice that \( tu_x^2/(t+1) - At \leq w(x,t) \), so that rearranging the above inequality gives

\[ u_x(x,t)^2 \leq (t+1)A + \frac{t+1}{2t} \max_{0 \leq x \leq L} g^2. \]

On the other hand, \( A = \max_Q (f_x^2 + |f||u|) \) by definition and by substituting the bound for \( |u| \) from part (1) we can bound \( A \) by an expression involving only \( g \) and \( f \), namely,

\[ A \leq \max f_x^2 + \max |f| (\max |g| + T \max |f|). \]
Inserting this into our upper bound for \( u^2 \) gives

\[
 u_x(x, t)^2 \leq (t + 1) (\max f_x^2 + \max |f| \max |g| + T \max f^2) + \frac{t + 1}{2t} \max g^2.
\]

Notice that this passes our sanity checks: the inequality is homogeneous of degree two (if I replace \((u, g, f)\) by \((cu, cg, cf)\) with \(c\) constant then \(c^2\) factors out of both sides), and the right side becomes infinite as \( t \to 0 \). ■

**Problem 2.3.**

The biggest issue I saw with this problem was unclear reasoning—it’s not enough for a problem like this to simply write down \( u_t \) and \( \Delta u \) and then state that \( u_t \leq \Delta u \). In this particular problem, writing down \( u_t \) and \( \Delta u \) is half the battle, but if you just wrote those down and did not argue further you did not receive full credit.

As you can see from the solution to this problem and to Problem 2, very often when you’re asked to prove that something is nonnegative there will be a way to write it so that it is obviously nonnegative. In this problem, for example, the difference \( \Delta u - u_t \) is a sum of squares. Most people found this expression and no one had an issue computing the derivatives.

**Solution.** Recall that \( w = t|\nabla^2 u|^2/(t + 1) + |\nabla u|^2/2 \), where \( u \) is a solution to the heat equation on \( \mathbb{R}^2 \). We will show that

\[
 \Delta w - w_t = \frac{2t}{t + 1} |\nabla^3 u|^2 + \left( 1 - \frac{1}{(t + 1)^2} \right) |\nabla^2 u|^2. \quad (2)
\]

The right side is clearly nonnegative for \( t \geq 0 \), so \( w \) is a subsolution to the heat equation. Here \( |\nabla^3 u|^2 \) is the sum of the squares of all third derivatives of \( u \):

\[
 |\nabla^3 u|^2 = u_{xxx}^2 + u_{yy}^2 + 3u_{xxy}^2 + 3u_{xy}^2.
\]

To prove the formula, we first collect some derivatives of \( w \):

\[
 w_t = \frac{1}{(1 + t)^2} |\nabla^2 u|^2 + \frac{2t}{t + 1} \mathrm{trace}(\nabla^2 u \nabla^2 u_t) + \nabla u \cdot \nabla u_t
 = \frac{1}{(1 + t)^2} |\nabla^2 u|^2 + \frac{2t}{t + 1} \mathrm{trace}(\nabla^2 u \nabla^2 \Delta u) + \nabla u \cdot \nabla \Delta u. \quad (3)
\]

Here \( \mathrm{trace} \) is just the trace of the \( 2 \times 2 \) matrix, that is, the sum of the diagonal entries. Notice that if \( A = (a_{ij}) \) and \( B = (b_{ij}) \) are two symmetric matrices then \( \mathrm{trace} AB \) is their “dot product”: it is the sum of all products \( a_{ij}b_{ij} \). So \( \mathrm{trace}(\nabla^2 u \nabla^2 \Delta u) \) is the sum of all products \( u_{ij}(\Delta u)_{ij} \), where the subscript \( ij \) means the mixed partial derivative \( \partial_i \partial_j \) (remember that the second derivative \( \nabla^2 f \) of a function \( f \) is a symmetric matrix of partial derivatives). And observe that \( \mathrm{trace}(\nabla^2 u \nabla^2 u) = |\nabla^2 u|^2 \).

Next we need the Laplacian of \( w \):

\[
 \Delta w = \frac{t}{t + 1} \Delta |\nabla^2 u|^2 + \Delta |\nabla u|^2. \quad (4)
\]

Notice that

\[
 \Delta |\nabla u|^2 = \frac{1}{2} \mathrm{div} \nabla |\nabla u|^2
 = \mathrm{div} \nabla^2 u \nabla u
 = |\nabla^2 u|^2 + \nabla \Delta u \cdot \nabla u.
\]
You should check these computations! Notice that $\nabla^2 u$ is a matrix and $\nabla u$ is a vector, so the product $\nabla^2 u \nabla u$ is a vector and it makes sense to take its divergence. (This is actually a special case of an important formula called Bochner’s formula.) On the other hand, $|\nabla^2 u|^2 = |\nabla u_x|^2 + |\nabla u_y|^2$, so exactly the same reasoning gives
\[
\Delta |\nabla^2 u|^2 = 2|\nabla u_x|^2 + 2\nabla u_x \cdot \nabla u_y + 2\nabla u_y \cdot \nabla u_y
\]
Putting everything together gives
\[
\Delta w = \frac{2t}{t+1} \left( |\nabla^3 u|^2 + \text{trace} (\nabla^2 u \nabla^2 u) \right) + |\nabla^2 u|^2 + \Delta u \cdot \nabla u. \tag{4}
\]
Finally, comparing (4) and (3) shows that (2) holds, as promised.

**Problem 2.4.**

**Solution.** Let me explain the strategy first. Suppose we can find a family of functions $\psi_R$ and constants $c_R > 0$, where $R$ varies over $\mathbb{R}$, with the following properties: (1) $w := \psi_R u_x^2 + c_R u^2$ is a subsolution to the heat equation, (2) $0 \leq \psi_R \leq \psi_R(2R)$ (it’s nonnegative and attains its max at $x = 2R$) and $\psi_R$ is zero outside the interval $[R, 3R]$, and (3) $c_R/\psi_R(2R) \to 0$ as $R \to \infty$. Let’s suppose for a moment that $\psi_R$ and $c_R$ can be chosen with these properties and see how to finish the proof.

Fix $R$. Since $w$ is a subsolution to the heat equation, we can apply the maximum principle to it on the compact strip $\overline{Q}_R = [R, 3R] \times [0, T]$ (really $\overline{Q}_R$ depends on $R$ too but I’m going to suppress that dependence to avoid cluttered notation). For $(x, t) \in \overline{Q}_R$, the (weak) maximum principle says that
\[
w(x, t) \leq \max_{\partial_r \overline{Q}_R} w,
\]
where $\partial_r \overline{Q}_R$ is the parabolic boundary of $\overline{Q}_R$ (consisting of the initial segment $[R, 3R] \times \{0\}$, the left boundary $\{R\} \times [0, T]$ and the right boundary $\{3R\} \times [0, T]$). Now on the sides $\{R\} \times [0, T]$ and $\{3R\} \times [0, T]$, we have $w = c_R u^2$ (where $C$ is the bound for $|u|$ assumed in the problem statement) because $\psi_R(R) = \psi_R(3R) = 0$ by assumption. Therefore we may write
\[
\psi_R(x)u_x(x, t)^2 \leq w(x, t) \leq \max_{\partial_r \overline{Q}_R} w
\]
\[
\leq \frac{\psi_R(3R)}{\psi_R(2R)} \max_{\partial_r \overline{Q}_R} w(x, t, 0)^2
\]
\[
\leq \frac{c_R}{\psi_R(2R)} \max_{\partial_r \overline{Q}_R} w(x, t, 0)^2.
\]
using in the last inequality that $\psi_R(2R)$ is the maximum value for $\psi_R$. Taking then $x = 2R$ on the left side and dividing through by $\psi_R(2R)$, we get the following inequality:
\[
u_x(2R, t)^2 \leq \frac{c_R}{\psi_R(2R)} \max_{\partial_r \overline{Q}_R} w(x, t, 0)^2. \tag{5}
\]
I claim that the right side of (5) converges to zero as $R \to \infty$. Indeed, the second term $\max_{R \leq y \leq 3R} u_x(y, 0)^2$ converges to zero because of the assumption from the problem statement that $\lim_{y \to \infty} u_x(y, 0) = 0$ (its convergence to zero is equivalent to this assumption). But the first term converges to zero as $R \to \infty$ because (A) $c_R/\psi_R(2R) \to 0$ by property (3) that we assumed about $c_R$ and $\psi_R$, and (B) $u^2$ is bounded by some constant on all of $\mathbb{R} \times [0, T]$ (this is assumed in the problem statement). So my claim is true and the right side of (5) does converge to zero as $R \to \infty$, meaning, of course, that the left side of (5) converges to zero also. But since $t$ is arbitrary, this is equivalent to the statement that $\lim_{R \to \infty} u_x(2R, t) = 0$ for all $t$, and that’s exactly
what we set out to prove. The other limit, as \( R \to -\infty \), is dealt with in exactly the same way (or you can apply the conclusion we just deduced to \( v(x, t) = u(-x, t) \), which also solves the heat equation and satisfies all the assumptions from the problem statement).

It remains only to find \( \psi_R \) and \( c_R \) with the required properties, and this is exactly what the hint does. Thus let \( \psi_R = \eta^2 = (x - R)^2(x - 3R)^2 \), so \( \psi_R \) is \( (x - R)^2(x - 3R)^2 \) when \( R \leq x \leq 3R \) and is zero otherwise. We will see in a moment why to choose \( c_R = 20R^2 \) as in the hint—this is chosen to make \( \psi_R u^2_x + c_R u^2 \) a subsolution to the heat equation (though as several of you found in your solutions, a smaller constant would actually do).

For clarity, I’ll suppress the dependence of \( c_R \) and \( \psi_R \) on \( R \) and just write \( c \) and \( \psi \) instead. Let’s compute the derivatives of \( w = \psi u^2_x + cu^2 \) and see what’s required for \( w \) to be a subsolution to the heat equation. Using the fact that \( u \) solves the heat equation, one finds that

\[
\begin{align*}
w_{xx} - w_t &= \psi_{xx} u^2_x + 4\psi_x u_x u_{xx} + 2\psi u^2_x + 2cu^2_x.
\end{align*}
\]

Now, on \((R, 3R)\), we have

\[
\psi_x = 4(x - R)(x - 2R)(x - 3R),
\]

and

\[
\psi_{xx} = 2(x - 3R)^2 + 2(x - R)^2 + 8(x - R)(x - 3R) \geq -8R^2.
\]

so we get

\[
\begin{align*}
w_{xx} - w_t &\geq -8R^2 u^2_x + 16(x - R)(x - 2R)(x - 3R)u_x u_{xx} + \psi u^2_x + 2cu^2_x \\
&\geq (2c - 40R^2)u^2_x,
\end{align*}
\]

where to obtain the second inequality we’ve used

\[
16(x - R)(x - 2R)(x - 3R)u_x u_{xx} \geq -32(x - 2R)^2 u^2_x - 2(x - R)^2(x - 3R)^2 u^2_{xx}
\]

\[
= -32(x - 2R)^2 u^2_x - \psi u^2_x
\]

\[
\geq -32R^2 u^2_x - \psi u^2_{xx}.
\]

According to the inequality \( \boxed{} \), \( w \) is a subsolution provided \( c \geq 20R^2 \), which is exactly the value of \( c \) suggested in the hint.

All that remains to check is that \( \psi \) and \( c \) satisfy our other conditions, that is, that \( \psi \) is nonnegative and attains its max at \( x = 2R \) (this is easy to check) and that \( c_R/\psi_R(2R) \to 0 \) as \( R \to \infty \). But for \( c_R = 20R^2 \) this second condition is clearly true since \( c_R/\psi_R(2R) = 20R^2/R^3 = 20R^2 \). So we’re done.

\section*{Problem 2.5.}

This was a challenging problem, and everyone who attempted it did a very nice job. There were a few key steps, and by far the most challenging was the construction of an appropriate function \( \varphi \) that bounds \( g \) on \( \Omega \). Here are a sequence of steps which, if completed, would have earned you full credit (everyone who received full credit followed these steps):

1. Notice that \(|\nabla u|^2\) is a subsolution to the heat equation and apply the maximum principle to it.

2. Use the barrier function \( \varphi(x) = a \log |x|/R \), where \( R \) is the radius of an exterior circle centered at the origin as in the hint and \( a \) is a constant to be chosen.

3. Choose a constant \( a \) in the definition of \( \varphi \) so that (1) \( \varphi \geq |g| \) on \( \Omega \) and (2) the gradient \( |\nabla \varphi(x_0)| \) at the boundary point \( x_0 \) is bounded by \( \max |g|, \max |\nabla g| \), and \( R \). This was the part that people had the most trouble with—to do it successfully you should bound \(|g|\) separately in the region near \( x_0 \) and far away from \( x_0 \).
4. Argue by using the comparison or maximum principle that $\varphi \geq |u|$ everywhere and then use this to conclude that $|\nabla \varphi(x_0)| \geq |\nabla u(x_0)|$.

Solution. We follow the hint and construct a “barrier” $\varphi$ which will allow us to bound the gradient of $u$.

First, notice that $|\nabla u|^2$ is a subsolution to the heat equation:

$$\partial_t |\nabla u|^2 = 2u \cdot \nabla u = 2u \cdot \nabla \Delta u = \Delta |\nabla u|^2 - |\nabla^2 u|^2 \leq \Delta |\nabla u|^2.$$ 

The maximum principle therefore applies to $|\nabla u|^2$, and it says that $|\nabla u|^2$ attains its max on the parabolic boundary of $\overline{Q_T} := \overline{\Omega} \times [0, T]$. This means we have a bound

$$\max_{\partial_t} |\nabla u|^2 \leq \max_{\Omega} |\nabla u(\cdot, 0)|^2 + \max_{0 \leq s \leq T} \max_{\partial \Omega} |\nabla u(\cdot, t)|^2 = \max_{\Omega} |\nabla g|^2 + \max_{0 \leq s \leq T} \max_{\partial \Omega} |\nabla u(\cdot, t)|^2.$$ 

The problem is then reduced to bounding $|\nabla u(x, t)|$ for $x$ in the boundary $\partial \Omega$ of $\Omega$.

Now suppose for a moment that we have constructed the function $\varphi$ from the hint near a point $x_0 \in \partial \Omega$: the function has the properties that $\varphi(x_0) = 0$, that $\varphi$ is rotationally symmetric around the center of an exterior sphere touching $\partial \Omega$ at $x_0$, that $\varphi \geq |g|$ in $\Omega$, and that $\Delta \varphi \leq 0$. I claim that in this situation $|\nabla u(x_0)| \leq |\nabla \varphi(x_0)|$. To see this, notice that $u(x, t) = u(x, t) - \varphi(x)$ is a subsolution to the heat equation. Therefore, it attains its maximum on the parabolic boundary of $\overline{Q_T}$. Since $\varphi \leq 0$ on the parabolic boundary (check this!), it follows that $u(x, t) \leq \varphi(x)$ for $(x, t) \in \overline{Q_T}$. If we replace $u$ with $-u$, we also find that $-u \leq \varphi$, hence in fact $|u| \leq \varphi$ on $\overline{Q_T}$.

Let $v$ be the outer unit normal to $\partial \Omega$ at the point $x_0$. Since the level sets of both $u$ and $\varphi$ are tangent to $\partial \Omega$ at $x_0$, we have $\nabla u(x_0) = \pm |\nabla u(x_0)| \nu$ and $\nabla \varphi(x_0) = |\nabla \varphi(x_0)| \nu$: the gradients of both functions are proportional to $\nu$. Then $|u(x_0 - \epsilon \nu)| = |\nabla u(x_0)| + O(\epsilon^2)$ and $\varphi(x_0 - \epsilon \nu) = \epsilon |\nabla \varphi(x_0)| + O(\epsilon^2)$ for small $\epsilon$, and from $|u| \leq \varphi$ it follows at once by sending $\epsilon \to 0$ that $|\nabla u(x_0)| \leq |\nabla \varphi(x_0)|$.

So it is enough to construct $\varphi$ at every point and show that every such $\varphi$ satisfies a uniform bound in terms of $g, |\nabla g|$ and the smallest radius of an exterior sphere touching $\partial \Omega$.

Now we construct $\varphi$ for a particular point $x_0 \in \partial \Omega$. As in the hint, consider the largest circle that lies outside of $\Omega$ and is tangent to $\partial \Omega$ at $x_0$. Translate the picture so that the center of this circle is the origin, and let $R$ denote its radius. We take $\varphi$ of the form $\varphi(x) = a \log(|x| / R)$, for a constant $a$ chosen so that $\varphi \geq |g|$ in $\Omega$. Notice that $\Delta \varphi = 0$ and that $\varphi(x_0) = 0$. (The function $\log |x|$ is called the fundamental solution to Laplace’s equation $\Delta w = 0$ on $\mathbb{R}^2$.)

It remains to pick $a$. Observe that $|\nabla \varphi(x_0)| = a / R$. Thus our bound on $|\nabla u|$ at the point $(x_0, t) \in \partial \Omega \times [0, T]$ is $a / R$. The challenge then in the choice of $a$ is that it must meet two competing criteria: (1) It must be large enough that $a \log(|x| / R) \geq |g|$ on $\Omega$, and (2) it must be small enough that we can bound it uniformly in terms of $R$, $\max |g|$, and $\max |\nabla g|$. Once we accomplish these two objectives we will have obtained a uniform bound on $|\nabla u|$ on all of $\partial \Omega \times [0, T]$.

To find $a$, we start by noticing that $|g(x)| \leq \text{dist}(x, \partial \Omega) \max |\nabla g|$ for $x \in \Omega$. This follows from the mean value inequality and the fact that $g = 0$ on $\partial \Omega$. Now choose $a$ to be the larger of $2R \max |\nabla g|$ and $\max |g| / \log 2$. I claim that with this choice of $a$ we have $\varphi \geq |g|$ on $\Omega$. We will check this separately inside the ball $B_{2R}$ of radius $2R$ centered at the origin and outside of the ball $B_{2R}$:

- For $x \in \Omega$ with $|x| \leq 2R$ we use the fact that $a \geq 2R \max |\nabla g|$ and that the ball $B_R$ of radius $R$ centered at the origin is outside of $\Omega$. Since $\varphi$ is a radial function that is nonnegative and radially increasing, we have (check the first inequality!)

$$\varphi(x) \geq \text{dist}(x, \partial B_R) \min_{x \in B_{2R} \setminus B_R} |\nabla \varphi| \geq \text{dist}(x, \partial B_R) a 2R \geq \text{dist}(x, \partial B_R) |\nabla g|.$$ 

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Finally, since $B_R$ is outside of $\Omega$ we have $\text{dist}(x, \partial B_R) \geq \text{dist}(x, \partial \Omega)$, so in sum

$$\varphi(x) \geq \text{dist}(x, \partial \Omega) \max |\nabla g| \geq |g(x)|$$

for $x \in B_{2R} \cap \Omega$.

- For $x \in \Omega$ with $|x| \geq 2R$ we use $a \geq \max |g|/\log 2$ and $\log |x|/R \geq \log 2$. Together these two facts imply that $\varphi(x) \geq \max |g|$ for $|x| \geq 2R$.

This finishes the construction of $\varphi$. In sum, we have shown that

$$|\nabla u(x_0, t)| \leq |\nabla \varphi(x_0)| = \frac{a}{R} \leq 2 \max |\nabla g| + \frac{\max |g|}{R \log 2}.$$

Combining this with the earlier argument and taking the maximum of the right side over all points $x_0$ in $\partial \Omega$ gives

$$\max_{\partial \Omega} |\nabla u|^2 \leq 4 \max_{\Omega} |\nabla g|^2 + \frac{1}{(R_0 \log 2)^2} \max_{\Omega} |g|^2,$$

where $R_0$ is the minimum over $x_0$ of the radius of the largest external circle tangent to $\partial \Omega$ at the point $x_0$. ■