You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

**Problem 1.** Let \( g(x), h_1(t), h_2(t), f(x,t) \) be smooth functions and let \( \alpha \geq 0 \) be a non-negative constant. Then, the following Cauchy-Robin problem to the diffusion equation has at most one smooth solution.

\[
\begin{align*}
    u_t(x,t) &= u_{xx}(x,t) + f(x,t), \quad \text{for } 0 \leq x \leq L, 0 \leq t, \\
    -u_x(0,t) + \alpha u(0,t) &= h_1(t), \quad \text{for } 0 \leq t, \\
    u_x(L,t) + \alpha u(L,t) &= h_2(t), \quad \text{for } 0 \leq t, \\
    u(x,0) &= g(x), \quad \text{for } 0 \leq x \leq L.
\end{align*}
\]

Notice that if \( \alpha = 0 \) then it is a Cauchy-Neumann problem.

**Problem 2.** Given smooth \( g(x) \), find the all solutions to the following Cauchy-Neumann problem;

\[
\begin{align*}
    u_t(x,t) &= u_{xx}(x,t), \quad \text{for } 0 \leq x \leq L, 0 \leq t, \\
    -u_x(0,t) &= -1, \quad \text{for } 0 \leq t, \\
    u_x(L,t) &= 2, \quad \text{for } 0 \leq t, \\
    u(x,0) &= g(x), \quad \text{for } 0 \leq x \leq L.
\end{align*}
\]

Hint: Refer the uniqueness result in the previous result, and consider a function \( \tilde{u}(x,t) = u(x,t) + ax^2 + bx + ct \) satisfying \( \tilde{u}_t = \tilde{u}_{xx} \) and \( \tilde{u}_x(0,t) = \tilde{u}_x(L,t) = 0 \).

**Problem 3.** Suppose that \( u(x,t) \) is the smooth solution to the following Cauchy-Neumann problem;

\[
\begin{align*}
    u_t(x,t) &= u_{xx}(x,t), \quad \text{for } 0 \leq x \leq L, 0 \leq t, \\
    u_x(0,t) &= 0, \quad \text{for } 0 \leq t, \\
    u(x,0) &= g(x), \quad \text{for } 0 \leq x \leq L,
\end{align*}
\]

where \( g(x) \) is smooth. Then, for \( p \geq 2 \) prove the following inequality

\[
\frac{d}{dt} \int_0^L |u(x,t)|^p dx \leq 0,
\]

namely \( \sqrt[p]{\int_0^L |u(x,t)|^p dx} \) monotonically decreases.

Notice that a smooth function \( u(x,t) \) satisfies

(\(^*)\)

\[
\lim_{p \to \infty} \left( \int_0^L |u(x,t)|^p dx \right)^{\frac{1}{p}} = \sup_{0 \leq x \leq L} |u(x,t)|.
\]
Hence, the result in Problem 3 shows that \( \sup_{0 \leq x \leq L} |u(x,t)| \) monotonically decreases. We will prove it again in class by using another method without (*)

**Problem 4.** Let \( u(x,t) \) be the smooth solution to the following Cauchy-Neumann problem:

\[
\begin{align*}
&u_t(x,t) = u_{xx}(x,t), \quad \text{for } 0 \leq x \leq L, 0 \leq t, \\
&u_x(0,t) = u_x(L,t) = 0, \quad \text{for } 0 \leq t, \\
&u(x,0) = g(x), \quad \text{for } 0 \leq x \leq L,
\end{align*}
\]

where \( g(x) \) is smooth. Show the following inequality

\[
\frac{d}{dt} \int_0^L \left| u_x(x,t) \right|^2 + \left| u_t(x,t) \right|^2 dx \leq 0.
\]