18.152 Practice problems for the midterm exam

The midterm exam will take place on March 19th Tuesday 9:35-10:50.

As an open book exam, during the exam you can see
1. the textbook : Partial Differential Equations in Action by Sandro Salsa,
2. notes, copies, and scratch papers.

However, the following are NOT allowed to use
1. electronic devices including Smartpads
2. the other books except the textbook.

\textbf{Problem 1.} Given a smooth bounded domain $\Omega \subset \mathbb{R}^n$ and a smooth function $f(x)$, a smooth function $u$ satisfies $\Delta u = f$ in $\Omega$ and $u = 0$ on $\partial \Omega$.

(1) Show that the following inequality holds for any $\epsilon > 0$

$$\int_{\Omega} |\nabla u|^2 dx \leq \frac{1}{\epsilon} \int_{\Omega} f^2 dx + \frac{\epsilon}{4} \int_{\Omega} u^2 dx.$$  

(2) (easy bonus) Suppose that we have the following inequality

(1) $$\int_{\Omega} |u|^2 dx \leq C \int_{\Omega} |\nabla u|^2 dx,$$

for some constant $C$. Show that

$$\int_{\Omega} |\nabla u|^2 dx \leq C \int_{\Omega} f^2 dx,$$

holds for some large constant $C$.

Note that the inequality (1) is called the Poincaré inequality.

The following two problems are related to the problem 1 in the first set.

\textbf{Problem 2.} Let $u(x,y)$ be a smooth harmonic defined on $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ satisfying the following boundary conditions

$$u(0,y) = 0, \quad u(1,y) = 0, \quad -u_y(x,0) = g(x), \quad u_y(x,1) = h(x),$$

where $g,h$ are smooth functions. Suppose that there exists two smooth non-zero function $v(x)$ and $w(y)$ such that $v(x)w(y)$ is a harmonic function on $\Omega$ and $v(0) = v(1) = 0$.

(1) Show that $v'' = -\lambda v$ and $w'' = \lambda w$ hold for some $\lambda \in \mathbb{R}$.

(2) Show that $\lambda$ would be $(\pi n)^2$ where $n \in \mathbb{N}$, and determine $v_n(x)$ for each $n$.

(3) For $\lambda = (\pi n)^2$, we know that $w_n(y) = A_n \cosh \pi ny + B_n \sinh \pi ny$ for some constant $A_n, B_n$ by ODE theory. We define $u(x) = \sum_{n=1}^{\infty} v_n(x)w_n(y)$, and determine $B_n$ to satisfies $-u_y(x,0) = g(x)$.

(4) Determine $A_n$ by using $u_y(x,1) = h(x)$. 

(5) Show that the \( u(x, y) = \sum_{n=1}^{\infty} v_n(x)w_n(y) \) is the only harmonic function satisfying the given boundary conditions.

Problem 3. Let \( u(x, y) \) be a smooth harmonic defined on \( \Omega = B_1(0) \subset \mathbb{R}^2 \) satisfying the following boundary conditions
\[ u(\cos \theta, \sin \theta) = g(\theta), \]
where \( g \) is a smooth function. Suppose that there exists two smooth non-zero function \( R(r) \) and \( \Theta(\theta) \) such that \( R(r)\Theta(\theta) = v(r \cos \theta, r \sin \theta) \) is a harmonic function on \( \Omega \) and \( \lim_{r \to 0} R(r) = 0 \).

(1) Show that \( \Theta'' = -\lambda \Theta \) and \( r^2 R'' + r R' = \lambda R \) hold for some \( \lambda \in \mathbb{R} \).
(2) Find possible \( \lambda \), and determine \( v_{\lambda}(r) \) for each \( \lambda \).
(3) Given possible \( \lambda \), find \( R_{\lambda}(r) \). We know that \( R(r) = C r^p \) or \( R(r) = C \log r \) by ODE theory.
(4) Define \( u = \sum_{\lambda} R_{\lambda}\Theta_{\lambda} \) and determine coefficients.
(5) Show that the function \( u = \sum_{\lambda} R_{\lambda}\Theta_{\lambda} \) is the only harmonic function satisfying the Dirichlet condition.

Problem 4. Suppose that a smooth function \( u(x) \) satisfy \( \Delta u + u = 0 \) in \( \Omega \subset \mathbb{R}^2 \).

(1) Show that the maximum principle does not hold for the solution \( u \).
(2) Suppose that there exists a positive smooth function \( w \) satisfying \( \Delta w + w = 0 \) in \( \Omega \). Prove that
\[ \max_{\Omega} \frac{u}{w} \leq \max_{\partial \Omega} \frac{u}{w}. \]

Problem 5. Let \( u(x, y, t) \) be a solution to the heat equation on \( \mathbb{R}^2 \times [0, T] \) satisfying \( |u(x, y, t)| \leq C \) on \( \mathbb{R}^2 \times [0, T] \) for some constant \( C \) and also satisfying
\[ \lim_{|x,y| \to +\infty} \sqrt{x^2 + y^2} |\nabla u(x, y, 0)| = 0. \]

(1) Show that the following holds for each \( t \geq 0 \)
\[ \lim_{|x,y| \to +\infty} \sqrt{x^2 + y^2} |\nabla u(x, y, t)| = 0. \]

(2) Suppose that \( \int_{\mathbb{R}^2} u^2(x, y, t)dy \leq C \) for \( t \in [0, T] \). Prove that the following holds for \( t \in [0, T] \).
\[ \int_{\mathbb{R}^2} |u(x, y, t)|^2dy \leq \int_{\mathbb{R}^2} |u(x, y, 0)|^2dy. \]

Hint(1): Modify the proof of the problem 4 in the second set.
Problem 6. $\Omega$ is a smooth bounded domain in $\mathbb{R}^2$ and $g(x)$ is a smooth function defined on $\partial \Omega$. Suppose that the graph of a smooth function $u(x)$ has the minimum area within the graph of smooth functions satisfying the Dirichlet boundary condition $u = g$ on $\partial \Omega$. Namely,

$$\int_{\Omega} \sqrt{1 + |\nabla u|^2} dx \leq \int_{\Omega} \sqrt{1 + |\nabla v|^2} dx,$$

holds if $v : \Omega \to \mathbb{R}$ is a smooth function satisfying $v = g$ on $\partial \Omega$. Then, the following equation holds in $\Omega$

(2) \[ \text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0. \]

Note that the left hand side of (2) is called the mean curvature of the graph. Namely, the area minimizer has the zero mean curvature.