## 1. Weak convergence

**Definition 1.** Given a Hilbert space $(H, \langle , \rangle)$, a sequence $\{x_k\} \subset H$ converges weakly to $x \in H$ and we write

$$x_k \rightharpoonup x$$

if the following holds for every $y \in H$

$$\lim_{k \to +\infty} \langle x_k, y \rangle = \langle x, y \rangle.$$

**Theorem 2** (Theorem 6.57)**.** *Every bounded sequence in a separable Hilbert space $H$ contains s subsequence which is weakly convergent to an element $x \in H$.*

**Theorem 3** (Theorem 6.56)**.** *Let $\{x_k\} \subset H$ such that $x_k \rightharpoonup x$. Then,*

$$\|x\| \leqslant \liminf_{k \to \infty} \|x_k\|.$$

## 2. Compactness

**Theorem 4** (Rellich-Kondrachov compactness)**.** *Every bounded sequence in $H^1_0(\Omega)$ contains a subsequence which is strongly convergent to an element $x \in L^2(\Omega)$.*

See Evans' PDE, pp 272 for the proof.

*Proof in the $n = 1$ case.* We quickly check the proof in the simplest case $\Omega = (0, 1) \subset \mathbb{R}$.

We claim that given $\epsilon > 0$ there exists a finite set of $L^2$-functions $\mathcal{F} = \{F_1, \cdots, F_{m_\epsilon}\}$ such that for each $u \in H^1_0(0, 1)$ with $\|u\|_{H^1} = 1$ there exists $F_i \in \mathcal{F}$ such that $\|u - F_i\|_{L^2} \leqslant 10\epsilon$. By the theorem 5, we have $u^* \in C^{0,\frac{1}{2}}[0, 1]$ such that $\|u - u^*\|_{H^1} = 0$ and

$$|u^*(x) - u^*(y)| \leqslant C_1 \|u^*\|_{H^1} |x - y|^{\frac{1}{2}} = C_1 |x - y|^{\frac{1}{2}}.$$

Since $u^*$ is continuous on $[0, 1]$, the Sobolev inequality yields $\sup_{[0,1]} |u^*| \leqslant C_2$.

We choose two large numbers $N, M \in \mathbb{N}$ such that $C_1 N^{-\frac{1}{2}} \leqslant \frac{1}{2}\epsilon$ and $M^{-1} \leqslant \frac{1}{2}\epsilon$. Then, given $k \in \mathbb{N}$ with $k \leqslant N$ there exists some integer $m$ such that $|u^*(\frac{k}{N}) - \frac{m}{M}| \leqslant M^{-1}$. Then, for $x \in [\frac{k-1}{N}, \frac{k}{N}]$ we have

$$|u^*(x) - \frac{m}{M}| \leqslant |u^*(x) - u^*(\frac{k}{N})| + |u^*(\frac{k}{N}) - \frac{m}{M}| \leqslant C_1 |x - kN^{-1}|^{\frac{1}{2}} + M^{-1} \leqslant \epsilon.$$

Therefore,

$$\int_{\frac{k-1}{N}}^{\frac{k}{N}} |u^*(x) - \frac{m}{N}|^2 dx = \int_{\frac{k-1}{N}}^{\frac{k}{N}} |u^*(x) - \frac{m}{N}|^2 dx \leqslant \int_{\frac{k-1}{N}}^{\frac{k}{N}} \epsilon^2 dx = \frac{\epsilon^2}{N}.$$

Now, we consider the set $\mathcal{F}$ which consists of the following $L^2$ functions

$$F(x) = m_k M^{-1}, \quad \text{on each interval } \frac{k-1}{N} < x \leqslant \frac{k}{N},$$

where $k \in \mathbb{N}$, $k \leqslant N$ $m_k \in \mathbb{Z}$ and $|m_k| \leqslant (C_2 + 1)M$.

Then, we can choose $F \in \mathcal{F}$ such that $|u^*(\frac{k}{N}) - F(\frac{k}{N})| \leqslant M^{-1}$, and so

$$\|u - F\|_{L^2}^2 = \|u^* - F\|_{L^2}^2 = \int_0^1 |u^*(x) - F(x)|^2 dx = \sum \int_{\frac{k-1}{N}}^{\frac{k}{N}} |u^*(x) - F(x)|^2 dx \leqslant N \cdot \frac{1}{N} \cdot \epsilon^2.$$

This completes the proof of the claim.

Now, we consider a bounded sequence $\{u_k\} \subset H_0^1(0,1)$ with $\|u_k\|_{H^1} \leqslant 1$. Then, by the claim there exists some $L^2$ function $F_1$ such that a subsequence $\{u_j^1\}$ satisfies $\|u_k^1 - F_1\|_{L^2} \leqslant \frac{1}{2}$. Next, we obtain a subsequence $\{u_i^2\}$ of the sequence $\{u_j^1\}$ such that $\|u_k^2 - F_2\|_{L^2} \leqslant \frac{1}{4}$ for some $F^2 \in L^2$ with $\|F_1 - F_2\|_{L^2} \leqslant \frac{1}{2}$. We iterate this process so that we obtain a Cauchy sequence $\{F^k\} \subset L^2$, namely $F^k \to \bar{F}$ in $L^2$. We observe that the subsequence $\{u_s^s\}_{s=1}^\infty$ converges to $\bar{F}$ strongly. $\square$

**Theorem 5** (Morrey's inequality)**.** *Suppose $u \in W^{1,p}(\Omega)$ for some $p > n$. Then, there exists a function $u^* \in C^{0,\gamma}(\overline{\Omega})$ with $\gamma = 1 - \frac{n}{p}$ such that $u = u^*$ almost everywhere and the following holds*

$$\|u^*\|_{C^{0,\gamma}(\overline{\Omega})} \leqslant C\|u\|_{W^{1,p}(\Omega)}$$

*for some constant $C$ depending on $p$, $n$, and $\Omega$.*

**Corollary 6.** *Every bounded sequence $\{u_k\} \subset H_0^1(\Omega)$ has a subsequence $\{u_{k_m}\}$ such that*

$$u_{k_m} \rightharpoonup \bar{u} \quad \text{in } H_0^1(\Omega), \qquad\qquad u_{k_m} \to \bar{u} \quad \text{in } L^2(\Omega),$$

*for some $\bar{u} \in H_0^1(\Omega)$.*

We will verify the corollary in the next class.

## 3. Eigen decomposition I

Suppose that a function $u \in H_0^1(\Omega)$ and a real number $\lambda$ satisfy

$$\int_\Omega \nabla u \cdot \nabla v dx = \lambda \int_\Omega uv dx,$$

for all $v \in H_0^1(\Omega)$. Then, we call $u$ and $\lambda$ as an Dirichlet eigenfunction and an eigenvalue for the Laplacian.

**Lemma 7.** *Suppose that $(u_1, \lambda_1)$ and $(u_2, \lambda_2)$ are pairs of eigenfunctions and eigenvalues. If $\lambda_1 \neq \lambda_2$ then*

$$\langle u_1, u_2 \rangle_{H^1} = 0.$$

*Proof.* Since $u_1, u_2 \in H_0^1$, we have

$$\lambda_1 \int u_1 u_2 dx = \int \nabla u_1 \cdot \nabla u_2 dx = \lambda_2 \int u_1 u_2 dx.$$

Thus, if $\lambda_1 \neq \lambda_2$ then

$$0 = \int u_1 u_2 dx = \int \nabla u_1 \cdot \nabla u_2 dx.$$

□

**Proposition 8.** *There exists a positive constant $C$ depending on $\Omega$ such that*

$$\int_\Omega |\nabla u|^2 dx \geqslant C \int_\Omega u^2 dx,$$

*for all $u \in H_0^1(\Omega)$.*

*Proof.* The Sobolev inequality. □

The proposition above implies that

$$\inf_{\|u\|_{L^2} \neq 0, u \in H^1} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} = \lambda_1 > 0.$$

**Theorem 9.** *There exists a function $w_1 \in H_0^1$ with $\|w_1\|_{L^2} \neq 0$ such that*

$$\frac{\int_\Omega |\nabla w_1|^2 dx}{\int_\Omega w_1^2 dx} = \lambda_1$$

*Proof.* There exists a sequence $\{u_j\} \subset H_0^1$ such that $\|u_j\|_{L^2} = 1$ and $\lim \int |\nabla u_j|^2 dx = \lambda_1$. We observe $\|u_j\|_{H^1}^2 = \int u_j^2 dx + \int |\nabla u_j|^2 dx \leqslant 1 + \lambda_1 + \epsilon \leqslant C$ for large $j$, namely $\{u_j\}$ is a bounded sequence in $H_0^1$. Hence, the corollary 6 guarantees that there exists a subsequence $\{u_{j_m}\}$ such that

$$u_{j_m} \rightharpoonup w_1 \quad \text{in } H_0^1(\Omega), \qquad\qquad u_{j_m} \to w_1 \quad \text{in } L^2(\Omega),$$

for some $w_1 \in H_0^1(\Omega)$.

Since $\|u_{j_m}\|_{L^2} = 1$, we can observe $\|w_1\|_{L^2} = 1$. Moreover, the theorem 3 shows

$$\|\bar{w}_1\|_{H^1}^2 \leqslant \liminf \|u_{j_m}\|_{H^1}^2 = \liminf \int |u_{j_m}|^2 + |\nabla u_{j_m}|^2 = 1 + \lambda_1.$$

This completes the proof. □

**Lemma 10.** *The ratio minimizer $w_1$ is an egienfunction and $\lambda_1$ is the corresponding eigenvalue.*

*Proof.* Without loss of generality, we assume $\|w_1\|_{L^2} = 1$. Given $v \in H_0^1$, we define a functional

$$I(t) = \frac{\int_\Omega |\nabla w_1 + t\nabla v|^2 dx}{\int_\Omega |w_1 + tv|^2 dx}.$$

Since $w_1 + tv \in H_0^1$, we have $I(t) \geqslant I(0)$. Differentiating $\log I$ yields

$$\frac{I'(t)}{I(t)} = \frac{\int_\Omega 2\nabla w_1 \cdot \nabla v + 2t|\nabla v|^2 dx}{\int_\Omega |\nabla w_1 + t\nabla v|^2 dx} - \frac{\int_\Omega 2w_1 v + 2tv^2 dx}{\int_\Omega |w_1 + tv|^2 dx}.$$

Hence,

$$0 = \frac{I'(0)}{2I(0)} = \frac{\int_\Omega \nabla w_1 \cdot \nabla v dx}{\int_\Omega |\nabla w_1|^2 dx} - \frac{\int_\Omega w_1 v dx}{\int_\Omega |w_1|^2 dx} = \frac{1}{\lambda_1} \int_\Omega \nabla w_1 \cdot \nabla v dx - \int_\Omega w_1 v dx.$$

□

Next, we define $X_2 = \text{span}\{w_1\}^\perp \subset H_0^1(\Omega)$, and then obtain a pair of eigenfunction $w_2 \in X_2$ and an eigenvalue $\lambda_2 \geqslant \lambda_1$ satisfying

$$\frac{\int_\Omega |\nabla w_2|^2 dx}{\int_\Omega w_2^2 dx} = \lambda_2 = \inf_{u \in X_2, \|u\|_{L^2} \neq 0} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}.$$

In addition, $w_2 \in X_2$ implies

$$0 = \langle w_1, w_2 \rangle_{H^1} = \int \nabla w_1 \cdot \nabla w_2 + w_1 w_2 dx = (\lambda_1 + 1) \int w_1 w_2 dx.$$

Since $\lambda_1 > 0$, we have $\langle w_1, w_2\rangle_{L^2} = 0$.

By iterating this process, we can obtain a sequence of triples $(w_j, \lambda_j, X_j)$ such that $X_{j+1} = \mathrm{span}\{w_1, \cdots, w_j\}^{\perp}$, $\lambda_{j+1} \geqslant \lambda_j$, $\langle w_i, w_j\rangle_{H^1} = \langle w_i, w_j\rangle_{L^2} = 0$ if $i \neq j$, and

$$\frac{\int_\Omega |\nabla w_j|^2 dx}{\int_\Omega w_j^2 dx} = \lambda_j = \inf_{u\in X_j, \|u\|_{L^2}\neq 0} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}.$$

Furthermore, $w_j$ are eigenfunctions and $\lambda_j$ are the corresponding eigenvalues.

**Theorem 11.**

$$\lim_{j\to+\infty} \lambda_j = +\infty.$$

*Proof.* Suppose $\lambda_j \leqslant M$ for some constant $M$. Without loss of generality, we assume $\|w_1\|_{L^2} = 1$. Then,

$$\|w_j\|_{H^1}^2 = \int |\nabla w_j|^2 + w_j^2 dx = \int (\lambda_j + 1) w_j^2 dx = \lambda_j + 1 \leqslant M + 1.$$

Hence, $\{w_j\}$ is a bounded sequence in $H_0^1$. By the compactness theorem 4, there exists a subsequence $\{w_{j_m}\}$ which is strongly convergent in $L^2$. However, $\langle w_{j_m}, w_{j_n}\rangle_{L^2} = 0$ for $j_m \neq j_n$. Contradiction. □