1. LAPLACE EQUATION

We call $\Delta u = 0$ the Laplace equation, and we call its solution $u$ a harmonic function. Given a smooth function $f(x)$, we call $\Delta u = f$ the Poisson’s equation.

Let $\Omega$ denote a bounded open subset in $\mathbb{R}^2$ with a smooth boundary curve $\partial \Omega$.

2. HARMONIC FUNCTION

Let us investigate several properties of harmonic functions.

**Theorem 1** (Uniqueness). *Given a smooth function $g : \partial \Omega \to \mathbb{R}$, there exists a unique smooth harmonic function $u : \Omega \to \mathbb{R}$ satisfying $u = g$ on $\partial \Omega$.*

**Proof.** Suppose that we have two solutions $u, v$, and define $w = u - v$. Then, we have $\Delta w = 0$ in $\Omega$ and $w = 0$ on $\partial \Omega$. Thus,

$$\int_{\Omega} |Dw|^2 \, dx = \int_{\partial \Omega} w \, \nu \, \mathbf{d}x - \int_{\Omega} w \Delta u \, dx = 0.$$

Therefore, $w$ is a constant, and thus $w = 0$ by the boundary condition. $\square$

**Theorem 2** (Mean value property). *A harmonic function $u$ satisfies*

$$u(0) = \frac{1}{\pi r^2} \int_{B_r(0)} u(x) \, dx = \frac{1}{2\pi} \int_{\partial B_r(0)} u(x) \, ds.$$

**Proof.** We begin by defining

$$h(r) = \frac{1}{2\pi r} \int_{\partial B_r(0)} u(x) \, ds$$

for $r > 0$. Then, we have

$$h(r) = \frac{1}{2\pi r} \int_0^{2\pi} u(r \cos \theta, r \sin \theta) \, rd\theta = \frac{1}{2\pi} \int_0^{2\pi} u(r \cos \theta, r \sin \theta) \, d\theta$$

Hence,

$$h'(r) = \frac{1}{2\pi} \int_0^{2\pi} \langle \nabla u(r \cos \theta, r \sin \theta), (\cos \theta, \sin \theta) \rangle \, d\theta$$

$$= \frac{1}{2\pi} \int_{\partial B_r(0)} \left\langle \nabla u(r \cos \theta, r \sin \theta), (\cos \theta, \sin \theta) \right\rangle \frac{1}{r} \, ds$$

$$= \frac{1}{2\pi r} \int_{\partial B_r(0)} \frac{\partial u}{\partial r} \, ds = \frac{1}{2\pi r} \int_{B_r(0)} \Delta u \, dx = 0.$$
Since \( h(r) \) is a constant and \( \lim_{r \to 0} u(0) \), we have \( u(0) = h(r) \) the second formula. The first one can be obtained as follows.
\[
\frac{1}{\pi r^2} \int_{B_r(0)} u(x) \, dx = \frac{1}{\pi r^2} \int_{0}^{r} \int_{\partial B_r(0)} u(x) \, ds \, dt = \frac{1}{\pi r^2} 2\pi t u(0) \, dt = u(0).
\]
\( \square \)

We call a function \( u \) subharmonic [resp. superharmonic] if it satisfies \( \Delta u \geq 0 \) [resp. \( \Delta u \geq 0 \)].

**Proposition 3.** A subharmonic function satisfies
\[
u \leq \frac{1}{\pi r^2} \int_{B_r(0)} u(x) \, dx,
\]
\[
u \leq \frac{1}{2\pi r} \int_{\partial B_r(0)} u(x) \, ds.
\]

A superharmonic function satisfies
\[
u \geq \frac{1}{\pi r^2} \int_{B_r(0)} u(x) \, dx,
\]
\[
u \geq \frac{1}{2\pi r} \int_{\partial B_r(0)} u(x) \, ds.
\]

**Proof.** Remind that \( \Delta u \geq 0 \) implies \( h'(r) \geq 0 \) in the proof of MVP. One can easily modify the proof above. \( \square \)

### 3. Maximum principle

We establish the maximum principle for a general class of linear elliptic PDEs. A simple proof of the maximum principle for harmonic functions is provided in the textbook chapter 3.3.

In this subsection, we consider \( a_{ij}(x), b_i(x), c(x) \) are smooth functions defined on \( \overline{\Omega} \) satisfying
\[
a_{ij}(x)\xi_i\xi_j \geq \lambda |\xi|^2, \tag{1}
\]
for some constant \( \lambda > 0 \), where \( \xi \in \mathbb{R}^n \). In addition, \( a_{ij}(x) \) is a symmetric matrix at each \( x \), namely \( a_{ij}(x) = a_{ji}(x) \). We define a linear differential operator \( L \) by
\[
Lu = a_{ij}\partial_{ij}u + b_i\partial_iu + cu. \tag{2}
\]
We recall the eigenvalue decomposition from Linear algebra. For each \( x \) there exists real numbers \( \lambda_1(x), \ldots, \lambda_n(x) \) and unit vectors \( \vec{q}_1(x), \ldots, \vec{q}_n(x) \in \mathbb{R} \) such that \( \lambda_i \geq \lambda, \langle \vec{q}_i, \vec{q}_j \rangle = 0 \) for \( i \neq j \), and

\[
a_{ij} = \sum_{k=1}^{n} \lambda_k \vec{q}_i^k \vec{q}_j^k, \tag{3}
\]

where \( \vec{q}_k = (q_1^k, \ldots, q_n^k) \).

**Lemma 4.** Suppose that \( Lu > 0 \) and \( c(x) \leq 0 \) hold in \( \Omega \). Then, the smooth subsolution \( u \) satisfies

\[
\max_{\Omega} u \leq \max_{\partial \Omega} u_+,
\]

where \( u_+ = \max\{0, u\} \).

**Proof.** Assume that \( u \) attains its maximum at an interior point \( x_0 \in \Omega \) and \( u(x_0) > 0 \). Then, at \( x_0 \) we have

\[
0 < Lu = a_{ij} u_{ij} + b_i u_i + c u \leq a_{ij} u_{ij},
\]

by \( u_i(x_0) = 0, c \leq 0, \) and \( u(x_0) > 0 \). In addition, by (3).

\[
0 < a_{ij} u_{ij} = \sum_{i,j,k=1}^{n} \lambda_k (q_i^k q_j^k u_{ij}).
\]

However, a function \( h(t) = u(x_0 + t \vec{q}_k(x_0)) \) attains its maximum at \( t = 0 \). Hence

\[
0 \geq h''(0) = q_i^k q_j^k u_{ij}(x_0), \tag{4}
\]

namely \( 0 < a_{ij} u_{ij} \leq 0 \). Contradiction. \( \square \)

**Theorem 5** (Weak maximum principle). Suppose that \( Lu \geq 0 \) and \( c(x) \leq 0 \) hold in \( \Omega \). Then, the smooth subsolution \( u \) satisfies

\[
\max_{\Omega} u \leq \max_{\partial \Omega} u_+,
\]

where \( u_+ = \max\{0, u\} \).

**Proof.** We define \( w = u + \epsilon e^{-\alpha x_1} \) for \( \epsilon > 0 \) and \( \alpha \in \mathbb{R} \). Then,

\[
Lw = Lu + \epsilon Le^{-\alpha x_1} \geq \epsilon Le^{-\alpha x_1}.
\]
Moreover,
\[ Le^{-\alpha x_1} = e^{-\alpha x_1}\left[a^2a_{11} + ab_1 + c\right] \geq e^{-\alpha x_1}\left[\lambda a^2 + ab_1 + c\right]. \]

Since \( \lambda > 0 \) and \( b, c \) are bounded, we can choose sufficiently large \( \alpha \) depending on \( \lambda, b, c \) such that \( Le^{-\alpha x_1} > 0 \). Then, we have \( Lw > 0 \). Thus, Lemma 4 yields
\[
\max_{\Omega} u \leq \max_{\Omega} w \leq \max_{\partial^+ \Omega} w_+ \leq \max_{\partial^+ \Omega} u_+ + \varepsilon \max_{\partial^+ \Omega} e^{-\alpha x_1}.
\]

Passing \( \varepsilon \to 0 \) yields the desired result. \( \square \)

**Lemma 6** (Hopf). *Suppose that \( Lu \geq 0 \) and \( c(x) \leq 0 \) hold in an open ball \( B \). Moreover, there exists a boundary point \( x_0 \in \partial B \) satisfying \( u(x_0) \geq 0 \) and \( u(x_0) > u(x) \) for \( x \in B \). Then, the following holds*
\[ \partial_+ u(x_0) > 0. \]

**Proof.** By translating the ball \( B \), we may assume \( x_0 \in \partial B_r(0) \) and \( B_r(0) \subset B \). Next, we define
\[ \Omega = B_r(0) \cap B_{r/2}(x_0). \]

We consider a function \( v = u + eh \), where \( h(x) = e^{-\alpha |x|^2} - e^{-ar^2} \). Then, in \( \Omega \)
\[ Lh = e^{-\alpha |x|^2}\left[4a^2a_{11}x_i x_j - 2a \sum_{i=1}^n a_{ii} + 2b_i x_i + c\right] - ce^{-ar^2} \geq e^{-\alpha |x|^2}\left[4a^2\lambda |x|^2 - 2a \sum_{i=1}^n (a_{ii} + |b_i||x|) + c\right]. \]

Since \( |x|^2 \geq \frac{\varepsilon^2}{4} \) and \( |x| \leq r \) holds in \( \Omega \), we have \( Lh > 0 \) by choosing a sufficiently large \( \alpha \). Namely, we have \( Lv = Lu + eLh > 0 \), and thus Lemma 4 yields
\[
\max_{\Omega} v \leq \max_{\partial^+ \Omega} v_+. \quad (5)
\]

Now, we claim that there exists a small enough \( \varepsilon \) such that \( v(x_0) = \max_{\partial \Omega} v_+ \). First of all, on the portion \( \partial B_r(0) \cap B_{r/2}(x_0) \subset \partial \Omega \) we have \( h = 0 \). Hence, \( v_+ = u_+ \leq u(x_0) = v(x_0) \). Next, the other portion \( B_r(0) \cap \partial B_{r/2}(x_0) \subset \partial \Omega \) is a compact subset of the open set \( B \), where \( u(x) < u(x_0) \) holds. Hence, there exists a small \( \delta > 0 \) such that \( u(x) \leq u(x_0) - \delta \) holds on \( B_r(0) \cap \partial B_{r/2}(x_0) \). Since \( h \) is bounded over \( \Omega \), we can choose small enough \( \varepsilon \) such that \( e\varepsilon \leq \delta \). Then, \( v = u + e\varepsilon \leq u + \delta \leq u(x_0) = v(x_0) \) holds on \( B_r(0) \cap \partial B_{r/2}(x_0) \).
In conclusion, we have $v(x_0) = \max_{\partial \Omega} v_+ = \max_{\overline{\Omega}} v$. Thus,

$$\partial_\nu u(x_0) = \partial_\nu v(x_0) - \epsilon \partial_\nu h(x_0) \geq -\epsilon \partial_\nu h(x_0) > 0.$$  \hspace{1cm} (6)

\[\square\]

**Theorem 7** (Strong maximum principle). Suppose that $Lu \geq 0$ and $c(x) \leq 0$ hold in $\Omega$. Then, the smooth subsolution $u$ is a constant in $\overline{\Omega}$ or

$$u(x) < \max_{\partial \Omega} u_+,$$

holds for $x \in \Omega$.

**Proof.** Assume that $u$ attains its maximum at an interior point $x_0 \in \Omega$ and $u(x_0) = M \geq 0$. We define a set $\Sigma = \{x \in \overline{\Omega} : u(x) = M\}$. Since $u$ is a continuous function, $\Sigma$ is a closed set. Towards contradiction, we assume $\Omega$ is not contained in $\Sigma$. Then, there exists a point $y_0 \in \Omega \setminus \Sigma$ such that $d(y_0, \partial \Omega) > d(y_0, \Sigma)$, where $d(y_0, A)$ denotes the distance from $y_0$ to the set $A$. There exists a small $r > 0$ such that $B_r(y_0) \subset \Omega \setminus \Sigma$, because $\Omega \setminus \Sigma$ is an open set. Next, we define $R$ by

$$R = \sup \{r : B_r(y_0) \cap \Omega \setminus \Sigma\}.$$

Then, there exists a point $z_0 \in \Sigma \cap \partial B_R(y_0)$ and $z_0 \in \Omega$. Since $u(z_0) = \max u$ and $z_0 \in \Omega$, we have $Du(z_0) = 0$. However, by the Hopf’s Lemma, we have $\partial_\nu u(z_0) > 0$ where $\nu$ is the outward pointing direction of $\partial B_R(y_0)$. Contradiction. \[\square\]