1. 1D Heat Equation

Suppose that a smooth solution $u(x, t)$ satisfies the following differential equation

$$u_t = u_{xx} \quad \text{(Heat equation)},$$

(1)

in $\{ (x, t) : 0 \leq x \leq L, 0 \leq t \}$. Then, $u(x, t)$ can represent the temperature under the heat flow on a rod located in $\{ 0 \leq x \leq L \}$. In order to solve the equation, we need the initial data

$$u(x, 0) = g(x) \quad \text{(Cauchy condition)},$$

(2)

and one of the following boundary data

$$u(0, t) = h_1(t), \quad u(L, t) = h_2(t) \quad \text{(Dirichlet condition)},$$

(3)

$$-u_x(0, t) = h_1(t), \quad u_x(L, t) = h_2(t) \quad \text{(Neumann condition)},$$

(4)

$$-u_x(0, t) + \alpha u(0, t) = h_1(t), \quad u_x(L, t) + \alpha u(L, t) = h_2(t) \quad \text{(Robin condition)}.$$  

(5)

2. Uniqueness

We establish the following uniqueness theorem.

**Theorem 1** (Uniqueness). Given smooth functions $g(x), h_1(t), h_2(t)$, the heat equation (1) has at most one smooth solution $u(x, t)$ satisfying (2) on $\{ 0 \leq x \leq L \}$ and (3) on $\{ t \geq 0 \}$.

**Proof.** Suppose that $u(x, t)$ and $v(x, t)$ are solutions satisfying the conditions. Then, the smooth function $w(x, t) = u(x, t) - v(x, t)$ satisfies

$$w_t = u_t - v_t = u_{xx} - v_{xx} = w_{xx}.$$  

(6)

Moreover, we can observe

$$w(0, t) = w(L, t) = 0.$$  

(7)

Next, we define an energy

$$E(t) = \int_0^L w^2(x, t)dx.$$  

Then, (6) shows

$$\frac{d}{dt}E(t) = \int_0^L 2ww_tdx = 2 \int_0^L w_{xx}dx.$$
Using the integration by part and (7),
\[
\frac{d}{dt}E(t) = 2ww_x|_{0}^{L} - 2 \int_{0}^{L} |w_x|^2 dx = -2 \int_{0}^{L} |w_x|^2 dx \leq 0.
\]
(8)
Therefore,
\[
0 \leq E(t) \leq E(0),
\]
for all \( t \geq 0 \). However, we have \( w(x,0) = 0 \) by definition, namely \( E(0) = 0 \). Thus, \( E(t) = 0 \) and \( w(x,t) = 0 \). Hence, the smooth solution is unique. \( \square \)

Remark. If \( h_1(t) = h_2(t) = 0 \), then we can modify the proof above to show
\[
\frac{d}{dt} \int_{0}^{L} u^2(x,t) dx \leq 0.
\]
(10)
Then, it would be a natural question to prove \( \lim_{t \to +\infty} \sup_{0 \leq x \leq L} |u(x,t)| = 0 \). We will prove this next week, but it’d be good to try to prove it yourself.

3. Review: Fourier series

We recall the Fourier series. In this class, we will use the following fact without proofs.

Given a smooth function \( f : [-L, L] \to \mathbb{R} \) with \( f(-L) = f(L) \), the following holds
\[
\lim_{N \to +\infty} \sup_{|x| \leq L} |f(x) - S_N(x)| = 0,
\]
for the partial sums \( S_N(x) \) of Fourier series,
\[
S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(m\pi x/L\right) + \sum_{m=1}^{\infty} b_m \sin\left(m\pi x/L\right),
\]
where
\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx, \quad a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(m\pi x/L\right) dx, \quad b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(m\pi x/L\right) dx.
\]
Suppose that \( f : [0, L] \to \mathbb{R} \) is a smooth function satisfying \( f(0) = 0 \). Then,
\[
\lim_{N \to +\infty} \sup_{0 \leq x \leq L} |f(x) - S_N(x)| = 0,
\]
holds for the partial sums $S_N(x)$ of Fourier sine series,

$$S_N(x) = \sum_{m=1}^{\infty} b_m \sin(m\pi x/L),$$

where

$$b_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx.$$

Suppose that $f : [0, L] \to \mathbb{R}$ is a smooth function satisfying $f'(0) = 0$. Then,

$$\lim_{N \to \infty} \sup_{0 \leq x \leq L} |f(x) - S_N(x)| = 0,$$

holds for the partial sums $S_N(x)$ of Fourier cosine series,

$$S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\pi x/L),$$

where

$$a_0 = \frac{2}{L} \int_0^L f(x) dx, \quad a_m = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx.$$

4. Review: ODE

We recall the some well-known results in ODEs. We will also use them without proofs.

Suppose that a function $u(x)$ satisfies the following differential equation

$$u''(x) + \mu^2 u(x) = 0. \quad (11)$$

Then,

$$u(x) = c_1 \sin(\mu x) + c_2 \cos(\mu x), \quad (12)$$

for some constants $c_1, c_2$ depending on initial (or boundary data). For example, if $u(x)$ satisfies $u(0) = 0$ and $u'(0) = 1$, then the constants must be $c_1 = \mu^{-1}$ and $c_2 = 0$.

Suppose that a function $u(x)$ satisfies the following differential equation

$$u'(x) = \lambda u(x). \quad (13)$$
Then, 

\[ u(x) = ce^{\lambda x}, \tag{14} \]

for some constant \( c \) depending on the initial data.

### 5. Separation of Variables

In this section, we will SOLVE the Cauchy-Dirichlet problem with the vanishing Dirichlet data. Namely, given smooth \( g(x) \), we will find the solutions to the heat equation (1) under the conditions (2) and (3), where \( h_1(t) = h_2(t) = 0 \).

To begin with, we remind that by the uniqueness theorem there exists at most one solution. Hence, if we find a solution, then it is the only solution.

Next, we want find a function \( u(x,t) = v(x)w(t) \) satisfying (1) and (3) with \( h_1 = h_2 = 0 \). (Notice that in this step we do not consider (2), yet.) Then, (1) implies

\[ w_t v = u_{xx} = w v_{xx}. \]

Dividing by \( vw \) yields

\[ \frac{w_t(t)}{w(t)} = \frac{v_{xx}(x)}{v(x)}. \]

The left hand side only depends on \( t \), while the right hand side only depends on \( x \). Therefore, there exists some constant \( \lambda \in \mathbb{R} \) such that

\[ \frac{w_t}{w} = \frac{v_{xx}}{v} = \lambda. \]

We consider the three cases that \( \lambda > 0, \lambda = 0, \) and \( \lambda < 0 \).

**Case 1:** \( \lambda > 0 \). In this case, by using the Dirichlet condition \( v(0) = v(L) = 0 \) we can obtain

\[ 0 \leq \lambda \int_0^L v^2 dx = \int_0^L v(\lambda v)dx = \int_0^L vv_{xx}dx = v|v_x|_0^L - \int_0^L |v_x|^2 dx = -\int_0^L |v_x|^2 dx \leq 0, \tag{15} \]

namely \( v = 0 \). Thus, \( u = 0 \).

**Case 2:** \( \lambda = 0 \). In this case, \( v_{xx} = 0 \) implies \( v(x) = ax + b \). Hence, the Dirichlet condition \( v(0) = v(L) = 0 \) guarantees \( v = 0 \). Thus, \( u = 0 \).
Case 3: $\lambda = -\mu^2 < 0$. In this case, the equation $v_{xx} + \mu^2 v = 0$ has non-trivial solutions. By the results in ODE, $v(x) = A \cos(\mu x) + B \sin(\mu x)$ holds for the constants $A, B$ satisfying the boundary conditions

$$
0 = v(0) = A \cos 0 + B \sin 0 = A,
0 = v(L) = A \cos(\mu L) + B \sin(\mu L) = B \sin(\mu L).
$$

Hence, we have $\sin(\mu L)$, and thus $\mu L = m\pi$ for a natural number $m$. Namely, given $m \in \mathbb{N}$ we have

$$
v_m = c \sin(m\pi x/L),
$$

for some constant $c$. In addition, $\lambda = -\mu^2 = (m\pi/L)^2$ gives

$$
\frac{d}{dt}w_m = -\mu^2 w_m = -(m\pi/L)^2 w_m.
$$

Hence, the ODE result says

$$
w_m = c \exp(-(m\pi/L)^2 t)
$$

for some constant $c$. In conclusion, for each $m \in \mathbb{N}$ and any constant $B_m \in \mathbb{R}$

$$
u_m(x, t) = B_m \exp(-(m\pi/L)^2 t) \sin(m\pi x/L)
$$

satisfies (1) and (3) with $h_1 = h_2 = 0$.

By the result in the last case, we know that

$$
u = \sum_{m=1}^{\infty} B_m \exp(-(m\pi/L)^2 t) \sin(m\pi x/L),
$$

satisfies (1) and (3) with $h_1 = h_2 = 0$.

Now, we define the coefficients $B_m$ by

$$
B_m = \frac{2}{L} \int_0^L g(x) \sin(m\pi x/L) dx.
$$

Then, by the Fourier series theorem above, the function $u(x, t)$ in (17) satisfies (2). Namely, it is the desired solution.