Problem 1. Smooth vector fields \( w_1, w_2, w_3 \) are defined by
\[
\begin{align*}
  w_1 &= x^2 \frac{\partial}{\partial x_3} - x^3 \frac{\partial}{\partial x_2}, \\
  w_2 &= x^3 \frac{\partial}{\partial x_1} - x^1 \frac{\partial}{\partial x_3}, \\
  w_3 &= x^1 \frac{\partial}{\partial x_2} - x^2 \frac{\partial}{\partial x_1}.
\end{align*}
\]
Given a fixed unit vector \( a = (a^1, a^2, a^3) \in \mathbb{R}^3 \), draw integral curves of the unit sphere \( S^2 \) with respect to the vector fields \( \sum_{i=1}^3 a^i w_i \). We do not need to verify your answer.

Problem 2. Suppose that a function \( f \in C^\infty(\mathbb{R}^3) \) satisfies \( L_v f = 0 \), where \( v = x^1 \frac{\partial}{\partial x_2} - x^2 \frac{\partial}{\partial x_1} \). Namely, the Lie derivative of \( f(x) \) along the vector field \( v \) is zero. Show that there exists a function \( h : \mathbb{R}^2 \to \mathbb{R} \) such that \( h(r, z) = f(r \cos \theta, r \sin \theta, z) \) for all \( r, z, \theta \in \mathbb{R} \).

Problem 3. Given charts \( (\varphi, U) \) of the unit sphere \( S^2 \), draw the region \( \varphi(U) \) and evaluate the following integral
\[
\int_U \sqrt{g_{11}g_{22} - (g_{12})^2} \, dx,
\]
where \( g_{ij} = \langle \partial_i \varphi, \partial_j \varphi \rangle \).

1. \( U = \mathbb{R}^2 \) and
   \[
   \varphi(x^1, x^2) = \left( \frac{2x^1}{1 + \|x\|^2}, \frac{2x^2}{1 + \|x\|^2}, \frac{-1 + \|x\|^2}{1 + \|x\|^2} \right).
   \]
2. \( U = \{ x \in \mathbb{R}^2 : \|x\| < 1 \} \) and
   \[
   \varphi(x^1, x^2) = \left( x^1, x^2, \sqrt{1 - \|x\|^2} \right).
   \]