You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

**Problem 1.** Complete the proof of Theorem 3 in Note 0917 by proving the Claim in page 2.

**Problem 2.** Let $M$ be a 2-manifold in $\mathbb{R}^3$ of class $C^\infty$. Show that given any coordinate chart $(\varphi, U)$ the metric tensor $g_{ij}$ is invertible.

**Problem 3.** $\varphi \in C^\infty(\mathbb{R}^2)$ is defined by
$$\varphi(x, y) = (\cosh x \cos y, \cosh x \sin y, x)^T.$$ 
Then, the image $M = \varphi(\mathbb{R}^2) \subset \mathbb{R}^3$ is a 2-manifold. Prove the following.

1. Given a point $p \in M$, there exists an open set $U \subset \mathbb{R}^2$ such that $p \in \varphi(U)$ and $(\varphi, U)$ is a coordinate chart.
2. Calculate $g_{ij}$, $g^{ij}$, and $\Gamma^k_{ij}$.
3. Show that $\sum_{ij} g^{ij} \nabla_i \nabla_j \varphi = 0$, where
$$\nabla_i \nabla_j \varphi = \frac{\partial^2 \varphi}{\partial x^i \partial x^j} - \sum_{k=1}^{2} \Gamma^k_{ij} \frac{\partial \varphi}{\partial x^k}.$$ 

We call the manifold $M$ a catenoid, which is a minimal surface.

Google images of minimal surfaces; Helicoid, Scherk surfaces, Costa’s minimal surfaces, Riemann’s minimal surfaces, Gyroid and so on.

**Problem 4.** Solve the problem 6 in page 7 of the course notes of prof. Guillemin.

**Problem 5.** Let $(\varphi, U)$ be the chart of the unit sphere in $\mathbb{R}^3$ defined by $U = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \subset \mathbb{R}^2$ and $\varphi(\theta, \psi) = (\cos \theta \cos \psi, \cos \theta \sin \psi, \sin \theta)$. Find all the geodesics $\varphi \circ x$ such that $x : [0, \frac{1}{2}\pi] \to U$ satisfies $x(0) = (0, 0)$ and $\|\frac{d}{dt} \varphi \circ x\|(0) = 1$. 

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Problem 6 (bonus). Let \((\varphi, U)\) be a coordinate chart of a 2-manifold \(M\) of class \(C^\infty\) in \(\mathbb{R}^3\). Suppose that a smooth map \(x : [a, b] \to U\) satisfies that the smooth curve \(\varphi \circ x([a, b])\) has the minimal length within all the curves in \(M\) connecting the two points \(\varphi(x(a)), \varphi(x(b)) \in M\). Show that \(x(t)\) satisfies

\[
\frac{d^2 x^k(t)}{dt^2} + \sum_{i,j} \Gamma^k_{ij}(x(t)) \frac{dx^i(t)}{dt} \frac{dx^j(t)}{dt} = 0
\]

for each \(k = 1, 2\).