Let (φ, U) be a coordinate chart of a n-manifold $M \subset \mathbb{R}^N$. Given a smooth map $x : [a,b] \to U$, namely of class $C^\infty([a,b])$, we define the length L of curve $\varphi \circ x([a,b])$ by

$$L = \int_a^b \|\tfrac{d}{dt}\varphi \circ x\| dt = \int_a^b \sqrt{\sum_{i=1}^n \sum_{j=1}^n g_{ij}(x) \tfrac{dx^i}{dt} \tfrac{dx^j}{dt}}\, dt, \tag{1}$$

where the metric tensor $g_{ij}(x)$ is defined by

$$g_{ij}(x) = \langle \partial_i \varphi(x), \partial_j \varphi(x) \rangle, \tag{2}$$

for each $i, j \in \{1, \cdots, n\}$. Notice that $\partial_i \varphi(x) = \frac{\partial}{\partial x^i} \varphi(x)$.

Moreover, we denote the inverse matrix of the matrix (g_{ij}) by (g^{ij}). For example, if $n = 2$ then

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}. \tag{3}$$

Problem 1. *Let M be a 2-manifold in $\mathbb{R}^3$. Show that given any coordinate chart (φ, U) the metric tensor g_{ij} is invertible.*

Next, we define $\Gamma_{ij,l}(x)$ by

$$\Gamma_{ij,l}(x) = \tfrac{1}{2}\big(g_{jl,i}(x) + g_{il,j}(x) - g_{ij,l}(x)\big). \tag{4}$$

where

$$g_{ij,k}(x) = \tfrac{\partial}{\partial_k} g_{ij}(x). \tag{5}$$

Then, we define the Christoffel symbols by

$$\Gamma_{ij}^k(x) = \sum_{l=1}^n g^{kl}(x) \Gamma_{ij,l}(x). \tag{6}$$

Suppose that a map $x : [a,b] \to U$ is of class C^2 and satisfies

$$\frac{d^2 x^k(t)}{dt^2} + \sum_{i,j} \Gamma_{ij}^k(x(t)) \frac{dx^i(t)}{dt} \frac{dx^j(t)}{dt} = 0, \tag{7}$$

for each $k = 1, \cdots, n$. Then, we say that the curve $\varphi \circ x([a,b])$ is a geodesic.