We recall the implicit function theorem.

**Theorem 1** (Implicit Function Theorem). Let $A$ be open in $\mathbb{R}^{k+n}$, and let $f : A \to \mathbb{R}^n$ be of class $C^r$. We write $f$ in the form $f(x,y)$ for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$. Suppose that $(a,b) \in A$ satisfies $f(a,b) = 0$ and $\det(\partial f/\partial y)(a,b) \neq 0$. Then, there exists a open ball $B \subset \mathbb{R}^k$ containing $a$ and a unique continuous function $g : B \to \mathbb{R}^n$ such that $g(a) = b$ and $f(x,g(x)) = 0$ for all $x \in B$. Moreover, $g \in C^r(B)$.

And we define a $k$-manifold in $\mathbb{R}^n$ of class $C^r$ as follows.

**Definition 2.** Let $k > 0$. Suppose that $M \subset \mathbb{R}^n$ have the following property: For each $p \in M$, there exist a relatively open set $V$ in $M$ containing $p$, a open set $U \subset \mathbb{R}^k$, and a function $\varphi : U \to V$ of class $C^r$ such that $\varphi$ is one-to-one and onto, the inverse function $\varphi^{-1} : V \to U$ is continuous, and $D\varphi(x)$ has rank $k$ for each $x \in U$. Then, $M$ is called a $k$-manifold in $\mathbb{R}^n$ of class $C^r$.

We now prove a simple theorem.

**Theorem 3.** Let $A \subset \mathbb{R}^{k+1}$ be open, and let $f : \mathbb{R}^{k+1} \to \mathbb{R}$ be of class $C^r$. Suppose that $Df(p) \neq 0$ for each $p$ satisfying $f(p) = 0$. Then, the level set $M = \{p \in \mathbb{R}^{k+1} : f(p) = 0\}$ is a $k$-manifold in $\mathbb{R}^{k+1}$ of class $C^r$.

**Proof.** Given a point $p_0 \in M$, we rotate the coordinate system to have $Df(p_0) \parallel e_{n+1}$, namely $Df(p_0) = \|Df(p_0)\|e_{n+1}$ or $Df(p_0) = -\|Df(p_0)\|e_{n+1}$.

We write $f(p)$ in the form $f(x,y)$ where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}$. Then, $\frac{\partial f}{\partial y}(p_0) = \langle Df(p_0), e_{n+1} \rangle = \pm\|Df(p_0)\| \neq 0$. Therefore, by the implicit function theorem, there exists an open ball $U \subset \mathbb{R}^k$ containing $x_0$ (where $p_0 = (x_0,y_0)$) and a unique continuous function $g : U \to \mathbb{R}$ such that $g(x_0) = y_0$ and $f(x,g(x)) = 0$ for all $x \in U$. Moreover, $g \in C^r(B)$.

Then, we define $\varphi : U \to \mathbb{R}^{k+1}$ by $\varphi(x) = (x,g(x))$, which is of class $C^r$. Then, we have $f(\varphi(x)) = f(x,g(x)) = 0$, namely $V = \varphi(U) \subset M$. Then, $\varphi : U \to V$ is one-to-one and onto function of class $C^r$. In addition, $D\varphi(x) = \left[ I_k \ Dg \right]$ has rank $k$.

Since $\varphi : U \to V$ is one-to-one and onto, given two points $p,q \in V$ there exist $x,y \in U$ such that $p = (x,g(x))$ and $q = (y,g(y))$. Hence, the inverse function $\varphi^{-1} : V \to U$ satisfies

$$\|\varphi^{-1}(p) - \varphi^{-1}(q)\|^2 = \|x - y\|^2 \leq \|x - y\|^2 + |g(x) - g(y)|^2 = \|p - q\|^2.$$ 

This implies the continuity of $\varphi^{-1}$ on $V$.

Therefore, we only need to show the following claim to complete the proof.
Claim: There exists an open set $\bar{U} \subset \mathbb{R}^k$ such that $0 \in \bar{U} \subset U$ and $\bar{V} = \varphi(\bar{U})$ is relatively open in $M$. Namely, there exists an open set $V'$ in $\mathbb{R}^{k+1}$ such that $M \cap V' = \bar{V}$.

We leave the proof of claim for homework. □