We recall the implicit function theorem.

Theorem 1 (Implicit Function Theorem)**.** *Let A be open in $\mathbb{R}^{k+n}$, and let $f : A \to \mathbb{R}^n$ be of class C^r. We write f in the form $f(x, y)$ for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$. Suppose that $(a, b) \in A$ satisfies $f(a, b) = 0$ and* $\det(\partial f/\partial y)(a, b) \neq 0$. *Then, there exists a open ball $B \subset \mathbb{R}^k$ containing a and a unique continuous function $g : B \to \mathbb{R}^n$ such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in B$. Moreover, $g \in C^r(B)$.*

And we define a k-manifold in $\mathbb{R}^n$ of class C^r as follows.

Definition 2. Let $k > 0$. Suppose that $M \in \mathbb{R}^n$ have the following property: For each $p \in M$, there exist a relatively open set V in M containing p, a open set $U \subset \mathbb{R}^k$, and a function $\varphi : U \to V$ of class C^r such that φ is one-to-one and onto, the inverse function $\varphi^{-1} : V \to U$ is continuous, and $D\varphi(x)$ has rank k for each $x \in U$. Then, M is called a k-manifold in $\mathbb{R}^n$ of class C^r.

We now prove a simple theorem.

Theorem 3. *Let $A \subset \mathbb{R}^{k+1}$ be open, and let $f : \mathbb{R}^{k+1} \to \mathbb{R}$ be of class C^r. Suppose that $Df(p) \neq 0$ for each p satisfying $f(p) = 0$. Then, the level set $M = \{p \in \mathbb{R}^{k+1} : f(p) = 0\}$ is a k-manifold in $\mathbb{R}^{k+1}$ of class C^r.*

Proof. Given a point $p_0 \in M$, we rotate the coordinate system to have $Df(p_0) \parallel e_{n+1}$, namely $Df(p_0) = \|Df(p_0)\|e_{n+1}$ or $Df(p_0) = -\|Df(p_0)\|e_{n+1}$.

We write $f(p)$ in the form $f(x, y)$ where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}$. Then, $\frac{\partial f}{\partial y}(p_0) = \langle Df(p_0), e_{n+1}\rangle = \pm\|Df(p_0)\| \neq 0$. Therefore, by the implicit function theorem, there exists a open ball $U \subset \mathbb{R}^k$ containing x_0 (where $p_0 = (x_0, y_0)$) and a unique continuous function $g : U \to \mathbb{R}$ such that $g(x_0) = y_0$ and $f(x, g(x)) = 0$ for all $x \in U$. Moreover, $g \in C^r(B)$.

Then, we define $\varphi : U \to \mathbb{R}^{k+1}$ by $\varphi(x) = (x, g(x))$, which is of class C^r. Then, we have $f(\varphi(x)) = f(x, g(x)) = 0$, namely $V = \varphi(U) \subset M$. Then, $\varphi : U \to V$ is one-to-one and onto function of class C^r. In addition, $D\varphi(x) = \begin{bmatrix} I_k & Dg \end{bmatrix}$ has rank k.

Since $\varphi : U \to V$ is one-to-one and onto,given two point $p, q \in V$ there exist $x, y \in U$ such that $p = (x, g(x))$ and $q = (y, g(y))$. Hence, the inverse function $\varphi^{-1} : V \to U$ satisfies

$$\|\varphi^{-1}(p) - \varphi^{-1}(q)\|^2 = \|x - y\|^2 \leqslant \|x - y\|^2 + |g(x) - g(y)|^2 = \|p - q\|^2.$$

This implies the continuity of φ^{-1} on V.

Therefore, we only need to show the following claim to complete the proof.

Claim : There exists an open set $\bar{U} \subset \mathbb{R}^k$ such that $0 \in \bar{U} \subset U$ and $\bar{V} = \varphi(\bar{U})$ is relatively open in M. Namely, there exists an open set V' in $\mathbb{R}^{k+1}$ such that $M \cap V' = \bar{V}$.

We leave the proof of claim for homework. □