Name :

Email (MIT email if applicable) :

1. (10 points) Determine whether the following statements are true or false. Verify your answer.

(A) \( U \) is an open set in \( \mathbb{R}^n \) and \( f : U \to \mathbb{R}^n \) is a smooth function. Suppose that \( Df \) is singular at a point \( x_0 \in U \). Then, given any open set \( V \subset U \) containing \( x_0 \), the function \( f \) restricted on \( V \) is not one-to-one.

(B) Given an initial data \( x(0) = x_0 \in \mathbb{R}^n \), there exists a unique integral curve \( x : (-\infty, +\infty) \to \mathbb{R}^n \) to the ODE
\[
\frac{d^2}{dt^2} x(t) = x(t).
\]
2. (15 points) Determine whether the following set $M$ is a 1-manifold of class $C^r$ in $\mathbb{R}^2$ or not. Verify your answer.

(A) $M = \{(x, y) : x = y|y|\}$ and $r = 1$.

(B) $M = \{(x, y) : x = y|y|\}$ and $r = 2$.

(C) $M = \{(x, y) : x^4 = y^2\}$ and $r = \infty$. 
3. (10 points) Determine whether the one-form \( \omega = (2x + y)dx + xdy + ydz \) in \((T\mathbb{R}^3)^*\) has a smooth potential function \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) or not, and verify your answer. Also, calculate \( \int_\gamma \omega \) for the curve \( \gamma : [0, 1] \rightarrow \mathbb{R}^3 \), where \( \gamma(t) = (1, t, t^2) \).
4. (15 points) Let $M$ denote the unit sphere $\{p \in \mathbb{R}^3 : \|p\| = 1\}$, which is a smooth 2-manifold in $\mathbb{R}^3$.

(A) Show that given a chart $(\varphi, U)$ and a point $x_0 \in U$, there exists some $\epsilon \in \mathbb{R}$ and a unique solution $x : (-\epsilon, +\epsilon) \to U$ to the equation
\[
\frac{d}{dt} \varphi(x(t)) = e_3 - \langle e_3, \varphi(x(t)) \rangle \varphi(x(t)),
\]
satisfying the initial condition $x(0) = x_0$.

(B) Given a point $p_0 \in M$, there exists a unique smooth curve $\gamma : (-\infty, +\infty) \to M$ satisfying $\gamma(0) = p_0$ and
\[
\frac{d}{dt} \gamma(t) = e_3 - \langle e_3, \gamma(t) \rangle \gamma(t).
\]