1. (10 points) Determine whether the following statements are true or false. Verify your answer.

(A) $U$ is an open set in $\mathbb{R}^n$ and $f: U \rightarrow \mathbb{R}^n$ is a smooth function. Suppose that $Df$ is singular at a point $x_0 \in U$. Then, given any open set $V \subset U$ containing $x_0$, the function $f$ restricted on $V$ is not one-to-one.

**Proof.** False. For example, $f(x) = x^3$ satisfies $f'(0) = 0$, namely $Df$ is singular at $x = 0$. However, $f: \mathbb{R} \rightarrow \mathbb{R}$ is one-to-one. $\square$

(B) Given an initial data $x(0) = x_0 \in \mathbb{R}^n$, there exists a unique integral curve $x: (-\infty, +\infty) \rightarrow \mathbb{R}^n$ to the ODE
\[
\frac{d^2}{dt^2} x(t) = x(t).
\]

**Proof.** False. For example, given $x_0 \neq 0$ we have two integral curves $x(t) = x_0 e^t$ and $x(t) = x_0 e^{-t}$.

In addition, we can set $x_0 = 0$, and have infinitely many solutions $x(t) = a \sinh(t)$, where $a \in \mathbb{R}$.

Please, think about the reason we lost the uniqueness. $\square$
2. (15 points) Determine whether the following set $M$ is a 1-manifold of class $C^r$ in $\mathbb{R}^2$ or not. Verify your answer.

(A) $M = \{(x, y) : x = y|y| \}$ and $r = 1$.

Proof. It is a 1-manifold of class $C^1$.

We define $\varphi(t) = (t|t|, t)$. Then, we can calculate $D\varphi(t) = (2|t|, 1)$. Indeed, $(2|t|, 1)$ is continuous, namely $\varphi \in C^1$. We can also easily check $\varphi(\mathbb{R}) = M$, and $\varphi$ is one-to-one. Moreover, $|\varphi^{-1}(t|t|, t) - \varphi^{-1}(s|s|, s)| = |t - s|$ implies the continuity of $\varphi^{-1}$.

(B) $M = \{(x, y) : x = y|y| \}$ and $r = 2$.

Proof. It is NOT a 1-manifold of class $C^1$.

Suppose that $M$ is a manifold of class $C^2$. Then, there exists a $C^2$ chart $(\varphi, I)$ such that $I$ is an open interval in $\mathbb{R}$ containing 0 and $\varphi(0) = (0, 0)$.

Let’s denote $\varphi(t) = (x(t), y(t))$. At $t = 0$, we obtain

$$|x'(0)| = \left| \lim_{t \to 0} \frac{y(t)|y(t)|}{t} \right| \leq \liminf_{t \to 0} |t| \left| \frac{y(t)}{t} \right|^2 = 0 \cdot |y'(0)|^2 = 0.$$ 

Therefore, $x'(t) = 2|y(t)|y'(t)$. In particular, $y(0) = 0$ yields $x'(0) = 0$, and thus we obtain $y'(0) \neq 0$ due to $D\varphi(0) \neq 0$.

Since $x(0) = x'(0) = 0$, l’Hospital’s rule yields

$$\frac{1}{2} x''(0) = \lim_{t \to 0^+} \frac{x(t)}{t^2} = \lim_{t \to 0^+} \frac{y(t)|y(t)|}{t^2} = \lim_{t \to 0^+} \frac{y(t)}{t} \left| \frac{y(t)}{t} \right| = y'(0)|y'(0)|,$$

and

$$\frac{1}{2} x''(0) = \lim_{t \to 0^-} \frac{x(t)}{t^2} = \lim_{t \to 0^-} \frac{y(t)|y(t)|}{t^2} = -\lim_{t \to 0^-} \frac{y(t)}{t} \left| \frac{y(t)}{t} \right| = -y'(0)|y'(0)|.$$

However, $y'(0)|y'(0)| \neq -y'(0)|y'(0)|$ by $y'(0) \neq 0$.

(C) $M = \{(x, y) : x^4 = y^2 \}$ and $r = \infty$.

Two different solutions are provided below.

The first solution to (C). It is NOT a 1-manifold of class $C^\infty$.

Suppose that $M$ is a manifold of class $C^\infty$. Then, there exists an open interval $I \subset \mathbb{R}$ containing $0 \in \mathbb{R}$, a continuous one-to-one function $\varphi : I \to M$ such that $\varphi(I)$ is relatively open in $M$ and $\varphi(0) = (0, 0) \in M$.

We denote by $M_i$ the intersection between $M$ and the $i$-th quadrant; for example $M_1 = \{(x, y) \in M : x, y > 0\}$. Notice that each $M_i$ is relatively open in $M$ and $M_i \cap M_j = \emptyset$ holds for $i \neq j$.

Let $\epsilon$ be a small positive constant satisfying $\pm \epsilon \in I$, and with out loss of generality we assume $\varphi(\epsilon) \in M_1$.

Claim : $\varphi(t) \in M_1$ holds for all positive $t \in I$. 

Toward a contradiction, we assume that there exists a positive positive $e' \in I$ such that $\varphi(e') \not\in M_1$. Since $\varphi$ is continuous and $[e', e]$ (or $[e, e']$) is connected, $\varphi([e', e])$ is connected.

However, $0 \not\in [e', e]$ implies $(0, 0) \not\in \varphi([e', e])$, and thus $\varphi([e', e]) \subset M \setminus \{(0, 0)\}$. Therefore, $\varphi([e', e]) \subset M_1 \cup (\bigcup_{i=2}^4 M_i)$, $\varphi([e', e]) \cap M_1 \neq \emptyset$, and $\varphi([e', e]) \cap (\bigcup_{i=2}^4 M_i) \neq \emptyset$ imply that $\varphi([e', e])$ is disconnected. Thus, the contradiction proves the claim.

Similar argument to the claim yields that $\varphi(I \cap \mathbb{R}_+^+ \subset M_i$ for a certain $i \in \{1, 2, 3, 4\}$ if $\varphi(e) \in M_i$, and $\varphi(I \cap \mathbb{R}_-^+ \subset M_j$ for a certain $j \in \{1, 2, 3, 4\}$ if $\varphi(-e) \in M_j$. Hence, $\varphi(I)$ is contained in the union of at most two of $M_1, \ldots, M_4$ and $\{(0, 0)\}$. Therefore, $\varphi(I)$ is not relatively open in $M$. □

The second solution to (C). It is NOT a 1-manifold of class $C^\infty$.

Suppose that $M$ is a manifold of class $C^2$. Then, there exists a $C^\infty$ chart $(\varphi, I)$ of $M$ with $0 \in I$ and $\varphi(0) = (0, 0)$. Let’s denote $\varphi(t) = (x(t), y(t))$. At $t = 0$, we obtain

$$|y'(0)| = \left| \lim_{t \to 0} \frac{y(t)}{t} \right| \leq \liminf_{t \to 0} \frac{|x(t)|}{t} = \liminf_{t \to 0} |x(t)| \left| \frac{x(t)}{t} \right| = |x(0)||x'(0)| = 0.$$  

Since $D\varphi(0) \neq 0$, we have $x'(0) \neq 0$.

We choose an open interval $0 \in J$ such that $(s, s^2), (s, -s^2) \in M$ holds for $s \in J$. We observe

$$x'(0) = e^T D\varphi(0) = e_1^T \lim_{s \to 0} \frac{(s, s^2)}{\varphi^{-1}(s, s^2)} = \lim_{s \to 0} \frac{s}{\varphi^{-1}(s, s^2)}.$$  

Since $e_2^T \varphi(t) \in C^\infty(I)$ satisfies $e_2^T \varphi(0) = \frac{d}{dt} e_2^T \varphi(0) = 0$, we have

$$y''(0) = \frac{d^2}{dt^2} e_2^T \varphi(0) = \lim_{s \to 0} \frac{s^2}{\varphi^{-1}(s, s^2)|s|^2} = \left| \lim_{s \to 0} \frac{s^2}{\varphi^{-1}(s, s^2)} \right|^2 = |x'(0)|^2.$$  

Similarly, we can obtain

$$x'(0) = e_1^T D\varphi(0) = e_1^T \lim_{s \to 0} \frac{(s, -s^2)}{\varphi^{-1}(s, -s^2)} = \lim_{s \to 0} \frac{s}{\varphi^{-1}(s, -s^2)},$$  

and thus

$$y''(0) = \frac{d^2}{dt^2} e_2^T \varphi(0) = \lim_{s \to 0} \frac{-s^2}{\varphi^{-1}(s, -s^2)|s|^2} = - \left| \lim_{s \to 0} \frac{s^2}{\varphi^{-1}(s, -s^2)} \right|^2 = -|x'(0)|^2.$$

These contradict to $x'(0) \neq 0$. □
3. (10 points) Determine whether the one-form \( \omega = (2x + y)dx + xdy + ydz \) in \((T\mathbb{R}^3)^*\) has a smooth potential function \( f : \mathbb{R}^3 \to \mathbb{R} \) or not, and verify your answer. Also, calculate \( \int_\gamma \omega \) for the curve \( \gamma : [0,1] \to \mathbb{R}^3 \), where \( \gamma(t) = (1, t, t^2) \).

**Proof.** Then, by definition we have \( \partial_y f = x \) and \( \partial_z f = y \). However,

\[
0 = \partial_z x = \partial_x \partial_y f = \partial_y \partial_z f = \partial_y y = 1,
\]

yields a contradiction. Therefore, there is no smooth potential function.

We set \( \gamma(t) = (x(t), y(t), z(t)) = (1, t, t^2) \). Then,

\[
\int_\gamma \omega = \int_0^1 (2x(t) + y(t))x'(t) + x(t)y'(t) + y(t)z'(t) dt
\]

\[
= \int_0^1 (2 + t) \cdot 0 + 1 \cdot 1 + t \cdot (2t) dt = \int_0^1 1 + 2t^2 dt = t + \frac{2}{3}t^3 \bigg|_0^1 = \frac{5}{3}.
\]

\( \square \)
4. (15 points) Let $M$ denote the unit sphere $\{ p \in \mathbb{R}^3 : \| p \| = 1 \}$, which is a smooth 2-manifold in $\mathbb{R}^3$.

(A) Show that given a chart $(\varphi, U)$ and a point $x_0 \in U$, there exists some $\epsilon \in \mathbb{R}$ and a unique solution $x : ( -\epsilon, +\epsilon ) \to U$ to the equation

\begin{equation}
\frac{d}{dt} \varphi(x(t)) = e_3 - \langle e_3, \varphi(x(t)) \rangle \varphi(x(t)),
\end{equation}

satisfying the initial condition $x(0) = x_0$.

Proof. We begin by defining

\[ \nabla^1 \varphi = \frac{\varphi \times \partial_2 \varphi}{\langle \varphi \times \partial_2 \varphi, \partial_1 \varphi \rangle}, \quad \nabla^2 \varphi = \frac{\varphi \times \partial_1 \varphi}{\langle \varphi \times \partial_1 \varphi, \partial_2 \varphi \rangle}. \]

We have $\langle \varphi, \partial_i \varphi \rangle = 0$ for each $i = 1, 2$, and thus $\{ \varphi, \partial_1 \varphi, \partial_2 \varphi \}$ forms a basis of $\mathbb{R}^3$. Thus, $\langle \varphi \times \partial_2 \varphi, \partial_1 \varphi \rangle \neq 0$ everywhere. Hence, $\nabla^1 \varphi$ is a well-defined $C^\infty$ vector field. In the same manner, $\nabla^2 \varphi$ is also a well-defined $C^\infty$ vector field.

As like the practice problem, a solution $x(t)$ to (*) satisfies

\begin{equation}
\frac{d}{dt} x(t) = \left( \langle e_3, \nabla^i \varphi(x(t)) \rangle \right) = (\nabla \varphi(x(t)))^T e_3.
\end{equation}

The Picard-Lindelof theorem, given initial data $x(0) = x_0 \in \mathbb{R}^2$ there exists a unique integral curve $x(t)$ to the ODE. Thus, a solution to (*) is unique.

In addition, there exists an integral curve $x(t)$ to (**) by the Picard-Lindelof theorem, and it satisfies

\[ \frac{d}{dt} \varphi(x(t)) = \sum_{i=1}^2 \langle e_3, \nabla^i \varphi(x(t)) \rangle \partial_i \varphi(x(t)) = e_3 - \langle e_3, N(x(t)) \rangle N(x(t)). \]

Namely, we have the existence of a solution to (*).

(B) Given a point $p_0 \in M$, there exists a unique smooth curve $\gamma : (-\infty, +\infty) \to M$ satisfying $\gamma(0) = p_0$ and

\begin{equation}
\frac{d}{dt} \gamma(t) = e_3 - \langle e_3, \gamma(t) \rangle \gamma(t).
\end{equation}

There are three different solutions below.

The first proof. Given a point $p \in M$, we choose a rotation matrix $R_p$ such that $R_{p,p} = (0, 0, 1)$, and obtain a $C^\infty$ chart $(\varphi_p, B(0, \frac{1}{2}))$ such that

\[ \varphi_p(x, y) = R_p^{-1}(x, y, \sqrt{1 - x^2 - y^2}), \]

where $R_p^{-1}$ is the inverse matrix of $R_p$. 

First of all, we claim that there exists a unique smooth solution \( \gamma : (-\frac{1}{10}, \frac{1}{10}) \to M \) to (1) with \( \gamma(0) = p_0 \). We recall the chart \( \varphi_{p_0} \) and then obtain the unique integral curve \( x_{p_0} : (a, b) \to B(0, \frac{1}{2}) \subset \mathbb{R}^2 \) to

\[
\frac{d}{dt} x_{p_0}(t) = (\nabla \varphi_{p_0}(x(t)))^T e_3
\]

with \( x_{p_0}(0) = (0, 0) \) and \( 0 \in (a, b) \). Moreover, we suppose that \( (a, b) \) is the maximum open interval where \( x_{p_0}(t) \) is uniquely defined. Then, \( \gamma(t) = \varphi_{p_0}(x_{p_0}(t)) \) solves (1) and satisfies \( \gamma(p_0) \). Since \( \|\frac{d}{dt} \gamma(t)\| \leq \|e_3\| = 1 \), \( \varphi_{p_0}^{-1}(\gamma(t)) \in B(0, \frac{1}{2}) \) for \( |t| < \frac{1}{2} \). Therefore, by the Picard-Lindelof theorem we have \([-\frac{1}{10}, \frac{1}{10}] \subset (a, b) \).

Next, at the point \( p_{\frac{1}{10}} = \gamma(\frac{1}{10}) \subset \varphi_{p_0}(B(0, \frac{1}{2})) \subset M \), we recall the chart \( \varphi_{p_{\frac{1}{10}}} \) and repeat the process again. Then, we obtain the unique smooth solution \( \gamma(t) \) for \( t \in [0, \frac{2}{10}] \) regarding the given point \( \gamma(\frac{1}{10}) = p_{\frac{1}{10}} \) from the previous step. Therefore, we have the unique smooth solution \( \gamma(t) \) for \( t \in [0, \frac{2}{10}] \).

We inductively repeat this process for each point

\[
p_{\frac{m}{10}} = \gamma(\frac{m}{10}) \subset \varphi_{p_{\frac{m-1}{10}}}(B(0, \frac{1}{2})) \subset M
\]

and chart \( \varphi_{p_{\frac{m}{10}}} \) for \( m \in \mathbb{N} \). Hence, there exists a unique smooth solution \( \gamma(t) \) for all \( t \in [-\frac{1}{10}, +\infty) \). In the same manner, \( \gamma(t) \) can be extended for all \( t \in \mathbb{R} \). \( \square \)

Some students tried to do in this direction with the compatibility of charts. However, the this first order ODE for integral curves, we do not need to use the compatibility. It’d be required for higher order ODE for integral curves or for flows.

The second proof. As we discuss in the first proof, the uniqueness is obvious. To be specific, given any point \( \gamma(t_0) \in M \) we choose a chart \( (\varphi, U) \) such that \( \gamma(t_0) = \varphi(x_0) \) for a point \( x_0 \in U \). Then, \( x(t) = \varphi^{-1}(\gamma(t)) \) is an integral curve to (**) with the initial data \( x(t_0) = x_0 \). Hence, the uniqueness of integral curve \( x(t) \) for some interval \( t_0 \in I \) implies the uniqueness of \( \gamma(t) \) for \( t \in I \).

Next, we have the existence of smooth \( \gamma(t) \) for \( t \in (-\epsilon, \epsilon) \) by using a chart \( (\varphi, U) \) satisfying \( p_0 \in \varphi(U) \). Hence, there exists an interval \( I \) containing \((-\epsilon, \epsilon)\) such that \( \gamma(t) \) can be extended for \( t \in I \) while a smooth \( \gamma(t) \) does exists on any other interval \( J \geq I \) containing \( I \).

Let’s show that \( I \) is an open interval. If \( I \) has an end point \( a \), then we choose a chart \( (\psi, V) \) around \( \gamma(a) \). Again, the Picard-Lindelof theorem
allows us to extend $\psi \circ \gamma$ for $(a - \epsilon, a + \epsilon)$. Hence, there is another interval $J$ containing $I$ such that a smooth $\gamma$ is well-defined on $J$. Contradiction.

Now, we show that the open interval $I = (a, b)$, where $a \in \{-\infty\} \cup \mathbb{R}$ and $b \in \{+\infty\} \cup \mathbb{R}$, must be the entire line. Suppose $b < +\infty$. $\|\frac{d}{dt} \gamma\| \leq 1$ implies $\|\gamma(t) - \gamma(t')\| \leq |t - t'|$, namely it is a sort of Cauchy sequence. Hence, $\lim_{t \to b} \gamma(t) = B$ for a certain point $B \in \mathbb{R}^3$. Since $M$ is a closed set, we have $B \in M$. Therefore, we again choose a chart around $B$ and then apply the Picard-Lindelof theorem to derive a contradiction to the definition of the maximum interval $I$. In conclusion, we have $b = +\infty$ and also obtain $a = -\infty$ in the same manner.  

The third proof. As like the first paragraph in the second proof, we have the uniqueness. Here, we show the existence in the other way.

In the special case $p_0 = (0, 0, \pm 1)$, we have a solution $\gamma(t) = p_0$. So, we consider the case $p_0 \neq (0, 0, \pm 1)$. Then, there exists some $\alpha \in \mathbb{R}$ and $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that

$$
\gamma_p = (\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta).
$$

Then, we define a curve

$$
(3) \quad \gamma(t) = (\cos \alpha \cos \theta(t), \sin \alpha \cos \theta(t), \sin \theta(t))
$$

where $\theta(0) = \beta$

$$
\frac{d}{dt} \theta(t) = \cos \theta(t).
$$

By a pset problem, a smooth function $\theta : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ exists. Indeed, one can find a specific formula

$$
\theta(t) = 2 \arctan \left( \tanh \left( \frac{t + c}{2} \right) \right),
$$

where $c$ is the constant satisfying $\theta(0) = \beta$.

One can easily check that the curve $\gamma(t)$ in (3) solves (*).  
