Problem Set 5 (30 points), due 4/27 in class

Materials needed are available in lectures/lectures notes on 4/6, 4/11, and 4/13. Each part in a problem (optional or not) is worth 10 points. The maximum point is 30 points.

We will denote by $F$ a fixed non-archimedean local field.

1. In this problem we take a look at the notion of dual representations. All representations are on vector spaces over $\mathbb{C}$. In principle, all topological groups below should be \textit{totally disconnected} as $GL_n(F)$, but we won’t begin with this assumption (You may assume it if preferred). For such a topological group $G$, we say a representation $(\pi, V)$ is smooth if for any $v \in V$ there exits an open compact subgroup $U \subset G$ such that $\pi(g)v = v$ for any $g \in U$.

(a) Let $K$ be a compact topological group. Let $(\pi, V)$ be a smooth representation of $K$. Prove that $(\pi, V)$ is a direct sum of (possibly infinitely many) finite-dimensional irreducible representations. Conclude that we may decompose $V = V^K \oplus V'$ where $V^K$ is the fixed subspace of $\pi(K)$ and $V' \subset V$ is some other subrepresentation.

(b) Now let $G$ be a general Hausdorff topological group, and $(\pi, V)$ a smooth representation of $G$. The smooth dual of $V$ is defined to be $\hat{V} := \{\phi : V \to \mathbb{C} \text{ linear} \mid \exists \text{ open compact subgroup } K \subset G \text{ such that } \phi(\pi(g)v) = \phi(v) \text{ for all } g \in K\}$. Show that the smooth dual of a non-trivial smooth representation (i.e. not on a zero vector space) is also a non-trivial (smooth) representation. Show also that for any vector $v \in V$, there exists a $\phi \in \hat{V}$ such that $\phi(v) \neq 0$.

(c) Let $G$ be as above and $C(G)^\infty$ be the space of smooth vectors in $C(G)$, i.e. functions $f : G \to \mathbb{C}$ such that there exists an open subgroup $U_f$ for which $f(g) = f(gh)$ for any $g \in G, h \in U_f$. We again equip $C(G)^\infty$ with the right translation action of $G$. Show that any irreducible smooth representation of $G$ can be embedded into $C(G)^\infty$.

(d) (Optional) Show that we have the natural map $V \to \hat{V}$, and this is an isomorphism iff $V$ is admissible.

2. (Optional) Let $G = GL_2(F)$. In this problem we investigate the principal series $I^G_B(\sigma) = \text{Ind}_B^G(\sigma \otimes \Delta_{1/2}^B)$. Write $\sigma = \sigma_1 \otimes \sigma_2$ and $\sigma^*$ be the character of $T \cong (F^\times)^2$ by permuting $\sigma_1$ and $\sigma_2$ as in the lecture.

(a) Show that if $I^G_B(\sigma)$ contains a 1-dimensional subrepresentation, then $\sigma_2 = | \cdot | \otimes \sigma_1$. (Hint: suppose $\pi$ is the 1-dimensional subrepresentation. By twisting $\sigma_1$ and $\sigma_2$ at the same time we may assume $\pi$ is trivial. Then $\text{Hom}_G(\pi, I^G_B(\sigma)) = \text{Hom}_B(\text{Res}_B^G \pi, \sigma \otimes \Delta_{1/2}^B)$ is non-trivial. Thus $\text{Res}_B^G \pi \cong \sigma \otimes \Delta_{1/2}^B$. Argue that this is only possible if $\sigma \otimes \Delta_{1/2}^B$ is trivial.)
(b) Show that \( \text{Hom}_G(I^G_B(\sigma), I^G_B(\sigma^*)) \) is non-trivial. (Hint: construct \( \phi \in \text{Hom}_B(\text{Res}^G_B I^G_B(\sigma), \sigma^*) \) by \( \phi(f) = \int \sigma\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\right)f\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right)\frac{dx}{|x|} \). The integral may not converge, but argue that it may always be modified so that it converges.)

Note: This shows that if \( I^G_B(\sigma) \) and \( I^G_B(\sigma^*) \) are irreducible, then they are isomorphic.

(c) Show that for another character \( \sigma' \) of \( T \), which we also realize as a character of \( B \), the Hom-space \( \text{Hom}_G(I^G_B(\sigma), I^G_B(\sigma')) \) is trivial when \( \sigma' \not\cong \sigma, \sigma^* \).