Problem Set 3 (30 points), due 3/23 in class

Materials needed are available in lectures/lectures notes on 3/7, 3/9 and 3/16. Each part in a problem (optional or not) is worth 10 points. The maximum point is 30 points.

1. In this exercise we look closer into local zeta integrals and local $L$-factors. Let $F$ be a local field and $\psi$ a non-trivial additive character, inducing $j_F : F \tilde\to \hat{F}$. Let $\chi_0 : F \to \mathbb{C}^\times$ be a unitary character and $d^\times a$ a Haar measure on $F^\times$. Recall that for $f \in S(F)$, the local zeta integral is

$$Z(f, s, \chi_0) = \int_{F^\times} f(a) \chi_0(a)|a|^s d^\times a.$$ 

which we have shown to converge for $\text{Re}(s) > 0$. We also have the local $L$-factors $L(s, \chi_0)$ defined in the lecture.

(a) Explain how our global result (which is then based on Fourier theory on $\mathbb{A}_K$) implies

$$Z(f_1, s, \chi_0)Z(\hat{f}_2, 1-s, \chi_0^{-1}) = Z(f_2, s, \chi_0)Z(\hat{f}_1, 1-s, \chi_0^{-1})$$

for any $f_1, f_2 \in S(F)$. (Note that both sides were originally defined only for $0 < \text{Re}(s) < 1$.)

(b) (Optional) Give another purely local proof of (1).

(c) If $F$ is non-archimedean, show that $Z(f, s, \chi_0) / L(s, \chi_0)$ is always a polynomial in $q^s$ and $q^{-s}$, where $q$ is the order of the residue field of $F$.

(d) (Optional) We continue to assume $F$ non-archimedean. Show that $\{Z(f, s, \chi_0) \mid f \in S(F)\} = \mathbb{C}[q^s, q^{-s}]L(s, \chi_0)$.

Note: This should serve as a motivation for defining $L(s, \chi_0) = 1$ when $\chi_0$ is ramified.

2. (Optional) In the lecture we have the claim about Gauss sum that

$$\epsilon_v(\chi_0, \psi) := q^{-c_v/2} \sum_{x \in \mathcal{O}_v^{-c_v+e} \mathcal{O}_K^\times / (\mathcal{O}_v^{-c_v+e} + \mathcal{O}_v^{c_v} \mathcal{O}_K)} \chi_0^{-1}(x) \psi_v(x)$$

has absolute value 1 ($v$ a non-archimedean place here).

(a) Give a local (direct) proof of this.

(b) Give a proof using our global result (which is based on Fourier theory on $\mathbb{A}_K$.)
3. Prove that if \( \rho : G_\mathbb{Q} \to GL_n(\mathbb{C}) \) is an Artin representation that is unramified everywhere, then \( \rho \) is a direct sum of trivial representations.

4. (Optional) Consider the polynomial \( x^3 - 3 \) which has discriminant \(-3^5\) and its splitting field \( L/\mathbb{Q} \) for which \( \text{Gal}(L/\mathbb{Q}) \cong S_3 \). Let \( \rho' : \text{Gal}(L/\mathbb{Q}) \to GL_2(\mathbb{C}) \) be the 2-dimensional irreducible representation of \( S_3 \), which gives us a 2-dimensional irreducible Artin representation \( \rho : G_\mathbb{Q} \to \text{Gal}(L/\mathbb{Q}) \xrightarrow{\rho'} GL_2(\mathbb{C}) \).

   (a) Compute the Artin conductor \( C(\rho) \).

   (b) Compute the epsilon factor \( \epsilon(\rho) \).