11 Local Langlands correspondence

Let $F$ be our non-archimedean local field. We note that local class field theory can be interpreted as a bijection

$$\{1\text{-dimensional } \mathbb{C}\text{-representations of } F^\times \} \xrightarrow{\text{rec}} \{1\text{-dimensional Weil representations}\}.$$ 

The local Langlands correspondence is a generalization, which on the Galois side (RHS) have general $n$-dimensional Frobenius-semisimple Weil representations; say we fixed an integer $n > 0$ from now on. On the representation side (LHS) things become more complicated. Consider the topological group $GL_n(F)$, i.e. the subset

$$\{(A, B) \in M_n(F)^2 \mid AB = \text{Id}_n\}.$$ 

equipped with the subspace topology from $M_n(F)^2 \cong F^{2n^2}$. This is a special case of what people call $p$-adic Lie groups.

**Definition 11.1.** A smooth representation $(\pi, V)$ of $G = GL_n(F)$ is a (usually infinite-dimensional) vector space $V$ and a homomorphism $\pi : G \to GL(V)$ such that for any $v \in V$, there exists an open subgroup $K \subset G$ for which $\pi(g)v = v$ for any $g \in K$.

Let us look at the space $C^\infty(G)$ of $\mathbb{C}$-valued, locally constant functions on $G$ equipped with the right translation action of $G$. That is, $(g.f)(h) = f(h.g)$ for any $f \in C^\infty(G)$, $g \in G$. This is a smooth representation. This representation is like the regular representation; suppose $(\pi, V)$ is any irreducible smooth representation of $G$. Let $f : V \to \mathbb{C}$ be any non-trivial functional. Then $v \mapsto (g \mapsto f(g^{-1}.v))$ is a $G$-equivariant map from $V$ to $C^\infty(G)$. As this map is evidently non-trivial and $(\pi, V)$ is irreducible, this is an embedding. While $C^\infty(G)$ contains all irreducible smooth representations, it is not a smooth representation itself. This may be fixed (see Problem Set 5) by considering the subspace of smooth vectors in $C^\infty(G)$, i.e. those $f \in C^\infty(G)$ which are right invariant by some open subgroup $K \subset G$. Nevertheless, we would like to have “smaller” representations:

**Definition 11.2.** A smooth representation $(\pi, V)$ of $G$ is called admissible if for any open subgroup $K \subset G$, the fixed-point subspace $V^K := \{v \in V \mid \pi(g)v = v, \forall g \in K\}$ is finite-dimensional.

In fact, all “small” smooth representations are admissible. We have this highly non-trivial result

**Theorem 11.3.** Every smooth representation of $G$ of finite length is admissible.
Theorem 11.4. (Harish-Taylor, Henniart) There is a natural bijection

\[
\{\text{irreducible smooth representations of } GL_n(F)\} \xrightarrow{\sim} \{\text{Frobenius-semisimple n-dimensional Weil-Deligne representations}\}.
\]

It is not our goal (nor possible) to get close to the proof of this monumental result. Nevertheless we wish to give many examples of Theorem 11.4 to motivate the subject. First we need some basics about \( G = GL_n(F) \) and its representations. Let \( \mu \) be any left Haar measure on \( G \). For any \( g \in G \), write \( r_g : G \xrightarrow{\sim} G \) the right translation map \( h \mapsto hg \). By definition \((r_g)_*(\mu)\) is still a left Haar measure, and thus \((r_g)_*(\mu) = \Delta(g)\mu\) for some \( \Delta(g) \in \mathbb{R}_+^\times \), i.e. \( \mu(Eg^{-1}) = \Delta(g)\mu(E) \) for any Borel subset \( E \subset G \). This gives a continuous homomorphism (called the modulus character) \( \Delta : GL_n(F) \to \mathbb{R}_+^\times \). By explicit calculation one may check that the commutator \((GL_n(F), GL_n(F)) = SL_n(F)\) and \( \Delta \) factors as \( \Delta : GL_n(F)/SL_n(F) \cong F^\times \to R_+ \).

We claim that \( \Delta(c \cdot \text{Id}_n) = 1 \) for any \( c \in F^\times \). Indeed, \( g = c \cdot \text{Id}_n \) is in the center of \( G \), so that \( r_g = l_g \) is also the left translation map and thus preserves any left Haar measure. But then \( \Delta \) is trivial on \( F^\times n \) and thus trivial on \( F^\times \), i.e. \( \Delta \) is identically trivial, which means any left Haar measure \( \mu \) is also a right Haar measure. In this case we say \( G \) is unimodular and \( \mu \) is a Haar measure. We now fix any Haar measure \( \mu \) on \( G \).

There are two basic ways to form explicit smooth representations of \( G \): parabolic induction and compact induction. We begin with the former. Let \( B \subset G \) be the subgroup of invertible upper triangular matrices. The subgroup \( B \) is usually called the Borel subgroup. We note that \( B \) is not unimodular. For example when \( G = GL_2 \), we have \( \Delta_B(\begin{pmatrix} x & z \\ 0 & y \end{pmatrix}) = |\frac{x}{y}| \).

To see this, one observes \( \mu(E \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}^{-1}) = \mu(\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} E \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}^{-1}) \), but conjugation by \( \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \) sends \( \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ 0 & \mathcal{O}_F \end{pmatrix} \) to \( \begin{pmatrix} \mathcal{O}_F^\times & \frac{z}{y} \mathcal{O}_F \\ 0 & \mathcal{O}_F^\times \end{pmatrix} \). Without loss of generality we assume \( |x/y| \geq 1 \). Then

\[
\begin{pmatrix} \mathcal{O}_F^\times & \frac{z}{y} \mathcal{O}_F \\ 0 & \mathcal{O}_F^\times \end{pmatrix} = \bigcup_{u \in \mathcal{O}_F^\times/\mathcal{O}_F} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ 0 & \mathcal{O}_F \end{pmatrix}
\]

and thus \( \mu(E \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}^{-1}) = \mu(\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} E \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}^{-1}) = |\frac{x}{y}| \mu(E) \). By similar computation, one may prove that for general \( B \subset G = GL_n \) the subgroup of invertible upper triangular \( n \times n \) matrices, if \( b \in B \) have diagonal entries \( b_{11}, ..., b_{nn} \) (in order), then \( \Delta_B(b) = \prod_{i=1}^n |b_{ii}|^{n+1-2i} \).

Now let \( T \subset B \) be the subgroup of invertible diagonal matrices. We have \( T \cong (F^\times)^n \). We have a natural epimorphism \( B \twoheadrightarrow T \) (the kernel being the subgroup of upper triangular...
unipotent matrices). Let \( \sigma : T \to \mathbb{C}^\times \) be any continuous homomorphism (=irreducible smooth representation of \( T \)). We may view \( \sigma \) also as a representation of \( B \), and consider

\[
I_B^G(\sigma) := \text{Ind}_B^G(\sigma \otimes \Delta_B^{1/2}) := \{ f \in C^\infty(G) \mid f(bg) = \sigma(b)\Delta_B(b)^{1/2}f(g) \quad \forall b \in B \}
\]

which is again equipped with the right translation action by \( G \). We claim that this is a smooth representation.

**Lemma 11.5.** The quotient topological space \( B \backslash G \) is compact.

**Proposition 11.6.** \( I_B^G(\sigma) \) is admissible.

*Proof.* First we show that \( I_B^G(\sigma) \) is smooth. Let \( f \in I_B^G(\sigma) \subset C^\infty(G) \). Then by definition for any \( g \in G \), there exists an open subgroup \( K_g \) such that \( f(g) = f(gh) \) for any \( h \in K_g \). From definition this implies also \( f(bg) = f(bgh) \) for any \( b \in B \), \( h \in K_g \). We may pick finitely many \( g_1, g_2, ... \) such that the images of \( g_i K_{g_i} \) covers \( B \backslash G \). Then \( f \) is right translation invariant by \( \bigcap K_{g_i} \), thus \( I_B^G \sigma \) is smooth. Now let \( K \subset G \) be any open subgroup and \( f \in V^K \). Then the value of \( f \) is determined by its values on a set of representatives of \( B \backslash G / K \). But as \( K \) is open and \( B \backslash G \) is compact, \( B \backslash G / K \) is finite and thus \( \dim_C V^K < \infty \). \( \square \)

*Proof of Lemma 11.5.* For convenience we will instead prove that \( G / B \) is compact, while \( G / B \cong B \backslash G \) by the inverse map. The space \( G / B \) is usually called the flag variety. This is because \( G \) acts (from the left) on \( F^n \) identified as the space of column vectors, and thus on the space of flags \( \{ V_1 \subset V_2 \subset ... \subset V_n = F^n \mid V_i \text{ are } i \text{-dimensional subspace of } F^n \} \). In particular, \( B \) is the stabilizer of the flag for which \( V_i \) is the span of \( e_1, e_2, ..., e_i \). Now \( G / B \) is a projective variety, and thus \( (G / B)(F) \) is compact for any local field \( F \). We give another proof that does not use algebraic geometry explicitly below.

For any \( 1 \leq i \leq n \), we may find \( v_1, ..., v_i \) which spans \( V_i \) such that each \( v_j \in O_F^n \) and their reduction \( \bar{v}_1, ..., \bar{v}_i \) modulo \( \mathcal{O}_F \) are independent in \( F^n \). The collection of such \( \{v_1, ..., v_i\} \) forms a compact subset of \( (F^n)^i \), thus the space (Grassmannian) of possible \( i \)-dimensional subspace \( V_i \subset F^n \) is compact. The flag variety is a closed subspace of the product of different Grassmannians (for each \( i \)), and thus is also compact. \( \square \)

Such representations \( I_B^G \sigma \) are called **principal series**. Note as \( T \cong (F^\times)^n \), the character \( \sigma : T \to \mathbb{C}^\times \) may be viewed as a collection of characters \( \sigma = \bigotimes_{i=1}^n \sigma_i \) where each \( \sigma_i \) is trivial on all \( F^\times \) but the \( i \)-th one. By local class field theory, each \( \sigma_i \) corresponds to a 1-dimensional Weil representation \( \text{rec}(\sigma_i) : W_F \to \mathbb{C}^\times \). Then \( \bigoplus_{i=1}^n \text{rec}(\sigma_i) \) is an \( n \)-dimensional Weil representation. Now we can state a very rough form of a special case of the Langlands correspondence:

For almost every \( \sigma \), \( \text{rec}(I_B^G(\sigma)) = \bigoplus_{i=1}^n \text{rec}(\sigma_i) \). \( (1) \)
There are some hidden statements in (1). It says \( \text{rec}(I_B^G(\sigma)) \) is irreducible for “almost every” \( \sigma \), and that permuting \( \sigma_i \) does not change \( I_B^G(\sigma) \). Before studying these statements, let us look at (1) from another point of view. The direct sum \( \bigoplus_{i=1}^n \text{rec}(\sigma_i) \) is a completely reducible \( n \)-dimensional Weil representation, i.e. that has \( n \) Jordan-Hölder factors. There are nevertheless more \( n \)-dimensional Weil-Deligne representations with \( n \) Jordan-Hölder factors: recall a Weil-Deligne representation is (\( \rho, N \)) with \( \rho : W_F \to GL_n(\mathbb{C}) \) and \( N \in M_n(\mathbb{C}) \) nilpotent such that \( \rho(\tau)N\rho(\tau)^{-1} = |\tau|N \) for any \( \tau \in W_F \). If we compare the situation of \( \ell \)-adic representation using Deligne’s dictionary, then saying (\( \rho, N \)) has Jordan-Hölder factors \( \text{rec}(\sigma_1), ..., \text{rec}(\sigma_n) \) is to say \( \rho = \bigoplus_{i=1}^n \text{rec}(\sigma_i) \) has diagonal image (up to permutations of \( \sigma_i \)), and \( N \) is a (strictly) upper triangular nilpotent matrix. The condition \( \rho(\sigma)N\rho(\sigma)^{-1} = |\sigma|N \) ensures that, if a certain entry \( N_{ij} \neq 0 \), then \( \text{rec}(\sigma_i) = \text{rec}(\sigma_j) \otimes | \cdot | \Leftrightarrow \sigma_i = \sigma_j \otimes | \cdot | \). Conversely we also have

**Lemma 11.7.** If \( \rho = \bigoplus_{i=1}^n \text{rec}(\sigma_i) \) is a completely reducible Weil representation, then there exists \( N \neq 0 \) such that (\( \rho, N \)) is a Weil-Deligne representation iff \( \sigma_i \cong \sigma_j \otimes | \cdot | \) for some \( i, j \).

In case the condition of the lemma happens, one would guess that \( I_B^G(\sigma) \) is somewhat different. This is indeed the case:

**Theorem 11.8.** (Bernstein-Zelevinsky) The induction \( I_B^G(\sigma) \) is always of finite length, and is irreducible iff \( \sigma_i \not\cong \sigma_j \otimes | \cdot | \) for any \( i, j \).

Let us investigate the example when \( n = 2 \) and \( G = GL_2(F) \). In this case \( I_B^G(\sigma) \) essentially depends on \( \sigma_1/\sigma_2 \): Suppose we have a different \( \sigma' = (\sigma'_1, \sigma'_2) \) such that \( \sigma_1/\sigma_2 = \sigma'_1/\sigma'_2 \) (as characters on \( F^\times \)). Then there exists a character \( \chi \) on \( F^\times \) such that \( \sigma_i = \sigma'_i \otimes \chi \), \( i = 1, 2 \). We also have \( \sigma = \sigma' \otimes (\chi \circ \text{det}) \). Write \( \pi = I_B^G(\sigma) \) and \( \pi' = I_B^G(\sigma') \). Then one has \( \iota : \pi \rightarrow \pi' \otimes (\chi \circ \text{det}) \) given by \( \iota(f)(g) = \chi(\text{det}(g))^{-1} \cdot f(g) \). One has

\[
\iota(f)(bg) = \chi(\text{det}(g))^{-1} \chi(\text{det}(b))^{-1} f(bg) = \chi(\text{det}(g))^{-1} \chi(\text{det}(b))^{-1} \sigma(b) \Delta_B(b)^{1/2} f(g) \\
= \chi(\text{det}(g))^{-1} \sigma'(b) \Delta_B(b)^{1/2} f(g) = \sigma'(b) \Delta_B(b)^{1/2} \iota(f)(g).
\]

To see the map \( \iota \) is \( G \)-equivariant, one checks that \( \iota(\pi(g)f)(h) = \chi(\text{det}(h))^{-1} \cdot f(hg) = \chi(\text{det}(g)) \cdot \chi(\text{det}(hg))^{-1} \cdot f(hg) = \chi(\text{det}(g)) \cdot (\iota(f))(hg) = ((\pi' \otimes (\chi \circ \text{det}))(g) \iota(f))(h) \). In particular, the Jordan-Hölder constituents of \( \pi' \) is those of \( \pi \) twisted by \( \chi^{-1} \circ \text{det} \).

Now we suppose \( \sigma_1 \cong \sigma_2 \otimes | \cdot | \). Recall that \( \Delta_B \left( \begin{array}{cc} x & z \\ 0 & y \end{array} \right) = |x/y| \). Thus we may take \( \sigma = \Delta_B^{-1/2} \), i.e. \( \sigma_1 = | \cdot |^{-1/2} \) and \( \sigma_2 = | \cdot |^{1/2} \). In this case, \( I_B^G(\Delta_B^{-1/2}) = \text{Ind}_B^G(\text{triv}) = \{ f \in C^\infty(G) \mid f(gb) = f(g), \forall b \in B \} \) equipped with the left translation action. One sees that it has the 1-dimensional subspace of all constant functions on \( G \) as a subrepresentation, the trivial representation. The quotient is going to be an irreducible representation (we will not prove this) St, called the Steinberg representation. Under local Langlands correspondence,
the trivial representation corresponds to the case \( N = 0 \) with \( \rho = \text{rec}(\sigma_1) \oplus \text{rec}(\sigma_2) \), and the Steinberg representation corresponds to the case with the same \( \rho \) but \( N \neq 0 \).

What if \( \sigma_2 \cong \sigma_1 \otimes |\cdot| \) instead? Suppose \( \sigma = \Delta_B^{1/2} \), i.e. \( \sigma_1 = |\cdot|^{1/2} \) and \( \sigma_2 = |\cdot|^{-1/2} \). We have \( I_B^G(\Delta_B^{1/2}) = \text{Ind}_B^G(\Delta_B) = \{ f \in C^\infty(G) \mid f(bg) = \Delta_B(b)f(g), \forall b \in B \} \). We have

**Lemma 11.9.** Let \( G \) be a locally compact Hausdorff topological groups and \( H \subset G \) be a closed subgroup. Let \( V \) be the space of (Borel) measurable functions \( f : G \to \mathbb{C} \) that satisfies \( f(hg) = \Delta_H(h)\Delta_G(h)^{-1}f(g) \) for any \( h \in H \). Then there is a non-trivial integration map \( \int : V \to \mathbb{C} \) that is \( G \)-invariant, i.e. \( \int f = \int (g.f) \) for any \( g \in G, f \in V \). Such an integration map is unique up to a constant.

Since \( \Delta_G \equiv 1 \), the lemma gives us a \( G \)-equivariant map from \( I_B^G(\sigma) \) to the 1-dimensional trivial representation. In fact we can make this explicit as follows: we put as above \( V = \{ f \in C^\infty(G) \mid f(bg) = \Delta_B(b)f(g), \forall b \in B \} \). Define \( \int : V \to \mathbb{C} \) by \( \int f := \int_F f\left(\frac{1}{x} 0 1\right)dx \) (with any choice of a Haar measure on \( F \)). We first see that this is well-defined. We have for \( f \in V \):

\[
f\left(\frac{1}{x} 0 1\right) = f\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -x^{-1} & 0 \\ 1 & x \end{pmatrix} \end{pmatrix} = f\left(\begin{pmatrix} 0 & -x^{-1} \\ x & 1 \end{pmatrix} \right)
\]

\[
= f\left(\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & x^{-1} \end{pmatrix} \end{pmatrix} = |x|^{-2} f\left(\begin{pmatrix} 0 & -1 \\ 1 & x^{-1} \end{pmatrix} \right)
\]

and thus

\[
\int_F f\left(\frac{1}{x} 0 1\right)dx = \int_F f\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 1 & x^{-1} \end{pmatrix} \end{pmatrix} \cdot |x|^{-2} dx = \int_F f\left(\begin{pmatrix} 0 & -1 \\ 1 & x^{-1} \end{pmatrix} \end{pmatrix} d(x^{-1})
\]

(2) converges at a neighborhood of \( \infty \) (i.e. for \( |x| \) large enough) by the local constancy of \( f \) at \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). For \( g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) for any \( y \in F \) we evidently have \( \int f = \int g.f \) by \( dx = d(x+y) \).

Similarly using (2) we have the same invariance for \( g = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \). Lastly, for \( g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \) we have

\[
\int g.f = \int_F f\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \end{pmatrix} dx = \int_F f\left(\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} dx
\]

\[
= \int_F \frac{a}{b} \cdot f\left(\begin{pmatrix} 1 & 0 \\ ab^{-1}x & 1 \end{pmatrix} \end{pmatrix} dx = \int_F f\left(\begin{pmatrix} 1 & 0 \\ ab^{-1}x & 1 \end{pmatrix} \end{pmatrix} d(ab^{-1}x) = \int f.
\]

Since \( GL_2(F) \) can be generated by elements of the form \( \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \), \( \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \), we have \( \int g.f = \int f \) for any \( g \in G \).
The kernel of the linear map $f$ is also the Steinberg representation. To see this, let us first recall the Frobenius reciprocity: If $(\pi, V)$ is any (say smooth) representation of $G$ and $H$ is any closed subgroup. We have a canonical isomorphism

$$\text{Hom}_H(\pi, \sigma) \cong \text{Hom}_G(\pi, \text{Ind}_H^G(\sigma))$$

for any (smooth) representation of $H$. Indeed, let $\phi \in \text{Hom}_B(\pi, \sigma)$. Then we have $\phi' \in \text{Hom}_B(\pi, \text{Ind}_H^G(\sigma))$ given by $\phi'(v)(g) = \phi(\pi(g)v)$, so that $\phi'(v)(hg) = \phi(\pi(h)\pi(g)v) = \sigma(h)\phi(\pi(g)v) = \sigma(h)\phi'(v)(g)$.

Now we would like to show there is a $G$-equivariant map from $I_B^G(\Delta_B^{-1/2})$ to $I_B^G(\Delta_B^{1/2})$ that maps the quotient Steinberg of the former into the latter. That is we would like a non-trivial $\phi' \in \text{Hom}_G(I_B^G(\Delta_B^{-1/2}), I_B^G(\Delta_B^{1/2})) = \text{Hom}_G(\text{Ind}_B^G(\text{triv}), \text{Ind}_B^G(\Delta_B)) \cong \text{Hom}_B(\text{Ind}_B^G(\text{triv}), \Delta_B)$. To produce a $\phi \in \text{Hom}_B(\text{Ind}_B^G(\text{triv}), \Delta_B)$, we consider the integral

$$\phi(f) = \int_F |x|^{-2} \left( f\left(\begin{array}{cc} 1 & 0 \\ x & 1 \end{array}\right) - f\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) \right) dx.$$

and check that this map is $B$-equivariant as before.

More generally, let $\underline{n} = (n_1, n_2, \ldots, n_r)$ be any ordered partition of $n$ and $P = P_{\underline{n}}$ be the subgroup of $G$ consisting of invertible blockwise upper triangular matrices with blocks of sizes $n_1, n_2, \ldots, n_r$ in order. Let $M = M_{\underline{n}} \subset P$ be the subgroup of invertible blockwise diagonal matrices (with blocks of sizes $n_1, \ldots, n_r$ in order). There is a natural quotient map $P \to M$ whose kernel consists of blockwise upper triangular matrices that are identities on each diagonal block. Given any smooth representation $(\sigma, W)$ of $M$, we put

$$I_P^G(\sigma) := \{ f \in G \to W \text{ smooth} \mid f(hg) = \Delta_P(h)^{1/2}\sigma(h)f(g), \forall h \in P, g \in G \}$$

where by smoothness we mean again that there exists some open $K \subset G$ such that $f(gh) = f(g)$ for any $h \in K, g \in G$. Obviously $P \supset B$, thus $P \setminus G$ is also compact and it follows from the same proof that $I_P^G(\sigma)$ is admissible. We have again

**Theorem 11.10.** (Bernstein-Zelevinsky) If $\sigma$ is irreducible (or of finite length), then $I_P^G(\sigma)$ is of finite length.

We call $P$ a parabolic subgroup, and $I_P^G$ the (normalized) parabolic induction. We also remark here that $I_P^G$ is an exact functor, i.e. it preserves short exact sequences. Now suppose $\sigma$ is irreducible. As $M \cong GL_{n_1} \times \ldots \times GL_{n_r}$, we have $\sigma = \bigotimes_{i=1}^r \sigma_i$ where each $\sigma_i$ is a representation of $GL_{n_i}$. We write $\text{rec}(\sigma_i)$ the $n_i$-dimensional Weil representation that the LLC gives us (i.e. dropping the $N$-part). Then we have

**Theorem 11.11.** Let $\pi$ be any Jordan-Hölder constituent of $I_P^G(\sigma)$. Then $\text{rec}(\pi) = \bigoplus_{i=1}^r \text{rec}(\sigma_i)$. 

6
We say a representation is **supercuspidal** if it is not the Jordan-Hölder constituent of any $I_{P_{n}}^{G}(\sigma)$ for some $n$ with $r > 1$ and irreducible smooth representation $\sigma$ of $M_{n}$. The above theorem then suggests

**Theorem 11.12.** The local Langlands correspondence restricts to a bijection

$$\xrightarrow{\text{rec}} \left\{ \text{supercuspidal representations of } GL_{n}(F) \right\}$$

Note that if a Weil-Deligne representation $(\rho, N)$ have $\rho$ irreducible, then $N = 0$ as otherwise $\ker N$ will be a subrepresentation of $W_{F}$. Now if we have order partitions $n_{i} = (n_{i1}, ..., n_{is_{i}})$ for each $n_{i}$, $i = 1, ..., r$. Then we can consider $P_{n_{i}} \in GL_{n_{i}}(F)$ as a parabolic subgroup. Suppose we have irreducible representations $\sigma_{ij}$ of $GL_{n_{ij}}$ for each $i = 1, ..., r$, $j = 1, ..., s_{i}$. Then we may form $\sigma_{i} := I_{P_{n_{i}}}^{GL_{n_{i}}}(F) \bigotimes_{j=1}^{s_{i}} \sigma_{ij}$ and again $\pi = I_{P_{n}}^{G}(\bigotimes_{i=1}^{r} \sigma_{i})$.

Write $\lambda = (n_{11}, ..., n_{1s_{1}}, n_{21}, ..., n_{s_{1}}, ..., n_{rs_{r}})$. Then $P_{\lambda} \subset G$ is in fact the preimage of $\prod_{P_{n_{i}}} \prod_{GL_{n_{i}}F} \cong M_{n}$ under $P_{n} \rightarrow M_{n}$. Instead of the two-step induction as above, one has

$$\pi \cong I_{P_{\lambda}}^{G}(\bigotimes_{i,j} \sigma_{ij}).$$

In particular, this says that for a general induction $I_{P_{n}}^{G}(\bigotimes_{i=1}^{r} \sigma_{i})$, if any $\sigma_{i}$ is not supercuspidal we may further partition $n_{i}$ and find the original induction within (as a subquotient) the refined induction. Thus every representation is a Jordan-Hölder constituent of some $I_{P_{n}}^{G}(\bigotimes_{i=1}^{r} \sigma_{i})$ for some supercuspidal representations $\sigma_{i}$, somewhat reflecting the picture in Theorem 11.11 and 11.12.

### 12 Hecke algebra

From a global point of view, an interesting (i.e. coming from geometry) $\ell$-adic representation is unramified almost everywhere. Under Deligne’s recipe, an unramified $\ell$-adic Weil representation becomes a Weil-Deligne representation $(\rho, N)$ where $N = 0$ and $\rho$ is unramified, that is $I_{F} \subset \ker(\rho)$ (we call such Weil-Deligne representation **unramified**). A Frobenius-semisimple Weil-Deligne representation thus has to be a direct sum of characters of the Weil group, and therefore corresponds to some Jordan-Hölder constituent of a principal series $I_{B}^{G}(\sigma)$, where $\sigma$ is a character on $T \cong (F^{\times})^{n}$ that may be written as $\sigma = \bigotimes_{i=1}^{n} \sigma_{i}$. As $\text{rec}(\sigma_{i})$ is unramified, we have $\sigma_{i}(O_{F}^{\times}) = 1$ (in which case we also say $\sigma_{i}$ is unramified).

We claim that if $\sigma$ is unramified, i.e. $\sigma_{i}(O_{F}^{\times})$ for all $i = 1, ..., n$, then $I_{B}^{G}(\sigma)$ contains a vector fixed by the action of $GL_{n}(O_{F})$. every $g \in GL_{n}(F)$ can be written as $b_{g}h_{g}$ for $b_{g} \in B$, $h_{g} \in H$. Theorem 11.12 then suggests
\( h_g \in GL_n(\mathcal{O}_F) \). Such decomposition is not unique, but the valuation of the diagonal entries of \( b_g \) are uniquely determined. Thus \( (\sigma \otimes \Delta_B^{1/2})(b_g) \) is well-defined if \( \sigma \) is unramified, and 
\[
 f(g) := (\sigma \otimes \Delta_B^{1/2})(b_g) \in I_B^G(\sigma) \text{ is fixed by } GL_n(\mathcal{O}_F).
\]
However, in general it is not easy to decompose \( I_B^G(\sigma) \) when it is not irreducible (nor necessarily easy to deal with it even if it is irreducible). We would like a different description, which we will carry out via the machinery of Hecke algebras.

For a finite group \( H \), representations of \( H \) can be viewed as an \( \mathbb{C}[H] \)-module. To have an analogue for our \( p \)-adic group \( G \), let \( C_c^\infty(G) \) be the space of compactly supported, locally constant (\( \mathbb{C} \)-valued) functions on \( G \). For any \( f_1, f_2 \in C_c^\infty(G) \), we have the convolution product
\[
 (f_1 \ast f_2)(g) := \int_{h \in G} f_1(h)f_2(h^{-1}g)d\mu,
\]
which makes \( C_c^\infty(G) \) an associative algebra. We call it the (full) Hecke algebra of \( G \), written \( \mathcal{H}(G) \). Note that for any \( f \in C_c^\infty(G) \), by the compactness of support there exists an open compact subgroup \( K \subset G \) such that \( f(g) = f(gh) \) for any \( h \in K \). On the other hand, for any such open subgroup \( K \), the function \( f \) is supported on finitely many left \( K \)-cosets and thus we may put \( f = \sum_{i \in I} c_i1_{g_iK} \) for some finite index set \( I \) and \( c_i \in \mathbb{C}, g_i \in G \).

Now let \( (\pi, V) \) be any smooth representation of \( G \). We may define an action of any \( f \in \mathcal{H}(G) = C_c^\infty(G) \) on \( V \) as follows: for any \( v \in V \), by definition there exists an open subgroup \( U_v \) such that \( v \) is stabilized by \( U_v \). Let \( K \) be as in the previous paragraph. By shrinking \( K \) we may assume \( K \subset U_v \). Then we may write \( f = \sum_{i \in I} c_i1_{g_iK} \) and put
\[
 \pi(f)v := \mu(K)\sum_{i \in I} c_i \cdot \pi(g_i)(v).
\]
One checks that this defines an action of the algebra \( \mathcal{H}(G) \) on \( (\pi, V) \). It is nevertheless not true that all \( \mathcal{H}(G) \)-modules come from smooth representations. Instead, for an open compact subgroup \( K \subset G \) let \( e_K := \frac{1}{\mu(K)}1_K \in \mathcal{H}(G) \) be an idempotent; \( e_K \ast e_K = e_K \). We say a \( \mathcal{H}(G) \)-module \( V \) is smooth if every \( v \in V \) is fixed by some \( e_K \) (for an open compact subgroup \( K \)). We have

**Lemma 12.1.** Smooth \( \mathcal{H}(G) \)-modules correspond to smooth representations of \( G \). More precisely, if \( (\pi, V) \) is a smooth representation of \( G \), then for any open compact subgroup \( K \) we have \( v \in V^K \) iff \( e_Kv = v \), and thus the corresponding \( \mathcal{H}(G) \)-module is smooth. On the other hand, if \( V \) is a smooth \( \mathcal{H}(G) \)-module, then \( V \) arises from a representation of \( G \).

**Proof.** Let us begin with a smooth representation \( (\pi, V) \) of \( G \). If \( v \in V^K \), then \( e_Kv = \frac{\mu(K)}{\mu(K)}\pi(1)v = v \). On the other hand if \( e_Kv = v \), that is there exists an open subgroup \( K' \subset K \) such that \( v \in V^{K'} \) and \( v = \frac{\mu(K')}{\mu(K)}\sum_{h \in K/K'} \pi(h)v \). For any \( g \in K \) we then have
\[
 \pi(g)v = \frac{1}{[K:K']} \sum_{h \in K/K'} \pi(gh)v = \frac{1}{[K:K']} \sum_{h \in K/K'} \pi(h)v = v.
\]
For the other direction we begin with a $\mathcal{H}(G)$-module $V$. For any $v \in V$ we have $v \in e_KV$ for some open compact subgroup $K$. Then we may define $\pi(g)v = \frac{1}{\mu(K)}1_{gK}v$; one checks that this is independent of the choice of $K$. To check that this defines a representation, let $g' \in G$ be arbitrary and put $K' = gKg^{-1}$. Then $e_{K'}\pi(g)v = \frac{1}{\mu(K')}\pi(g) = \frac{1}{\mu(K)}1_{gK}v = \pi(g)v$. We then have $\pi(g')\pi(g)v = \frac{1}{\mu(K')}\frac{1}{\mu(K)}1_{g'gK}v = \pi(g'gv)$.

It will be desirable to fix an $K$ that plays the role is the above paragraphs. Fix $K$ an open compact subgroup of $G$. Let $\mathcal{H}(G, K) \subset \mathcal{H}(G)$ be the subalgebra given by those functions $f \in C_c^\infty(G)$ with $f(h_1gh_2) = f(g)$ for any $h_1, h_2 \in K, g \in G$. Every such $f$ can be written as $\sum_{i \in I}c_i1_{g_iK}$ for some finite index set $I$ and $c_i \in \mathbb{C}, g_i \in G$. They can also be written as $\sum_{i \in I}c_i1_{Kg_i}$; and also $\sum_{i \in I}c_i1_{Kg_iK}$. From the last expression we have $\mathcal{H}(G, K) = e_K\mathcal{H}(G)e_K$ because $e_K1_{Kg_iK}e_K = 1_{Kg_iK}$.

Let $(\pi, V)$ be a smooth representation of $G$. In particular $V$ is a (smooth) representation of $K$. As $K$ is compact, we may write $V = V^K \oplus V'$ where $V'$ is a direct sum of (usually infinitely) many non-trivial irreducible finite-dimensional representations of $K$ (see Problem set 5). Recall that we can see $V$ is a $\mathcal{H}(G)$-module.

**Lemma 12.2.** $V^K$ is a $\mathcal{H}(G, K)$-submodule of $V$. In fact, we have $V^K = e_KV$.

**Proof.** Indeed, for any $v \in V^K$ we have $v = e_Kv$ and thus $V^K \subset e_KV$. On the other hand, for any $v \in V$ we have $e_K(e_Kv) = e_Kv$ and thus $e_KV \subset V^K$.

**Lemma 12.3.** Let $V$ be an irreducible smooth $\mathcal{H}(G)$-module and suppose $V^K = e_KV$ is non-trivial. Then $V^K$ is an irreducible $\mathcal{H}(G, K)$-module.

**Proof.** Suppose on the contrary that $W \subsetneq V^K$ is a proper $\mathcal{H}(G, K)$-submodule. Then $W$ generates a $\mathcal{H}(G)$-submodule of $V$, and thus $V = \mathcal{H}(G)W$. But then as $W \subset V^K$, we have $V^K = e_K\mathcal{H}(G)W = e_K\mathcal{H}(G)e_KW = \mathcal{H}(G, K)W = W$, a contradiction.

**Proposition 12.4.** Suppose $V_1, V_2$ are irreducible smooth $\mathcal{H}(G)$-module with $V^K_1$ and $V^K_2$ non-trivial. Suppose we have an isomorphism $\phi : V^K_1 \cong V^K_2$ of $\mathcal{H}(G, K)$-module. Then $\phi$ extends (necessarily uniquely) to an isomorphism of $\mathcal{H}(G)$-module from $V_1$ to $V_2$.

**Proof.** Fix any non-zero $w \in V^K_1$. As $V_1$ is irreducible, $V_1 = \mathcal{H}(G)w$ and we are forced to define the extension by $\phi(f.w) = f.\phi(w)$, i.e. by the value of $\phi$ at $w$. It suffices to show that whenever $f.w = 0$ we have $f.\phi(w) = 0$. Suppose $\phi(f.w) = 0$. Then $0 = (e_K*f'*f)e_Kw = (e_K*f'*f)*e_Kw = (e_K*f)*e_Kw = 0$. But then $e_K*f'*f* e_K \in \mathcal{H}(G, K)$ and thus by assumption $(e_K*f'*f)e_K\phi(w) = (e_K*f'*f)e_K\phi(w) = (e_K*f)e_K\phi(w) = 0$. In other words, $v := f.\phi(w) \in V_2$ is such that $(e_K*f')v = 0$ for any $f' \in \mathcal{H}(G)$. If $v \neq 0$, then as $V_2$ is irreducible there exists some $f'$ such that $f'v$ is a non-trivial vector in $V^K_2$, a contradiction.
We now fix \( K = GL_n(\mathcal{O}_F) \) and write \( \mathcal{H}^o = \mathcal{H}(G, K) \). This is usually called the spherical Hecke algebra. Recall that our motivation was to study the following kind of representations:

**Definition 12.5.** An irreducible representation of \( G = GL_n(F) \) is called unramified if \( V^K \neq 0 \).

By Lemma 12.3 and Proposition 12.4, we see that the classification of unramified representations of \( G \) is equivalent to the classification of irreducible finite-dimensional \( \mathcal{H}^o \)-modules. To understand \( \mathcal{H}^o \), it is natural to first figure out a set of representative for \( K \setminus G/K \). What is an \( n \times n \) matrix up to row and column operations with coefficients in \( \mathcal{O}_F \)?

**Lemma 12.6.** Fix a choice of an uniformizer \( \varpi_F \in F \). Then the following is a set of representative for \( K \setminus G/K \):

\[
\mathcal{B} = \left\{ \begin{pmatrix} \varpi_F^{a_1} & 0 & \ldots & 0 \\ 0 & \varpi_F^{a_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \varpi_F^{a_n} \end{pmatrix} \mid a_1 \geq a_2 \geq \ldots \geq a_n \in \mathbb{Z} \right\}
\]

Consequently the characteristic functions \( 1_{KgK} \) for \( g \in \mathcal{B} \) make a basis for \( \mathcal{H}^o \). Now consider \( \iota : G \to G \) by \( \iota(g) = g' \). This is an anti-homomorphism, i.e. \( \iota(g_1g_2) = \iota(g_2)\iota(g_1) \), and \( \iota(K) = K \). Then we have \( \iota \) induces an anti-homomorphism of the spherical Hecke algebra \( f \mapsto \iota(f) \) where \( (\iota(f))(g) := f(\iota(g)) \). We still denote by this anti-automorphism by \( \iota \).

**Lemma 12.7.** The anti-homomorphisms \( \iota \) is the identity on \( \mathcal{H}^o \). In particular, \( \mathcal{H}^o \) is commutative, and any irreducible finite-dimensional \( \mathcal{H}^o \)-module is 1-dimensional.

**Proof.** As \( \iota(g) = g \) for any \( g \in \mathcal{B} \), we have \( \iota(1_{KgK}) = 1_{KgK} \) for any \( g \in \mathcal{B} \). The rest quickly follows. \( \square \)

We now assume that the Haar measure on \( G \) was normalized so that \( K = GL_n(\mathcal{O}_F) \) has measure 1. For \( j = 1, 2, \ldots, n \), put \( g_j \in GL_n(F) \) be the diagonal matrix whose \( i \)-th entry on the diagonal is \( \varpi_F \) for \( i \leq j \) and 1 for \( j < i \leq n \). We have

**Lemma 12.8.** The elements \( t_j := q_F^{-j(n-j)/2}1_{Kg,K} \) for \( j = 1, \ldots, n \) and \( t_j^{-1} \) generates \( \mathcal{H}^o \) as an algebra. In fact, \( \mathcal{H}^o \cong \mathbb{C}[t_1, \ldots, t_{n-1}, t_n, t_n^{-1}] \).

**Proof.** For any decreasing sequence of integers \( \mathbf{a} = (a_1, \ldots, a_n) \), let \( g_\mathbf{a} \) be the diagonal matrix with \( i \)-th entry on the diagonal equal to \( \varpi_F^{a_i} \), and \( f_\mathbf{a} = 1_{Kg_\mathbf{a}K} \). In particular \( t_j \) is a multiple of \( f_{\sum_{i=1}^j a_i} \). We have seen that \( \{ f_\mathbf{a} \} \) is a basis for \( \mathcal{H}^o \). For any such \( \mathbf{a} = (a_1, \ldots, a_n) \) and \( \mathbf{a}' = (a'_1, \ldots, a'_n) \). We say \( \mathbf{a} \succ \mathbf{a}' \) if \( a_1 + \ldots + a_n = a'_1 + \ldots + a'_n \), and \( a_1 + \ldots + a_i \geq a_1 + \ldots + a'_i \) for \( i = 1, 2, \ldots, n-1 \). By explicit calculation one has \( f_\mathbf{a} + f_\mathbf{a}' = c_{\mathbf{a} \mathbf{a}'} f_{\mathbf{a} + \mathbf{a}'} + \sum_{\mathbf{a} + \mathbf{a}' \succ \mathbf{a}} c_{\mathbf{a} \mathbf{a}'} f_{\mathbf{a}'} \), for some \( c_{\mathbf{a} \mathbf{a}'} \in \mathbb{Z}_{>0} \) and \( c_{\mathbf{a} \mathbf{a}'} \in \mathbb{Z} \). This shows that \( t_1, \ldots, t_{n-1}, t_n, t_n^{-1} \) generates \( \mathcal{H}^o \) and are algebraically independent. \( \square \)
In particular, an isomorphism class of irreducible $\mathcal{H}^o$-modules is equivalent to a maximal ideal of $\mathcal{H}^o$, namely an assignment $t_j \mapsto s_j$, where $s_1, ..., s_n \in \mathbb{C}$ and $s_n \neq 0$. We can now state our final result.

**Theorem 12.9. (Unramified local Langlands of $GL_n$)** The local Langlands correspondence matches the unramified representation $(\pi, V)$ for which $\mathcal{H}^o$ acts on $V^K$ by $(s_1, ..., s_n)$ to the unramified Weil representation for which the image of the Frobenius has characteristic polynomial $\lambda^n - s_1\lambda^{n-1} + ... + (-1)^n s_n$.

We will only prove a simplified case of the theorem. Suppose $n = 2$ (one can extends the following to general $n$ with a bit of work) and $\rho : W_F \to GL_2(\mathbb{C})$ is unramified, so that $\rho \simeq \text{rec}(\sigma_1) \oplus \text{rec}(\sigma_2)$ with $\sigma_1(\mathcal{O}_F^\times) \equiv \sigma_2(\mathcal{O}_F^\times) \equiv 1$. Let $\sigma = \sigma_1 \otimes \sigma_2 : T \simeq (F^\times)^2 \to \mathbb{C}^\times$ be as before. We furthermore assume $\sigma_1 \not\simeq \sigma_2 : |\cdot|^{\pm 1}$ so that $\rho$ cannot be paired with a non-trivial $N$-part, and $I_B^G(\sigma)$ is irreducible.

Recall that $I_B^G(\sigma) = \{ f \in C^\infty(G) \mid f(bg) = \sigma(b)\Delta_B^{1/2}(b)f(g) \}$ and we have seen that the $K$-fixed vector is (up to scalar) the unique function $f^o$ given by $f(bh) = \sigma(b)\Delta_B^{1/2}(b)$ for any $b \in B$, $h \in K$. Now we would like to know how $t_1, t_2$ acts on $f^o$. As $t_2$ really comes from the center, we easily see that $t_2 f^o = \left( \begin{smallmatrix} 1 & 0 \\ 0 & \varpi_F \end{smallmatrix} \right) f^o = \sigma_1(\varpi_F) \sigma_2(\varpi_F) f^o = s_2 f^o$. To see how $t_1 = q_F^{-1/2} K g_1 K$ acts on $f^o$, note that the double coset $K g_1 K$ can be decomposed as

$$K g_1 K = K \left( \begin{smallmatrix} \varpi_F & 0 \\ 0 & 1 \end{smallmatrix} \right) K = \left( \begin{smallmatrix} 1 & 0 \\ 0 & \varpi_F \end{smallmatrix} \right) K \sqcup \bigsqcup_{a \in k} \left( \begin{smallmatrix} \varpi_F & a \\ 0 & 1 \end{smallmatrix} \right) K,$$

where $k$ is the residue field of $F$. Since we know $1_{K g_1 K} f^o \in V^K$ is a scalar multiple of $f^o$, it suffices to compute

$$(1_{K g_1 K} f^o)(e) = f^o\left( \begin{smallmatrix} 1 & 0 \\ 0 & \varpi_F \end{smallmatrix} \right) + \sum_{a \in k} f^o\left( \begin{smallmatrix} \varpi_F & a \\ 0 & 1 \end{smallmatrix} \right) = q_F^{1/2} \sigma_1(\varpi_F) f^o + q_F^{1/2} \sigma_2(\varpi_F) f^o$$

and thus $t_1 f^o = s_1 f^o$ as $s_1 = \text{Tr}(\rho(\text{Frob})) = \sigma_1(\varpi_F) + \sigma_2(\varpi_F)$.