9 Artin representations

Let $K$ be a global field. We have enough for $G^b_K$. Now we fix a separable closure $K^{sep}$ and $G_K := \text{Gal}(K^{sep}/K)$, which can have many nonabelian simple quotients. An Artin representation $(\rho, V)$ is a continuous homomorphism $\rho : G_K \to GL(V)$ for some finite dimensional $\mathbb{C}$-vector space $V$. Here $GL(V)$ is the space of automorphisms of $V$ equipped with the natural topology from that of $\mathbb{C}$. We say $(\rho, V)$ and $(\rho', V')$ are equivalent if there is an isomorphism $f : V \sim V'$ such that $\rho' = f \circ \rho \circ f^{-1}$. We identify equivalent Artin representations. It is common to abbreviate $(\rho, V)$ to $\rho$ when there is no ambiguity.

For any decomposition of vector space $V = V_1 \oplus V_2$, we have an inclusion of groups $\iota : GL(V_1) \times GL(V_2) \hookrightarrow GL(V)$ where $GL(V_1)$ acts trivially on $V_2$ and vise versa. For $\rho_i : G_K \to GL(V_i), i = 1, 2$, we say $\rho = \rho_1 \oplus \rho_2$ if $\rho = \iota \circ (\rho_1, \rho_2)$ and say $\rho$ is the direct sum of the subrepresentations $\rho_1$ and $\rho_2$. If this happens for some $V_1, V_2$ non-trivial we say $\rho$ is reducible, and irreducible otherwise. Representation theory of the finite group $G_K/\ker(\rho)$ gives

**Lemma 9.1.** Every Artin representation is a unique direct sum (up to permutation) of irreducible subrepresentations.

We have also seen

**Lemma 9.2.** For any Artin representation $\rho$, $\text{Im}(\rho)$ is finite.

The kernel of $\rho$ is a finite index open normal subgroup, and thus $\ker \rho = G_L$ for some finite Galois extension $L/K$. Recall also that for any place $v$ for $K$, we can fixed an embedding of separable closures $K^{sep} \hookrightarrow K_v^{sep}$, which induces $\iota_v : G_{K_v} \hookrightarrow G_K$. Different choices simply conjugate $\iota_v$ by elements in $G_K$.

**Definition 9.3.** $\rho$ is said to be unramified (resp. tamely ramified) at a non-archimedean place $v$ iff the following equivalent condition holds:

(i) $L/K$ is unramified (resp. tamely ramified) at $v$.

(ii) The image of the inertia $I_v$ (resp. wild inertia $P_v$) of $G_{K_v}$ is contained in $\ker \rho$.

It is obvious that any Artin representation is unramified almost everywhere. When $\rho$ is unramified at $v$, the composition $G_{K_v} \hookrightarrow G_K \xrightarrow{\rho} GL(V)$ factors through $G_{K_v}/I_v = \langle \text{Frob}_v \rangle$. In general, let $V^{I_v} \subset V$ be the subspace fixed by $\rho(I_v)$, then $\text{Frob}_v$ acts on $V^{I_v}$.

**Definition 9.4.** Let $(\rho, V)$ be an Artin representation and $v$ a non-archimedean place of $K$. We define the Artin local $L$-factor

$$L_v(s, \rho) := \det(\text{Id}_{V^{I_v}} - q_v^{-s} \rho(\text{Frob}_v)|_{V^{I_v}})^{-1}.$$ 

Recall $\Gamma_\mathbb{R}(s) = \pi^{-s/2} \Gamma(s/2)$. 

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Definition 9.5. Let $(\rho, V)$ be an Artin representation and $v$ an archimedean place of $K$. We define the Artin local $L$-factor at $v$ as follows: either $K_v \cong \mathbb{R}$ and $\rho = (\text{triv})^{n_1} + (\text{sgn})^{n_2}$, or $K_v \cong \mathbb{C}$ and $\rho = (\text{triv})^{\dim V}$ is trivial. In the former case, we put $L_v(s, \rho) = \Gamma_{\mathbb{R}}(s)_{n_1} \Gamma_{\mathbb{R}}(s + 1)^{n_2}$. In the latter case put $L_v(s, \rho) = \Gamma_{\mathbb{R}}(s)^{\dim V} \Gamma_{\mathbb{R}}(s + 1)^{\dim V}$.

Remark 9.6. Despite that we chose a global language, one sees that all these definitions are purely local. For example, for $F$ a local field and $\rho : G_F \to GL(V)$ a continuous representation (i.e. homomorphism) we can define $L_F(s, \rho)$ likewise, so that $L_v(s, \rho) = L_{K_v}(s, \rho|_{G_{K_v}})$.

Definition 9.7. Let $(\rho, V)$ be an Artin representation. The Artin $L$-function is the Euler product

$$L(s, \rho) = \prod_v L_v(s, \rho)$$

The Euler product converges again for $\Re(s) > 1$. This is because as $\Im(\rho)$ is finite, $\text{Frob}_v$ is diagonalizable with eigenvalues being roots of unity. One sees directly that when $\dim V = 1$, our Artin representations become Dirichlet characters, and our Artin $L$-functions agrees with the Dirichlet $L$-functions we had last time.

In fact, they are compatible in even greater generality. Suppose $G$ is any group and $H \subset G$ a (say finite index) subgroup, and $\rho_H : H \to GL(V)$ a representation of $H$. One may think of $V$ as a left $\mathbb{C}[H]$-module, and form $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ as a left $\mathbb{C}[G]$-module. This gives a representation $\rho_G : G \to GL(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V)$ and we write $\text{ind}_H^G \rho_H := \rho_G$. For any finite separable extension $L/K$ and an Artin representation $\rho_L : G_L \to GL(V)$ for $L$, one checks that $\text{ind}_G^K \rho_L$ is an Artin representation for $K$.

Proposition 9.8. Let $L/K$ be a finite separable extension and $\rho_L$ be an Artin representation for $L$. Then

$$L_v(s, \text{ind}_G^K \rho_L) = \prod_w L_w(s, \rho_L)$$

where $w$ runs over places of $L$ above $v$.

Corollary 9.9. Let $L/K$ be a finite separable extension and $\rho_L$ be an Artin representation for $L$. Then

$$L(s, \text{ind}_G^K \rho_L) = L(s, \rho_L)$$

where the first Artin $L$-function is of the Artin representation $\text{ind}_G^K \rho_L$ for $K$.

To prove the proposition, we first establish its purely local part:
**Lemma 9.10.** Let $E/F$ be a finite separable extension of local fields and $\rho_E : G_E \to GL(V)$ a continuous representation. Then for

$$L_F(s, \text{ind}^{G_F}_{G_E} \rho_E) = L_E(s, \rho_E).$$

**Proof.** The case for $F$ archimedean is obvious; we assume $F$ non-archimedean. Let $E'$ be the maximal unramified subextension of $E/F$. Then we have from definition $\text{ind}^{G_F}_{G_E} \rho_E = \text{ind}^{G_E}_{G_E} \rho_E$, and the desired equality will follow from the two equalities $L_F(s, \text{ind}^{G_F}_{G_E} \rho_E) = L_{E'}(s, \text{ind}^{G_F}_{G_E} \rho_E) = L_E(s, \rho_E)$. In other words, it suffices to deal with the case $E' = E$ and $F = E'$.

Let us write $\rho_F = \text{ind}^{G_F}_{G_E} \rho_E$. Also write $I_F$ the inertia in $G_F$ and likewise $I_E = I_F \cap G_E$ the inertia in $G_E$. First we consider the case $E' = F$, so that $E/F$ is totally ramified and equivalently $G_F = I_F G_E$. In particular $G_F/I_F \cong G_E/I_E$ and they share the same Frobenius $\sigma$. Let $\{\theta_0 = 1, \theta_1, ..., \theta_{e-1}\}$ be a set of representatives of $G_F/G_E = I_F G_E/G_E \cong I_F/I_E$. Then

$$V' := \mathbb{C}[G_F] \otimes_{\mathbb{C}[G_E]} V = \bigoplus_{i=0}^{e-1} \theta_i V$$

and one has $V'^{I_E} \hookrightarrow V'^{I_F}$ by $v \mapsto (v, \theta_1(v), ..., \theta_{e-1}(v))$. This is actually an isomorphism; every element in $V'$ fixed by $I_F$, i.e. by $I_E$ and by $\theta_1, ..., \theta_{e-1}$ has to be of this form. Hence $L_F(s, \text{ind}^{G_F}_{G_E} \rho_E) = \det(\text{Id}_{V'^{I_F}} - \rho_F(\sigma)|_{V'^{I_F}})^{-1} = \det(\text{Id}_{V'^{I_E}} - \rho_E(\sigma)|_{V'^{I_E}})^{-1} = L_E(s, \rho_E)$.

Now suppose $E = E'$, so that $E/F$ is unramified and equivalently $I_F = I_E$. Let $\sigma \in G_F$ be any Frobenius. We have $\{1, ..., \sigma^{f-1}\}$ as a set of representatives of $G_F/G_E$, and $\sigma^f$ is a Frobenius in $G_E$. We have

$$V' := \mathbb{C}[G_F] \otimes_{\mathbb{C}[G_E]} V = \bigoplus_{i=0}^{f-1} \sigma^i V, \quad \text{and} \quad V'^{I_F} = \bigoplus_{i=0}^{f-1} \sigma^i V^{I_E}.$$  

By the elementary properties of determinants this gives $L_F(s, \text{ind}^{G_F}_{G_E} \rho_E) = \det(\text{Id}_{V'^{I_F}} - \rho_F(\sigma)|_{V'^{I_F}})^{-1} = \det(\text{Id}_{V'^{I_E}} - \rho_E(\sigma^f)|_{V'^{I_E}})^{-1} = L_E(s, \rho_E).$ \hfill $\square$

**Proof of Proposition 9.8.** Now we are ready to prove the global result. The idea is that if $\{\alpha_i\}$ is a set of representatives of $G_K/G_L$. Then we have $V' := \mathbb{C}[G_K] \otimes_{\mathbb{C}[G_L]} V = \bigoplus_i \alpha_i V$. Now Frob$_v$ acts on $V'$. The action is related to the action of $G_{K_v}$ on $G_K/G_L$. We have a $G_{K_v}$-equivariant (left action) bijection

$$G_K/G_L \cong \bigsqcup_{v|w} G_{K_v}/G_{L_w}.$$  

Thus one has $V' = \bigoplus_{v|w} \mathbb{C}[G_{K_v}] \otimes_{\mathbb{C}[G_{L_w}]} V$ as representations of $G_{K_v}$, which is enough for defining $L$-factors. The desired identity thus follows from Lemma 9.10. \hfill $\square$
Example 9.11. Let $L/K$ be any quadratic (separable) extension of global field. Let $\text{triv}_K$ (resp. $\text{triv}_L$) be the trivial representation of $G_K$ (resp. $G_L$) and $\chi_{L/K}$ be the non-trivial character of $\text{Gal}(L/K)$ pulled-back to $G_K$. One has $\text{ind}_{G_K}^{G_L} \text{ind}_L = \text{ind}_K \oplus \chi_{L/K}$, and thus by Corollary 9.9 we have

$$L(s, \text{ind}_L) = L(s, \text{ind}_{G_K}^{G_L} \text{ind}_K) = L(s, \text{ind}_K)L(s, \chi_{L/K})$$

which is well-known at least when $K = \mathbb{Q}$.

By the standard character theory of finite (or compact Hausdorff) groups, we know we can identify representations of $\text{Gal}(L/K)$ (or $G_K$) as characters, i.e. certain functions on $\text{Gal}(L/K)$ (or $G_K$), and one can talk about possibly negative integral linear combination of representations as virtual representations. Suppose now $\rho$ as a representation of $G = \text{Gal}(L/K)$ (or $G = G_K$) is an integral combination of certain $\text{ind}_{G_H}^{G} \chi$ for finite index open subgroups $H \subset G$ and 1-dimensional representations $\chi : H \rightarrow \mathbb{C}^\times$. Then by Corollary 9.9 and our results for $L$-functions of Dirichlet characters, we see that $L(s, \rho)$ is meromorphic for $s \in \mathbb{C}$. This is indeed the case by a theorem of Brauer

**Theorem 9.12.** (Brauer) Let $G$ be a finite group. Then any representation of $G$ is an integral linear combination of inductions of 1-dimensional characters of subgroups $H \subset G$. In fact, all $H$ can be chosen among subgroups for which each is a direct product of a cyclic group and a $p$-group.


As taking dual of a representation is a conjugate-linear action on the space of characters, we have a functional equation relating $L(s, \rho)$ and $L(1-s, \rho^\vee)$ where $\rho^\vee$ is the dual representation. By Brauer’s theorem we thus have

**Corollary 9.13.** Let $\rho$ be any Artin representation. Then $L(s, \rho)$ is meromorphic for $s \in \mathbb{C}$, and there exists constants $C(\rho), N(\rho) \in \mathbb{Q}_{>0}$ as well as $\epsilon(\rho) \in \mathbb{C}, \lvert \epsilon(\rho) \rvert = 1$ such that

$$L(1-s, \rho^\vee) = (C(\rho)N(\rho))^{1/2}\epsilon(\rho)L(s, \rho).$$

We would like to say a few words about $C(\rho), N(\rho)$ and $\epsilon(\rho)$. Recall that when $\dim(\rho) = 1$, we had $N = \prod_v q_v^{-n_v}$ where the product runs over non-archimedean places $v$ and $n_v$ is such that our chosen\(^1\) additive character $\psi_K$ is trivial on $\mathfrak{o}_{K_v}$ but non-trivial on $\mathfrak{o}_{K_v}^{-1}$. When $K$ is a number field, $N$ is the norm of the different to $\mathbb{Q}$. When $K$ contain $\mathbb{F}_q$ as its constant field, we have $N = q^{2g-2}$.

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\(^1\)When one scales the choice by some $x \in K^\times$, the product formula ensures that $N$ is scaled by $\lvert x \rvert = 1$, thus invariant.
Now we would like to put simply $N(\rho) = N^{\dim(\rho)}$; in particular $N(\rho) \in \mathbb{Z}$ when $K$ is a number field. It is then not an easy task to find a general formula for $C(\rho)$. This had also been carried out by E. Artin and involves substantial interplay with ramification groups and class field theory. Recall that for a finite Galois extension of local fields $E/F$ with $G = \text{Gal}(E/F)$, the lower ramification groups are defined by

$$G_i = \{\sigma \in G \mid \sigma(x) - x \in \varpi^{i+1} \mathcal{O}_E, \forall x \in \mathcal{O}_E\}, \ i \geq 0$$

so that $G_0$ is the inertia and $G_1$ is the wild inertia. We define $c_{E/F} : G \to \mathbb{C}$ as

$$c_{E/F}(g) = \begin{cases} 0, & g \notin G_0 \\ -(i + 1) \frac{|G_i|}{|G_0|}, & g \in G_i \setminus G_{i+1} \\ -\sum_{g \in G \setminus \{1\}} c_{E/F}(g), & g = 1 \end{cases}$$

One directly computes that for any representation $\rho$ of $G$ with character $\chi_{\rho} : G \to \mathbb{C}$ we have

$$(c_{E/F}, \chi_{\rho}) = \frac{1}{|G|} \sum_{g \in G} c_{E/F}(g) \chi_{\rho}(g) = \frac{1}{|G_0|} \sum_{i \geq 0} \sum_{\sigma \in G_i} \chi_{\rho}(1) - \chi_{\rho}(\sigma).$$

**Theorem 9.14. (Artin)** $(c_{E/F}, \chi_{\rho}) \in \mathbb{Z}_{\geq 0}$. In other words, $c$ is the character of some representation.

Now let $\rho$ be an Artin representation for our global field $K$ which factor through $\text{Gal}(L/K)$ for some $L/K$ finite Galois. For each non-archimedean place $v$, take any $w|v$ of $L$, let $G_v = \text{Gal}(L_w/K_v) \subset \text{Gal}(L/K)$ and let $\chi_{\rho_v}$ be the character of $\rho|G_v$. Let the Artin conductor $\mathfrak{c}(\rho)$ be

$$\mathfrak{c}(\rho) := \prod_v \mathfrak{p}_v^{(\epsilon_{L_w/K_v} \cdot \chi_{\rho_v})}.$$

and we also put $C(\rho) := N(\mathfrak{c}(\rho)) = [\mathcal{O}_K : \mathfrak{c}(\rho)]$. (When $K$ is a global field, we should think of $\mathfrak{c}(\rho)$ as a divisor and $C(\rho) = q^{\deg(\mathfrak{c}(\rho))}$.) In particular we see $C(\rho) \in \mathbb{Z}$.

**Theorem 9.15. (Artin)** Our $C(\rho)$ and $N(\rho)$ are what is needed in Corollary 9.13.

There remains the question of expressing $\epsilon(\rho)$. It will be desirable to write $\epsilon(\rho) = \prod_v \epsilon_v(\rho, \psi)$. This turns out to be difficult:

**Theorem 9.16. (Deligne)** There exist unique constants $\epsilon_F(\rho, \psi_F) \in \mathbb{C}^\times$ of absolute value 1 for any local field $F$, non-trivial additive character $\psi_F$ and virtual $\mathbb{C}$-representation $\rho$ of $G_F$, such that

(i) $\epsilon_F(\rho, \psi_F)$ is the same as what we had when $\rho$ is a 1-dimensional (actual) representation.

(ii) $\epsilon_F(\rho_1 \oplus \rho_2, \psi_F) = \epsilon_F(\rho_1, \psi_F)\epsilon_F(\rho_2, \psi_F)$.
(iii) Suppose $E/F$ is a finite extension, $\psi_E = \psi_F \circ \text{Tr}_{E/F}$, and $\rho_E$ is any virtual representation of degree 0 of $G_E$, we have

$$\epsilon_E(\rho_E, \psi_E) = \epsilon_F(\text{ind}_{G_E}^{G_F} \rho_E, \psi_F).$$

(iv) Let $K$ be our global field with chosen $\psi_K : \mathbb{A}_K \to \mathbb{C}$. Write $\psi_K^v := \psi_K|_{K_v}$. Then $\epsilon(\rho) = \prod_v \epsilon_{K_v}(\rho|_{G_{K_v}}, \psi_{K_v}).$

Lastly, we mention the Artin Conjecture, which is one of the many motivations for Langlands program:

**Conjecture 9.17.** Let $\rho$ be an irreducible non-trivial Artin representation. Then $L(s, \rho)$ is entire.

In the case when $K$ is a global function field (say the function field of the algebraic curve $X$), the conjecture was proved by Weil in the case when $K$ is a global function field. Nowadays this case is a standard application of Grothendieck-Lefschetz fixed point theorem, which allows us to express $L(s, \rho)$ in terms of the $\ell$-adic cohomology of the local system on $X$ associated to $\rho$. (In particular, the poles of $L(s, \rho)$ comes from even degree $\ell$-adic cohomology in $H^\text{even}_{\text{et}}(X, \rho) = H^0_{\text{et}}(X, \rho) \oplus H^2_{\text{et}}(X, \rho) \cong H^0_{\text{et}}(X) \oplus H^2_{\text{et}}(X, \rho^\vee) = 0$, thus there are no poles.)

## 10 $\ell$-adic representations

We fix a prime $\ell$. Artin representations are not enough; $\ell$-adic representations with infinite images naturally appear in geometry.

**Definition 10.1.** Let $K$ be any field. An $\ell$-adic Galois representation (for $K$) is a continuous homomorphism $\rho : G_K \to \text{GL}(V)$ where $V$ is a finite-dimensional vector space over $\mathbb{Q}_\ell$.

**Example 1.** There is a natural surjection $G_\mathbb{Q} \twoheadrightarrow G_\mathbb{Q}^{ab} \cong \hat{\mathbb{Z}}^\times$. Explicitly, $\mathbb{Q}^{ab} = \mathbb{Q}(\zeta_\infty)$ is the extension of $\mathbb{Q}$ with all roots of unity. The Galois group $G_\mathbb{Q}$ acts on the $m$-th roots of unity, giving a map $G_\mathbb{Q} \twoheadrightarrow (\mathbb{Z}/m\mathbb{Z})^\times$. Taking the (inverse) limit of all such maps gives $G_\mathbb{Q} \twoheadrightarrow \hat{\mathbb{Z}}^\times$. One may also concentrate at a prime $\ell$ by taking $\hat{\mathbb{Z}}^\times \twoheadrightarrow \mathbb{Z}_\ell^\times$. This amounts to looking at the action of $G_\mathbb{Q}$ on all $\ell^n$-th roots of unity. We call the Galois representation $\chi_{\text{cyc}} : G_\mathbb{Q} \to \mathbb{Z}_\ell^\times$ the ($\ell$-adic) cyclotomic character. We have a geometric interpretation of the cyclotomic character: Let $\mathbb{G}_m$ be the variety for which $\mathbb{G}_m(k) = \{(a, b) \in k^2 \mid ab = 1\}$ for any field $k$. Write $\mu_{\ell^n}$ the $\ell^n$-th roots of unity (as a $G_\mathbb{Q}$-module), one has $\lim_{\leftarrow} \mu_{\ell^n} \cong H^1_{\text{et}}(\mathbb{G}_m, \mathbb{Z}_\ell)$.

We may restrict the Galois representation to $G_{\mathbb{Q}_p}$ for any prime $p$. The resulting map $\chi_{\text{cyc}} : G_{\mathbb{Q}_p} \to \mathbb{Z}_\ell^\times$ is always still surjective. When $p \neq \ell$, it’s easy to see that the cyclotomic
character is unramified, i.e. trivial on the inertia in $G_{\mathbb{Q}_p}$. When $\ell = p$, this is no longer the case, but the geometry suggests no difference for $\ell$. This amounts to the phenomenon that when $\ell = p$, there is a notion in $p$-adic Hodge theory that corresponds to the usual unramified property, that $\chi_{cyc} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is crystalline.

Lemma 10.2. Any compact subgroup $H \subset GL_n(\overline{\mathbb{Q}}_\ell)$ is contained in $GL_n(E)$ for some $E/\mathbb{Q}_\ell$ finite.

Proof. The set $H$ is compact Hausdorff, and thus a Baire space; it is never a countable union of nowhere-dense closed subset. Let $E/\mathbb{Q}_\ell$ be any finite extension. Either the intersection $H \cap GL_n(E)$ has finite index in $H$, so that by adding finitely many elements into $E$ we have $H \subset GL_n(E)$, or $H \cap GL_n(E)$ has infinite index in $H$. We now assume the latter is the case for any $E/\mathbb{Q}_\ell$ finite. There are countably many (why?) finite extensions of $\mathbb{Q}_\ell$. Thus $H = \bigcup_E H \cap GL_n(E)$ as a countable union. However each $H \cap GL_n(E)$ cannot contain an open subgroup and thus are nowhere-dense, a contradiction.

Corollary 10.3. Any $\ell$-adic Galois representation is realized on a $E$-vector space for some $E/\mathbb{Q}_\ell$ finite.

Example 2. Another difference that (the category of) $\ell$-adic Galois representations is different from (that of) Artin representations or representations of compact groups is that an $\ell$-adic Galois representation need not be semisimple. For example, let $F$ be a non-archimedean local field with residue field $k$, char($k$) $\neq \ell$, and consider the composition $\chi : G_F \to \hat{\mathbb{Z}} \to \mathbb{Z}_\ell$ where the first map is the natural projection from $G_F$ to Gal($\overline{k}/k) \cong \hat{\mathbb{Z}$ and the second map is the natural projection from $\hat{\mathbb{Z}}$ to $\mathbb{Z}_\ell$. Consider the representation

$$\rho = \begin{pmatrix} 1 & \chi \\ 1 & 1 \end{pmatrix}, \text{ i.e. } \rho(\sigma) = \begin{pmatrix} 1 & \chi(\sigma) \\ 1 & 1 \end{pmatrix}, \forall \sigma \in G_F.$$ 

It’s obvious that $\rho$ is non-trivial, yet it fits into an exact sequence of $\ell$-adic representations

$$1 \to \text{triv} \to \rho \to \text{triv} \to 1,$$

and thus $\rho$ is not semisimple.

In general, for a $(\rho, V)$ be an $\ell$-adic Galois representation we will find a filtration $0 = V_0 \subset V_1 \subset \ldots \subset V_m = V$ of subrepresentations, so that each $V_i/V_{i-1}$ is an irreducible Galois representation. The semi-simplification of $V$ is $V^{ss} := \bigoplus_{i=1}^m V_i/V_{i-1}$.

Theorem 10.4. (Jordan-Hölder) The semi-simplification $V^{ss}$ is unique up to isomorphism.

Theorem 10.5. (Brauer-Nesbitt) Two representations $\rho$ and $\rho'$ of $G_K$ (in fact, any group) on a finite-dimensional vector space over any field have isomorphic semi-simplification if and only if $\rho(\sigma)$ and $\rho'(\sigma)$ have the same characteristic polynomial for all $\sigma \in G_K$. 

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Let \((\rho, V)\) be any \(\ell\)-adic Galois representation of \(G_K\), where by Lemma 10.2 we take \(V\) to be an \(E\)-vector space, \(E/\mathbb{Q}_\ell\) finite. Let \(\Lambda' \subset V\) be any lattice, i.e. a \(\mathcal{O}_E\)-submodule of rank equal to \(\dim_E V\). Let \(GL(\Lambda') := \{ g \in GL(V) \mid g\Lambda' \subset \Lambda'\}\). This is an open subgroup of \(GL(V)\), and thus \(H := \rho^{-1}(GL(\Lambda'))\) is open. Let \(\sigma_1, ..., \sigma_s\) be a set of representative for \(G_K/H\). Then \(\Lambda = \sum_{i=1}^s \rho(\sigma_i)\Lambda'\) is a \(\rho(G_K)\)-stable lattice.

Now let \(\Lambda\) be an arbitrary \(\rho(G_K)\)-stable lattice of \(V\). We may consider the quotient \(\tilde{\Lambda} := \Lambda/\varpi E\Lambda\). We have a natural induced action of \(G_K\) on \(\tilde{\Lambda}\).

**Corollary 10.6.** The semi-simplification of the \(G_K\)-representation \(\tilde{\Lambda}\) does not depends on the choice of \(\Lambda\).

**Proof.** The characteristic polynomial of any element in \(G_K\) on \(\tilde{\Lambda}\) comes from reducing the characteristic polynomial on \(V\) modulo \(\varpi E\), and thus independent of \(\Lambda\). The result then follows from Theorem 10.5. \(\square\)

**Example 3.** Let \(p\) be a prime with \(\ell \nmid p + 1\). We have a natural surjection \(\mathbb{Z}_{p^2}^\times \twoheadrightarrow F_\ell\). By taking any isomorphism \(F_\ell \cong \{ x \in \mathbb{Q}_\ell^\times \mid x^{\ell} = 1\}\) and \(\mathbb{Q}_p^\times \cong \mathbb{Z}_{p^2} \times \mathbb{Z}\), we have a character \(\chi' : \mathbb{Q}_p^\times \to \mathbb{Q}_\ell^\times\), which extends to a 1-dimensional \(\ell\)-adic Galois representation \(\chi : G_{\mathbb{Q}_{p^2}} \to \mathbb{Q}_\ell^\times\). Let \(\rho = \text{ind}_{G_{\mathbb{Q}_{p^2}}} G_{\mathbb{Q}_p} \chi : G_{\mathbb{Q}_p} \to GL_2(\mathbb{Q}_\ell)\). This is an irreducible representation, while \(\overline{\rho} \cong \text{triv} \oplus sgn_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}\).

We now fix some \(E/\mathbb{Q}_\ell\) finite and a non-archimedean local field \(F\) with residual characteristic \(p \neq \ell\). Let \(G_F \supset I_F \supset P_F\) be the absolute Galois group, the inertia, and the wild inertia, respectively. For an \(\ell\)-adic Galois representation \(\rho : G_F \to GL(V)\), we first look at how \(\rho|_{I_F}\) looks like. Recall that we have \(I_F/P_F \cong (\mathbb{Z}/p) := \prod_{p' \neq p} \mathbb{Z}_{p'}\). (Sketch of proof: By Hensel lemma style argument, any degree \(m\) extension of \(F^{ur}\) with \(p \nmid m\) is of the form \(F^{ur}(\sqrt[p]{\varpi_F})\) for a fixed uniformizer \(\varpi_F\).) Write \(t_\ell : I_F \to \mathbb{Z}_\ell^\times\) the composition \(I_F \twoheadrightarrow I_F/P_F \twoheadrightarrow \mathbb{Z}_\ell\).

**Definition 10.7.** Say a compact topological group \(G\) is prime to \(\ell\) if \(x \mapsto x^\ell\) is a homeomorphism.

We have \(\ker(t_\ell)\) is prime to \(\ell\). One easily proves the following (see also exercise)

**Lemma 10.8.** If \(G\) is a prime-to-\(\ell\) group and \(\rho : G \to GL_n(E)\) an \(\ell\)-adic representation, then the image of \(\rho\) is finite.

**Corollary 10.9.** (Grothendieck’s \(\ell\)-adic monodromy theorem) Let \(\rho : G_F \to GL_n(E)\) be an \(\ell\)-adic Galois representation. Then there exists a relatively open subgroup \(U \subset I_F\) and a necessarily unique \(n \times n\) nilpotent matrix \(N \in M_n(E)\) such that

\[
\rho(\tau) = \exp(t_\ell(\tau)N), \quad \forall \tau \in U.
\]
Proof. By Lemma 10.8, ker(\(\rho\)) contains an open subgroup of ker(\(t_\ell\)). By shrinking \(U\) we may assume \(\rho|_{U \cap \ker(t_\ell)}\) is trivial. We have seen that the image of \(\rho\) stabilize a lattice, and thus by conjugation we may assume Im(\(\rho\)) \(\subset GL_n(\mathcal{O}_E)\). By shrinking \(U\) again we may assume \(\rho(U) \subset K(2) := \{g \in GL_n(\mathcal{O}_E) \mid g \approx \text{Id mod } \ell^2\}\), so that it makes sense to talk about \(\log \rho(\tau)\) for \(\tau \in U\). We have seen that the image of \(\rho\) stabilize a lattice, and thus by conjugation we may assume \(\text{Im}(\rho) \subset GL_n(\mathcal{O}_E)\). By shrinking \(U\) again we may assume \(\rho(U) \subset K(2) := \{g \in GL_n(\mathcal{O}_E) \mid g \sim \text{Id mod } \ell^2\}\), so that it makes sense to talk about \(\log \rho(\tau)\) for \(\tau \in U\). We have \(t_\ell(U) \sim U/(U \cap \ker(t_\ell)) \to K(2) \xrightarrow{\log} \ell^2 M_n(\mathcal{O}_E)\). Since \(t_\ell(U) \sim \ell m \mathbb{Z}_\ell\) for some \(m\), the above composition has to be given by \(x \mapsto xN\) for some \(N \in M_n(E)\). This is the \(N\) we seek.

It remains to prove that \(N\) is nilpotent. Let \(\Phi \in G_F\) be any (lift of) arithmetic Frobenius. Then for any \(\tau \in I\), we have \(t_\ell(\Phi \tau \Phi^{-1}) = q_F \cdot t_\ell(\tau)\) where \(q_F\) is the order of the residue field of \(F\). Applying this to the definition of \(N\) we see \(\rho(\Phi)N\rho(\Phi)^{-1} = q_F \cdot N\). The eigenvalues of \(N\) thus have to be all zero and \(N\) has to be nilpotent.

**Example 4.** We have the following \(\ell\)-adic Galois representation \(\rho : G_F \to GL_2(\mathbb{Q}_\ell)\) given by

\[
\rho = \begin{pmatrix}
\chi_cyc & t_\ell \\
0 & 1
\end{pmatrix}.
\]

This is somewhat an example of Galois representation that is not semisimple, but the \(\rho(\Phi)\) is semisimple, i.e. diagonalizable over \(\bar{\mathbb{Q}}_\ell\). It is a general conjecture that Frobenius should always acts semisimply. In fact, this is probably the most important non-trivial Galois representation; it’s the Tate module of an elliptic curve that has semistable but not good reduction.

We have understood the difference between local Artin representations and \(\ell\)-adic Galois representations of \(G_F\) up to the inertia \(I_F\). To state the final result in a better way and to connect with Langlands program, we’d like to switch from the absolute Galois group \(G_F\) to the so-called Weil group. Recall that we have an exact sequence

\[
1 \longrightarrow I_F \longrightarrow G_F \longrightarrow \hat{\mathbb{Z}} \longrightarrow 1
\]

We also suppose that our local class field theory is normalized so that an arithmetic Frobenius \(\Phi\) is mapped to the inverse of a uniformizer. We then define the Weil group \(W_F\) of \(F\) to be the preimage of \(\mathbb{Z} \subset \hat{\mathbb{Z}}\), but equipped with a new topology similar to that of \(\mathbb{Z} \subset \hat{\mathbb{Z}}\). That is, the topology on \(W_F\) is such that \(I_F\) has the same subspace topology as before but is open in \(W_F\). Note also that by local class field theory we have \(W_F^{ab} \cong F^\times\).

We also denote by \(| \cdot | : W_F \to \mathbb{C}^\times\) the normalized norm, so that \(|\Phi| = q_F^{-1}\). An \(\ell\)-adic Weil representation for \(F\) is a homomorphism \(\rho : W_F \to GL(V)\). One easily sees that the result of Corollary 10.9 also holds for Weil representations. We now fix a Weil representation \((\rho, V)\) where \(V\) is an \(E\)-vector space. Let \(\Phi\) again be any (lift of) Frobenius. We have
Theorem 10.10.  (Deligne) Let \( t_\ell \) be as above and \( N \) as in Corollary 10.9. The formula
\[
\rho_\sharp(\Phi^{m\tau}) := \rho(\Phi)^m \rho(\tau) \exp(-t_\ell(\tau)N), \quad \forall m \in \mathbb{Z}, \quad \tau \in I_F
\]
defines a new representation of \( W_F \) on \( V \). Moreover, the isomorphism class of \((\rho_\sharp, V)\) does not depend on the choice of \( \Phi \).

Proof. Recall that we had \( t_\ell(\Phi \tau \Phi^{-1}) = q_F \cdot t_\ell(\tau) \Rightarrow \rho(\Phi)N \rho(\Phi)^{-1} = q_F \cdot N \). A similar argument gives \( t_\ell(\sigma \tau^{-1}) = |\sigma| \cdot t_\ell(\tau) \) and \( \rho(\sigma)N \rho(\sigma)^{-1} = |\sigma| \cdot N \) for any \( \sigma \in W_F \). Similarly we have \( \rho(\sigma) \exp(xN) \rho(\sigma)^{-1} = \exp(|\sigma|xN) \).

Now for the first statement of the theorem, suppose we have \( \Phi^{m_1\tau_1}\Phi^{m_2\tau_2} = \Phi^{m_1+m_2\tau} \), i.e. \( \tau = \Phi^{-m_2\tau_1}\Phi^{m_2\tau_2} \). Then
\[
\rho_\sharp(\Phi^{m_1+m_2\tau}) = \rho(\Phi^{m_1+m_2\tau}) \exp(-t_\ell(\tau)N)
\]
\[
= \rho(\Phi^{m_1\tau_1}) \rho(\Phi^{m_2\tau_2}) \exp(-t_\ell(\Phi^{-m_2\tau_1}\Phi^{m_2\tau_2})N) \exp(-t_\ell(\tau_2)N)
\]
\[
= \rho(\Phi^{m_1\tau_1}) \rho(\Phi^{m_2\tau_2}) \exp(-q_F^{-m_2}t_\ell(\tau_1)N) \exp(-t_\ell(\tau_2)N)
\]
\[
= \rho(\Phi^{m_1\tau_1}) \exp(-t_\ell(\tau_1)N) \rho(\Phi^{m_2\tau_2}) \exp(-t_\ell(\tau_2)N) = \rho_\sharp(\Phi^{m_1\tau_1}) \rho_\sharp(\Phi^{m_2\tau_2}).
\]

For the second statement, suppose \( \Phi' \) is a different choice of Frobenius, and \( \Phi = \Phi' \tau' \), with \( t_\ell(\tau') = s' \). Then the resulting \( \rho_\sharp' \) has \( \rho_\sharp'(\Phi) = \rho_\sharp'(\Phi' \tau') = \rho_\sharp(\Phi) \exp(-sN) \). On the other hand, we have \( \rho(\Phi) \exp(xN) \rho(\Phi)^{-1} = \exp(xN)^{q_F} \Rightarrow \rho(\Phi) = \exp(xN)^{q_F} \rho(\Phi) \exp(xN)^{-q_F} = \exp(xN)^{q_F} \rho(\Phi)(\exp(xN)^{q_F})^{-1} \). Taking \( x = \frac{s'}{q_F-1} \), we get \( \rho_\sharp(\Phi) = \rho(\Phi) \) is conjugate to \( \rho_\sharp(\Phi) \exp(-sN) = \rho_\sharp'(\Phi) \) by \( \exp(xN)^{q_F} \). Since for any \( \tau \in I \) we have \( \rho_\sharp(\tau) = \rho_\sharp(\tau') \) commutes with \( N \) and \( \exp(xN) \) by the formula, we have \( \rho_\sharp' \) is conjugate to \( \rho_\sharp \) by \( \exp(xN) \).

Remark 10.11. \( \rho_\sharp|_{I_F} \) is trivial on the open subgroup \( U \subset I_F \) in Corollary 10.9, and thus has finite image and is a semisimple representation. On the other hand, in Corollary 10.9 we have \( N = 0 \) iff \( |\rho_\sharp|_{I_F} \) has finite image, which is also equivalent to \( \rho = \rho_\sharp \).

Definition 10.12. A Weil-Deligne representation is a pair \( (\rho_\sharp, N) \) where \( \rho_\sharp : W_F \to GL_n(\mathbb{C}) \) is a continuous homomorphism, and \( N \in M_n(\mathbb{C}) \) is a nilpotent \( n \times n \) matrix in \( \mathbb{C} \) such that \( \rho_\sharp(\sigma)N\rho_\sharp(\sigma)^{-1} = |\sigma| \cdot N \) for any \( \sigma \in W_F \).

When \( N = 0 \), it’s also common to call it a Weil representation. We fix an isomorphism \( \iota : \mathbb{Q}_\ell \cong \mathbb{C} \).

Corollary 10.13. Upon the choice of \( \iota \), Theorem 10.10 gives a bijection between the set of isomorphism classes of \( \ell \)-adic Weil representations and the set of isomorphism classes of Weil-Deligne representations.
We can also talk about $L$-functions of Weil-Deligne representations.

**Definition 10.14.** Let $\rho = (\rho_2, N)$ be an $n$-dimensional Weil-Deligne representation. Let $V := \ker(N)^I_F$ be the subspace of $\ker(N)$ on which $\rho_2(I_F)$ acts trivially. Then $L(s, \rho) := \det(\text{Id}_V - q^{-s}\rho_2(\Phi)|_V)^{-1}$.

**Lemma 10.15.** Let $(\rho_2, N)$ be a Weil-Deligne representation. If $\rho_2(\Phi)$ is semisimple, then $\rho_2(\sigma)$ is semisimple for any $\sigma \in W_F$.

**Proof.** Let $U \subset I_F$ be as in Corollary 10.9. By shrinking $U$ we may assume $U$ is normalized by $W_F$. For any $\sigma = \Phi^m\tau \in W_F$, we may write $\sigma^a = \Phi^{am}\tau^a$ for any $a \in \mathbb{Z}$. The image of $\tau_a$ in $I_F/U$ has to be periodic, and thus $\tau_a \in U$ for some $a > 0$. This gives $\rho_2(\sigma)^a = \rho_2(\Phi)^am$ is semisimple. Since $\mathbb{C}$ has characteristic zero, $\rho_2(\sigma)$ is also semisimple. (If $m = 0$, then $\sigma \in I_F$ and the semisimplicity of $\rho_2(\sigma)$ is automatic as $\rho_2|_{I_F}$ has finite image.)

**Definition 10.16.** A Weil-Deligne representation is called Frobenius-semisimple if it satisfies the property in Lemma 10.15.

A general conjecture is that Weil-Deligne representations that correspond to interesting (i.e. those that come from geometry) $\ell$-adic Weil representations should be Frobenius-semisimple. What is very exciting is then that Frobenius-semisimple Weil-Deligne representations can be related to representation theory by the local Langlands conjecture.

**Remark.** Theorem 10.10, Definition 10.12 and all those follow also work for arbitrary reductive group. For example, let $V$ be any quadratic space over $\mathbb{Q}$. Then $SO(V)$ is a reductive group, and one may consider $\rho : W_F \to SO(V \otimes \overline{\mathbb{Q}})$ and $\rho_2 : W_F \to SO(V \otimes \mathbb{C})$. Then $N$ should lie in the Lie algebra of $SO(V \otimes \mathbb{C})$, namely a anti-self-adjoint nilpotent operator on $V$. 

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