5 Fourier Analysis on $\mathbb{A}_K$

We would like to prove the existence of isomorphisms $j_F : F \rightarrow \hat{F}$ for local fields $F$ and $j_K : \mathbb{A}_K \rightarrow \hat{\mathbb{A}}_K$ for global fields $K$ which restricts to $K \rightarrow \hat{\mathbb{A}}_K/K$. Before defining $j_F$ (resp. $j_K$), note that $j_F(1)$ (resp. $j_K(1)$) is by definition a unitary additive character on $F$ (resp. on $\mathbb{A}_K$ that is trivial on $K$). We begin by constructing such a character $\psi_F$ (resp. $\psi_K$).

Case $F = \mathbb{R}$: We put $\psi_{\mathbb{R}}(x) := e^{-2\pi ix}$.

Case $F = \mathbb{Q}_p$: Consider $\psi_{\mathbb{Q}}(x) = e^{2\pi ix}$. It is trivial on $\mathbb{Z}$. Now $\hat{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}$ satisfies $\hat{\mathbb{Q}} = \mathbb{Q} + \mathbb{Z}$ and $\mathbb{Q} \cap \hat{\mathbb{Z}} = \mathbb{Z}$. Hence we may extends $\psi_{\mathbb{Q}}$ to $\psi'_{\mathbb{Q}} : \hat{\mathbb{Q}} \rightarrow \mathbb{C}^\times$ by assigning $\psi'_{\mathbb{Q}}|_{\mathbb{Z}} \equiv 1$.

Case $F = \mathbb{F}_p((t))$: See exercise.

Case $K = \mathbb{Q}$: Let $\psi_{\mathbb{Q}} : \mathbb{A}_\mathbb{Q} \rightarrow \mathbb{C}^\times$ be defined as $\psi_{\mathbb{R}} \times \psi'_{\mathbb{Q}}$. From the definitions above we see that $\psi_{\mathbb{Q}|K} \equiv 1$.

Case $K$ number field: We put $\psi_K(x) := \psi_{\mathbb{Q}}(\text{Tr}_{K/\mathbb{Q}}(x))$.

Case $F$ or $K$ of positive characteristic: See Appendix.

We now define $j_F : F \rightarrow \hat{F}$ by $(j_F(x))(y) = \psi_F(xy)$. We have

**Theorem 5.1.** Let $F$ be a local field. Then $j_F : F \rightarrow \hat{F}$ is an isomorphism of topological groups.

**Proof.** (i) $j_F$ is a homomorphism: $(j_F(x_1 + x_2))(y) = \psi_F((x_1 + x_2)y) = \psi_F(x_1y) \cdot \psi_F(x_2y) = (j_F(x_1))(y) + (j_F(x_2))(y)$.

(ii) $j_F$ is injective: For every $x \neq 0$ there exists some $y$ such that $\psi(xy) \neq 1$.

(iii) $j_F$ is continuous: Let $\Delta \subset \mathbb{C}^\times$ be any open set and $V \subset F$ any compact subset, we have to prove that $j_F^{-1}\{\phi \in \hat{F} \mid \phi(V) \subset \Delta\} = \{x \in F \mid \psi(xy) \in \Delta, \forall y \in V\}$ is open. Fix any $x \in F$ such that $\psi(xy) \in \Delta$ for all $y \in F$. Since $\psi$ is continuous, for any $y \in V$ there exists $U_y \times V_y$ a neighborhood of $(x, y)$ such that $\psi(U_y \times V_y) \subset \Delta$. Since $V$ is compact, it is covered by finitely many $V_y$, and thus $\bigcap U_y$ for those $y$ is an open neighborhood of $x$ that we seek.

(iv) $j_F$ is surjective: We use the structure of the local fields. Let $\chi : F \rightarrow \mathbb{C}^\times$ be any additive unitary character. When $F = \mathbb{R}$, let $\Delta = \{z \in \mathbb{C} \mid |z - 1| < 1\}$, and $U := \chi^{-1}(\Delta)$ is open; suppose $(-a, a) \subset U$. For any $|x| < a$, we have $\chi(x) = e^{2\pi is(x)}$ for some $\phi(x) \in (-1/2, 1/2)$, and $\phi(x/2) = \phi(x)/2$. Also $\phi(x + y) = \phi(x) + \phi(y)$ whenever $x, y, x + y \in (-a, a)$. By continuity this shows $\phi(x) = sx$ is linear, and $\chi(x) = \psi(-sx)$. The $F = \mathbb{C}$ case follows as $\mathbb{C} = \mathbb{R} \times \mathbb{R}$. Lastly, when $F$ is non-archimedean, as $O_F$ is profinite we know that $\ker(\chi) \supset \varpi_F^m O_F$ for some $m$. Suppose this is the case, then one checks that it’s possible to find $s_m \in F$ such that $\chi \cdot j_F(s_m)$ is trivial on $\varpi_F^{m-1} O_F$. Likewise, one may
find $s_{m-1}$ such that $\chi \cdot j_F(s_m + s_{m-1})$ is trivial on $\varpi_F^{m-2}O_F$. One verifies that $s_m + s_{m-1} + ...$ converges in $F$, and $\chi = j_F(-s_m - s_{m-1} - ...)$. 

(v) $j_F^{-1} : j_F(F) \to F$ is continuous: It suffices to prove the continuity at the identity. Recall that we have a norm $|·|$ on $F$ such that $|xy| = |x||y|$, and the topology on $F$ is defined by $|·|$. As a homomorphism of topological groups it suffices to prove that $j_F^{-1}$ is continuous at the identity. That $F$ is locally compact implies $V = \{y \in F \mid |y| \leq 1\}$ is compact. If $|x| \to 0$, then $|xy| \to 0$ for any $y \in V$ and thus $\psi_F(xy) \to 1$. This proves the asserted continuity. 

For the global case, we likewise define $j_K : \mathbb{A}_K \to \hat{\mathbb{A}}_K$ by $(j_K(x))(y) = \psi_K(xy)$. Likewise we have

**Theorem 5.2.** Let $K$ be a local field. Then $j_K : \mathbb{A}_K \to \hat{\mathbb{A}}_K$ is an isomorphism of topological groups.

**Proof.** The same proof in (i),(ii),(iii) and (iv) above shows that $j_K$ is a continuous injective homomorphism with dense image. To prove the rest we need a lemma

**Lemma 5.3.** For almost all places $v$, $\psi_K|_{\mathcal{O}_K}$ is trivial on $\mathcal{O}_K$ but non-trivial on $\varpi_v^{-1}\mathcal{O}_K$.

**Proof of Lemma 5.3.** That $\psi_K$ is trivial on almost all $\mathcal{O}_K$ comes from the continuity of $K$. When $K$ is a number field, we have $\psi_K|_{\mathcal{O}_K}$ is non-trivial on $\varpi_v^{-1}\mathcal{O}_K$ for those finite places $v$ for which $K/Q$ is unramified. (This is because if $E/F$ is a finite unramified extension, then $\text{Tr}_{E/F}(\mathcal{O}_E) = \mathcal{O}_F$.) When $K$ is a global function field, this is because any non-trivial 1-form $\omega \in \Omega_1^{1, K/k}$ has finitely zeros and poles; see Appendix. 

Now we prove that $j_K$ is surjective: any additive unitary character $\phi : \mathbb{A}_K \to \mathbb{C}^\times$ restricts to $\phi_v : K_v \to \mathbb{C}^\times$. By construction $\phi_v \neq 0$. Thus by our results for local fields there exists $x_v \in K_v$ such that $\phi_v = j_K(x_v)|_{K_v}$. Now by continuity, $\phi_v|_{\mathcal{O}_K}$ is trivial for almost all $v$. This implies, thanks to Lemma 5.3, that $x_v \in \mathcal{O}_K$ for almost all $v$. We then have $x = (x_v) \in \mathbb{A}_K$ and $\phi = j_K(x)$.

Lastly, to show that $j_K^{-1}$ is continuous, let $S$ be the finite set of places consisting of all archimedean places as well as non-archimedean places such that $\psi_K|_{\mathcal{O}_K}$ is non-trivial on $\mathcal{O}_K$ (thanks to Lemma 5.3 again). For any basic neighborhood $U = \prod U_v \subset A_K \neq 0$, by enlarging $S$ we may assume $U_v = \mathcal{O}_K$ for all $v \not\in S$. By the continuity of $j_K^{-1}$, that we proved earlier, for every $v \in S$ there exists compact $C_v$ and open $I$ such that $\psi_K|x_vy_v \in I$ for every $v \in S$, $x_v \in U_v$ and $y_v \in C_v$. Let $C = \prod_{v \in S} C_v \times \prod_{v \not\in S} \mathcal{O}_K$, then $\psi_K(xy) \in I$ for any $x \in U$, $y \in C$, and thus $j_K(U) \supset \{\phi \in \hat{A}_K \mid \phi(C) \subset I\}$. This shows the required continuity. 

Since $\psi_K$ is trivial on $K$, we see from definition that restricting $j_K$ to (the diagonally embedded) $K$ gives $j_K|_K : K \to \mathbb{A}_K/K$. We have
Theorem 5.4. $j_K|_K$ is an isomorphisms from $K$ to $\hat{A}_K/K$.

Proof. Note that it suffices to prove it algebraically, since $\hat{A}_K/K$ is compact $\Rightarrow \hat{A}_K/K$ is discrete, and also $K$ is discrete. Now $j_K^{-1}(\hat{A}_K/K)$ is a discrete subgroup of $\hat{A}_K$ (note $j_K$ is a homeomorphism). But $K$ is already a cocompact discrete subgroup of $\hat{A}_K$. Thus $[j_K^{-1}(\hat{A}_K/K) : K] = [\hat{A}_K/K : j_K(K)]$ is finite. On the other hand, $j_K$ induces a $K$-vector space structure on $\hat{A}_K$, and it's easy to check both $\hat{A}_K/K$ and $j_K(K)$ are $K$-subspaces. Thus $[\hat{A}_K/K : j_K(K)] = 1$ and $j_K(K) = \hat{A}_K/K$.

Now we apply Poisson summation formula to the LCA groups we have worked on! Let's set up some language.

Definition 5.5. Suppose $j_B : B \sim \hat{B}$ is a topological isomorphism between an LCA group $B$ and its dual. We say a Haar measure $\mu$ on $B$ is self-dual (with respect to $j_B$) if the dual measure $\hat{\mu} = (j_B)_*(\mu)$.

Since $\widehat{c \mu}$ is $c^{-1}\hat{\mu}$, self-dual measure always exist. We have

Proposition 5.6. If we have a short exact sequence $0 \to A \to B \to C \to 0$ of LCA groups with $j_B : B \sim \hat{B}$ which restricts to $j_A : A \sim \hat{C}$, and a self-dual measure $\mu_B$ for $B$. Let $\mu_A$ be any Haar measure for $A$. Then we have for any nice function $f$ on $B$

$$\int_A f(a)d\mu_A = \int_C \hat{f}(a)d(j_A)_*\mu_A.$$  \hspace{1cm} (1)

In particular, if $A$ is discrete and $\mu_A$ is the counting measure, then $\mu_C(C) = 1$.

Proof. Equation (1) is really the Poisson summation formula, and the first assertion is that the measure $\hat{\mu}_C$ on $\hat{C}$ is given by $(j_A)_*\mu_A$. We know that the two sides of (1) differ by a constant $c \in \mathbb{R}_+\setminus\{1\}$ for any $f$. By replacing $f$ with $\hat{f}$ (or rather $j_B^{-1}(\hat{f})$), one gets the two sides of (1) differ by $c^{-1}$, thus $c^2 = 1 \Rightarrow c = 1$. To show the last assertion, suppose $\hat{\mu}_C = (j_A)_*\mu_A$ is the counting measure. Let $h$ be the function on $\hat{C}$ that takes value 1 at the identity $\phi_0$ and zero elsewhere. Fourier inversion gives $h(\phi_0) = \int_C \phi_0^{-1}(c)\hat{h}(c)d\mu_C = \int_C 1 d\mu_C = \mu_C(C)$ and thus $\mu_C(C) = 1$.

Now let $K$ be a global field, $j_K : \mathbb{A}_K \to \hat{\mathbb{A}}_K$ as constructed above and $\mu$ a self-dual measure on $\mathbb{A}_K$. We equip $\mathbb{A}_K/K$ with the quotient measure $\bar{\mu}$ given by $\mu$ and the counting measure on $K$. The proposition above says $\mathbb{A}_K/K$ has measure 1. We say $\bar{\mu}$ is a Tamagawa measure. The usual norm $|\cdot|$ on $\mathbb{A}_K$ has the property that for any $x \in \mathbb{A}_K$. If we denote by $t_x = (y \mapsto xy)$ the automorphism on $\mathbb{A}_K/K$, then $(t_x)_*(\bar{\mu}) = |x|\bar{\mu}$ (see exercise).

We now notationally identify $\hat{\mathbb{A}}_K$ with $\mathbb{A}_K$. For every archimedean place $v$ of $K$, let $S(K_v)$ be the space of Schwartz functions on $K_v$, i.e. smooth functions with all derivatives
having exponential decay. On the other hand, for \( v \) non-archimedean, let \( S(K_v) = C^\infty_c(K_v) \) be the space of compactly supported functions on \( K_v \) that are locally constant, i.e. invariant under translation by some open subgroup. Now let \( S(\mathbb{A}_K) \) be the linear space of functions generated by those of the form \( \bigotimes_v f_v \) with \( f_v \in S(K_v) \) and \( f_v = 1_{\mathcal{O}_{K_v}} \) for almost all \( v \).

One checks that Fourier transform preserves \( S(\mathbb{A}_K) \); the main point is that for almost all \( v \), \( \hat{1}_{\mathcal{O}_{K_v}} = 1_{\mathcal{O}_{K_v}} \). This functions are nice in that they are limits of compactly supported continuous functions for our analytical need. In particular (5.6) holds for \( f \in S(\mathbb{A}_K) \) with \( A = K \) (equipped with the counting measure), \( C = \mathbb{A}_K/K \) and \( \hat{C} = \mathbb{A}_K/K = K \), giving

\[
\sum_{x \in K} f(x) = \sum_{x \in K} \hat{f}(x).
\]

For any \( a \in \mathbb{A}_K^\times \), if we put \( f_a(x) = f(ax) \), then one easily checks \( \hat{f_a}(x) = |a|^{-1} \hat{f}(a^{-1}x) \). Plugging \( f_a \) into the above equation gives

\[
\sum_{x \in K} f(ax) = \frac{1}{|a|} \sum_{x \in K} \hat{f}(a^{-1}x). \tag{2}
\]

### 5.1 Appendix

Here we develop the theory differential 1-forms and their residues on \( \text{Spec}(F) \) or \( \text{Spec}(K) \), the algebraic geometry objects underlying \( F \) or \( K \) when they are local and global function fields. Such 1-forms will turn out to be equivalent to additive characters.

Let \( L \supset k \) be any extension of fields. The space of relative differentials 1-forms \( \Omega^1_{L/k} \) is the \( L \)-vector space generated by \( df \) with \( f \in L \), with the relation \( df = 0 \) whenever \( f \in k \), and \( d(f_1f_2) = f_1df_2 + f_2df_1 \) for any \( f_1, f_2 \in L \).

Now suppose \( F \) and \( K \) are local and global function fields containing a finite field \( k \) as its constant field. One may check\(^{1}\) \( \dim_F \Omega^1_{F/k} = 1 \) (resp. \( \dim_K \Omega^1_{K/k} = 1 \)). In the local case, we may identify \( F \cong k((t)) \). For any 1-form \( \omega \in \Omega^1_{F/k} \) we may write \( \omega = \sum_i a_i t^i dt \) with \( a_i \in k, a_i = 0 \) for \( i \ll 0 \). We then define the residue of \( \omega \) as \( \text{res} \omega := a_{-1} \in k \). This is a similar to the theory of residue for Riemann surfaces, where residues should be defined (coordinate-freely) for 1-forms instead of functions.

It is then necessarily to check that residue does not depends on the choice of the identifcation \( F \cong k((t)) \) (which is made by the single choice of uniformizer \( t \in F \)). I however cannot find any smart non-cohomological proof; for example, in Serre’s *Algebraic groups and class fields* (see pp. 19-21; the book can be downloaded with MIT IP) he proves it with two

\(^{1}\)I made the remark in class that \( \dim_{L/k} \Omega^1_{L/k} \) has dimension equal to the transcendental degree. This is only true when \( L/k \) is finitely generated and separable, and thus applies only to \( K \) but not \( F \).
pages of elementary calculation, using the property that \(\text{res}(df/f) = \text{val}(f)\) is independent of the choice. We assume such calculation has been done.

We now fix a non-trivial additive character \(\psi : k \to \mathbb{C}^\times\). For any \(\omega \in \Omega^1_{F/k}\), we define \(\psi_F\) on \(F\) by \(\psi_F,\omega(f) = \text{res}(f\omega)\). When \(\omega\) is non-zero, \(\psi_F,\omega\) is non-trivial. It’s obvious that \(\psi_F,\omega\) is an additive unitary (continuous) character. We would like to explain why this point of view is natural. Let \(E/F\) be a finite separable\(^2\) extension, and write \(k_E/k_F\) their respective constant field. Denote by \(\iota_{E/F} : \Omega^1_{F/k} \to \Omega^1_{E/k}\) the natural map. Then we have the following equation

\[
\text{Tr}_{k_E/k_F}(\text{res}(f \cdot \iota_{E/F}(\omega))) = \text{res}(\text{Tr}_{E/F}(f) \cdot \omega), \forall f \in E, \omega \in \Omega^1_{F/k}.
\]

It’s easy to verify (3) when \(E/F\) is unramified, so that one may identify \(E = k_E((t)) \supset F = k_F((t))\), and \(\text{Tr}_{E/F}\) pretty much just comes from \(\text{Tr}_{k_E/k_F}\). It’s also easy to verify (3) when \(E/F\) is totally ramified of the form \(k_F((t^{1/m}))/k_F((t))\). In general, when \(E/F\) is separable but tamely ramified, things can get pretty tricky; see Serre pp. 23-25.

Now the point is that (3) is equivalent to saying \(\psi_F,\omega \circ \text{Tr}_{E/F} = \psi_{E, \iota_{E/F}}(\omega)\). Thus it seems appropriate to identify \(\Omega^1_{F/k}\) as \(\hat{F}\) by sending \(\omega\) to \(\psi_F,\omega\) (that this is an algebraic isomorphism is proved in Theorem 5.1).

Next, we look at the global case. For any place \(v\) of \(K\), we denote by \(k_v\) the constant field of \(\mathbb{F}_q\) in \(K_v\) (i.e. the algebraic closure of \(\mathbb{F}_q\) in \(K_v\)). There is a natural map \(\iota_v : \Omega^1_{K_0/k_v} \to \Omega^1_{K/k_v}\). For any \(\omega \in \Omega^1_{K/\mathbb{F}_q}\), we define \(\text{res}_v(\omega) = \text{Tr}_{k_v/\mathbb{F}_q}(\text{res}(\iota_v(\omega)))\). Now \(K\) should be thought of as the field of meromorphic functions on a projective smooth curve over \(\mathbb{F}_q\) - an \(\mathbb{F}_q\)-analogue of a compact Riemann surface, and our experiences about the latter will suggest

\[
\sum_v \text{res}_v(\omega) = 0. \tag{4}
\]

**Proof of equation (4).** First we reduce to the case of \(K = k(t)\). We know that \(K\) is always a finite separable extension of some \(K_0 \cong k(t)\). We wish to reduce (4) from \(K\) to \(K_0\). Denote by \(\iota : \Omega^1_{K_0/k} \to \Omega^1_{K/k}\) the natural map. For any \(f \in K, \omega \in \Omega^1_{K_0/k}\), and a place \(w\) of \(K_0\) we have

\[
\sum_v \text{res}_v(f \cdot \iota(\omega)) = \text{res}_w(\text{Tr}_{K_0/K}(f) \cdot \omega) \tag{5}
\]

where \(v\) runs over places of \(K\) above \(w\). This is essentially (3). Summing up (5) for all \(w\) gives

\[
\sum_v \text{res}_v(f \cdot \iota(\omega)) = \sum_w \text{res}_w(\text{Tr}_{K_0/K}(f) \cdot \omega) \tag{6}
\]

\(^2\)Equation (3) is also true when \(E/F\) is not separable, in which case both sides are identically zero.
where \( v \) runs over places of \( K \) and \( w \) runs over places of \( K_0 \). Observe that (5) implies \( \iota : \Omega_{K_0/k}^1 \to \Omega_{K/k}^1 \) is non-trivial, as one may choose \( \omega \in \Omega_{K_0/k}^1 \) so that \( \text{res}_w(\omega) \neq 0 \) and \( f \in K \) such that \( \text{Tr}_{K/K_0}(f) \neq 0 \). Equation (5) then implies \( \text{res}_v(f \cdot \iota(\omega)) \neq 0 \). As \( \Omega_{K/k}^1 \) is 1-dimensional, every 1-form in \( \Omega_{K/k}^1 \) can be written as \( f \cdot \iota(\omega) \), and thus (6) reduces the proof of (4) to \( K_0 \).

Now \( K = K_0 = k(t) \). The set of places are indexed by monic irreducible polynomials in \( p(t) \in k[t] \) together with \( \infty \). Any 1-form \( \omega \in \Omega_{K/k}^1 \) may be written as

\[
\omega = f(t)dt + \sum_i \frac{c_i}{p_i(t)^{k_i}}dt
\]

for some \( c_i \in \mathbb{F}_q, k_i \in \mathbb{Z}_{>0}, p(t) \in \mathbb{F}_q[t] \) monic irreducible, and \( f(t) \in \mathbb{F}_q[t] \) arbitrary. It suffices to check the identity (4) for each term. Firstly, \( f(t)dt \) has no pole and thus no residue at all finite places, and if we write \( f(t) = \sum_{i \geq 0} a_i t^i \), then \( f(t)dt = \sum_{i \leq -2} -a_{-i-2}s^i ds \) where \( s = 1/t \) has at least double poles but still no residue.

For the terms \( \frac{1}{p(t)^k} \), write \( v_p \) the place of \( K \) associated to the prime ideal \( (p(t)) \). We may, by using (3) with the case \( E/F \) is unramified, assume \( k \) is replaced by a finite extension so that \( p(t) \) is linear. Then \( \text{res}_{v_p} \frac{dt}{p(t)^k} = 1 \) if \( k = 1 \) and 0 if \( k > 1 \). Also \( \text{res}_v \frac{dt}{p(t)^k} = 0 \) for all other finite \( v \neq v_p \) as the 1-form has no pole there. Lastly

\[
\text{res}_\infty \frac{dt}{p(t)^k} = \text{res}_\infty \frac{(p(t)^{-1})^{k-2}}{d(p(t)^{-1})}
\]

has a simple pole of residue \(-1\) when \( k = 1 \) and higher order pole of zero residue when \( k > 1 \), thus the sum of residue is zero and this completes the proof.

Recall that we fixed an additive character \( \psi : k \to \mathbb{C}^\times \). For any \( \omega \in \Omega_{K/k}^1 \), we define \( \psi_{K,\omega} \) on \( \mathbb{A}_K \) by putting \( \psi_{K,\omega}(x) = \sum_v \text{res}_v(x_v \omega) \) for any \( x = (x_v) \in K_v \subset \mathbb{A}_K \). When \( x \in K \), \( x \omega \in \Omega_{K/k}^1 \) and it follows from equation (4) that \( \psi_{K,\omega} \) is trivial on \( K \). Moreover, suppose \( L/K \) is a finite separable extension, and we write \( \iota_{L/K} : \Omega_{K/k}^1 \to \Omega_{L/k}^1 \) the natural map, then by the same reason as in (3) and (6) we have \( \psi_{L,\iota_{L/K}(\omega)} = \psi_{K,\omega} \circ \text{Tr}_{L/K} \). This explains that the map \( \omega \mapsto \psi_{K,\omega} \) is a natural map from \( \Omega_{K/k}^1 \) to \( \mathbb{A}_K/K \).

6