1 Topological groups

Recall that a topological group $G$ is a group $G$ endowed with a topology so that the group operations $m : G \times G \to G$ and (inverse) $i : G \times G$ are continuous. Suppose one has an inverse system of topological groups $\{G_i\}_{i \in I}$ indexed by poset $I$, i.e. with continuous homomorphisms $\phi_{ji} : G_j \to G_i$ for $i < j$ such that $\phi_{ki} = \phi_{ji} \circ \phi_{kj}$ whenever $i < j < k$. One may form the inverse limit

$$\lim_{i \in I} G_i = \{ (a_i) \in \prod_{i \in I} G_i | \phi_{ji}(a_j) = a_i \text{ for } i < j \}$$

and equip it with the topology given by the subspace topology of the product topology. In other words, a basis of open sets is given by $\{(a_i)_{i \in I} \in \lim G_i | a_j \in U_j \}$ for any $j \in I$, $U_j$ open in $G_j$. One checks that $\lim G_i$ is closed in $\prod_{i \in I} G_i$. Hence if all $A_i$ are compact (resp. totally disconnected), then $\lim G_i$ is compact (resp. totally disconnected) as well. This is particularly the case if all $G_i$ are finite and discrete.

Let $G$ be a topological group, if $N' \subset N \subset G$ are both finite index open normal subgroups of $G$, then we have a natural projection of discrete quotients $G/N' \twoheadrightarrow G/N$. All such $G/N$ form an inverse system of (discrete) topological groups, and the profinite completion of $G$ is

$$\hat{G} = \lim_{N < G \text{ open with } [G:U] < \infty} G/N$$

and is again compact and Hausdorff. Projections $G \twoheadrightarrow G/N$ induce a natural map $\phi : G \to \hat{G}$.

Lemma 1.1. There is a natural bijection between finite index open subgroups $U \leq G$ and (finite index) open subgroups $\hat{U} \leq \hat{G}$.

Proof. The direction $U \mapsto \hat{U}$ is given by $U \mapsto \text{cl}(\phi(U))$, and $\hat{U} \mapsto U$ by $\hat{U} \mapsto \phi^{-1}(\hat{U})$. Begin with $U \subset G$ open of finite index. Let $U_1, \ldots, U_n$ be all conjugate of $U$; $n < \infty$ as $[G : U] < \infty$. Then $N := \cap U_i$ is normal and open with finite index. Let $\hat{\pi}_N : \hat{G} \twoheadrightarrow G/N$ be the natural projection. We have $\phi(U) \subset \hat{\pi}_N^{-1}(U/N)$. The latter subgroup is open, thus closed, and so $\text{cl}(\phi(U)) \subset \hat{\pi}_N^{-1}(U/N)$. We have $\phi^{-1}(\hat{\pi}_N^{-1}(U/N)) = U \Rightarrow \phi^{-1}(\text{cl}(\phi(U))) \subset U$. The other inclusion is obvious.

For the other direction, begin with $\hat{U} \subset \hat{G}$ an open subgroup (necessarily finite index as $\hat{G}$ is compact). It contains a basic open set, which is of the form $\hat{\pi}_N^{-1}\{e\}$ as above. We then have $\hat{U} = \hat{\pi}_N^{-1}(H)$ for some $H \subset G/N$. Let $\pi_N = \phi \circ \hat{\pi}_N : G \twoheadrightarrow G/N$ be the other natural projection, so that $\phi^{-1}(\hat{U}) = \phi^{-1}\hat{\pi}_N^{-1}(H) = \pi_N^{-1}(H)$. Put $U = \pi_N^{-1}(H)$. We claim that $\phi(U)$ is dense in $\hat{U}$. If not, we have an open subset of $\hat{U}$ disjoint from $\phi(U)$. By replacing this open subset by a possibly smaller one, we may assume it is of the form $\hat{\pi}_N^{-1}\{\bar{g}\}$ for some $N' \subset G$ open with finite index that is also contained in $N$, and $\bar{g} \in G/N'$. We have
\[ \hat{\pi}_{N'}^{-1}\{\bar{g}\} \subset \hat{U} \Rightarrow gN' \subset U. \] But this means we can choose a lift \( g \in G \) of \( gN' \) so that \( \phi(U) \supset \hat{\pi}_{N'}^{-1}\{\bar{g}\} \), a contradiction. \[ \square \]

In particular, we see that \( \text{clos}(\phi(G)) = \hat{G} \), i.e \( \phi \) has dense image. One also note that this bijection preserves the index.

## 2 Review of class field theory

Let \( K \) be a field. Fix a separable closure \( K^{\text{sep}} \) of \( K \). Recall that a maximal abelian extension of \( K \) (in \( K^{\text{sep}} \)) is

\[
K^{\text{ab}} := \bigcup_{L \subset K^{\text{sep}}, L/K \text{ abelian}} L
\]

The Galois group \( G_K^{\text{ab}} := \text{Gal}(K^{\text{ab}}/K) \) is defined to the inverse limit topological group

\[
\text{Gal}(K^{\text{ab}}/K) = \lim_{\text{L/K finite abelian}} \text{Gal}(L/K)
\]

The topological group \( G_K := \text{Gal}(K^{\text{sep}}/K) \) is defined similarly, and the topology on \( G_K^{\text{ab}} \) agrees with the quotient topology it receives from \( G_K \).

Now the local reciprocity map is

**Theorem 2.1.** For any local field \( F \) there exists a canonical continuous injective\(^1\) homomorphism

\[ \theta_F : F^{\times} \rightarrow G_F^{\text{ab}} \]

such that

(i) When \( F \) is non-archimedean, the restriction of \( \theta_F \) to \( \mathcal{O}_F^{\times} \) induces an isomorphism between \( \mathcal{O}_F^{\times} \) and \( I_F^{\text{ab}} \), the inertia in \( G_F^{\text{ab}} \). In fact, \( \theta_F \) identifies \( G_F^{\text{ab}} \) as the profinite completion of \( F^{\times} \) (as a topological group!) by sending a uniformizer in \( F^{\times}/\mathcal{O}_F^{\times} \) to \( \text{Frob} \in G_F^{\text{ab}}/I_F^{\text{ab}} \).

(ii) The map \( \theta_{\mathbb{R}} \) is the only possible non-trivial map with kernel \( \mathbb{R}_{>0} \). The map \( \theta_{\mathbb{C}} \) is the necessarily trivial map.

(iii) For any finite separable extension \( E/F \), we have the following commutative diagram:

\[
\begin{array}{ccc}
E^{\times} & \xrightarrow{\theta_E} & G_E^{\text{ab}} \\
\downarrow{N_{E/F}} & & \downarrow{\pi_E^{\text{ab}}} \\
F^{\times} & \xrightarrow{\theta_F} & G_F^{\text{ab}}
\end{array}
\]

\(^1\)Warning: the topology on \( F^{\times} \) will not be the subspace topology from \( G_F^{\text{ab}} \).
Corollary 2.3. The following three classes of objects are in canonical bijections:

\[ \theta \text{ to that} \]

\[ (\text{topological groups}) \]

The map \( \theta \) comes from the product of \( G_{\text{ab}} \). Now the global reciprocity map comes from the product of \( \theta_{K_v} \):

**Theorem 2.2.** The map

\[ \prod_{v} \theta_{K_v} : \mathbb{A}^\times_K \to G_{\text{ab}}^\times \]

is a continuous map such that

(i) \( \prod_{v} \theta_{K_v} \) is trivial on \( K^\times \). We denote the induces map on \( \mathbb{A}^\times_K/K^\times \) by \( \theta_K \).

(ii) When \( K \) is a number field, the kernel of \( \theta_K \) is the smaller possible, namely the connected component \( U \) of the identity in \( \mathbb{A}^\times_K/K^\times \), and \( \theta_K \) induces an isomorphism of topological groups from \( U \setminus \mathbb{A}^\times_K/K^\times \cong G_{\text{ab}}^\times \).

(iii) When \( K \) is a global function field, \( \theta_K \) is injective and has dense image. Write \( (\mathbb{A}^\times_K)^1 = \{(x_v) \in \mathbb{A}^\times_K \mid \prod_{v} |x_v| = 1\} \subset \mathbb{A}^\times_K \). Write \( k \) the constant field of \( K \) and \( \mathcal{T}_K^\text{ab} \) the kernel of the projection \( G_{\text{ab}}^\times \to G_k := \text{Gal}(\bar{k}/k) \). Then \( \theta_K \) induces an isomorphism of topological groups \( (\mathbb{A}^\times_K)^1/K^\times \cong \mathcal{T}_K^\text{ab} \).

In fact, by investigating the structure of \( \mathbb{A}^\times_K/K^\times \), one checks (ii) and (iii) are equivalent to that \( \theta_K \) identifies \( G_{\text{ab}}^\times \) as the profinite completion of \( \mathbb{A}^\times_K/K^\times \).

**Corollary 2.3.** The following three classes of objects are in canonical bijections:

(i) Finite abelian extensions of \( K \).

(ii) Finite index open\(^3\) subgroups of \( G_{\text{ab}} \).

\(^{2}\)To well-define this map, check that \( \prod_{v} \theta_{K_v} \) induces a map from \( \mathbb{A}^\times_K \) to \( \text{Gal}(L/K) \) for any \( L/K \) finite abelian by using Theorem 2.1(i) and that \( L/K \) is unramified everywhere.

\(^{3}\)We will be isomorphic to a product of \( \mathbb{R}^\times_{>0} \) and \( \mathbb{C}^\times \).

\(^{4}\)This is the algebraic closure of \( F_p \) in \( K \).

\(^{5}\)The adjective “open” here is necessarily; see exercise.
(iii) Finite index open subgroup of $\mathbb{A}_K^\times/K^\times$.

Proof. (i)$\iff$(ii) is standard infinite Galois theory. (ii)$\iff$(iii) is given by Theorem 2.2(ii),(iii).

Here to use (ii), note that in the case of a number field, every finite index subgroup of $\mathbb{A}_K^\times$ has to contain the connected component of the identity.

We would nevertheless like to describe (i)$\iff$(iii). Fix a finite abelian extension $L/K$, the corresponding finite index open subgroup of $G_{K_{L_w}}^{ab}$ is the image of $\pi_{L/K} : G_{L}^{ab} \to G_{K_{L_w}}^{ab}$. As $\theta_L$ has dense image, it is equal to the closure of

$$\pi_{L/K}\theta_L(\mathbb{A}_L^\times/L^\times) = \pi_{L/K}(\prod_w \theta_{L_w})(\mathbb{A}_L^\times) = \pi_{L/K}(\prod_w (\theta_{L_w}L_w^\times)) = \prod_w (\pi_{L_w/K_v}\theta_{L_w}L_w^\times)$$

where $v$ is the place of $K$ below $w$ and $\pi_{L_w/K_v}$ is the natural map $G_{L_{w}}^{ab} \to G_{K_{w}}^{ab}$. But by Theorem 2.1(iii), $\pi_{L_w/K_v}\theta_{L_w}L_w^\times = \theta_{K_v}N_{L_w/K_v}L_w^\times$. Hence the finite index subgroup of $G_{K_w}^{ab}$ corresponding to $L$ is the closure of

$$\prod_w \theta_{K_v}N_{L_w/K_v}L_w^\times = (\prod_v \theta_{K_v})(\prod_w N_{L_w/K_v}L_w^\times) = (\prod_v \theta_{K_v})N_{L/K}\mathbb{A}_L^\times = \theta_KN_{L/K}(\mathbb{A}_L^\times/L^\times).$$

Now notice that almost all $\nu$ are unramified for $L/K$ and for such $\nu$ we have $N_{L_{w}/K_v}L_w^\times \supset \mathcal{O}_{K_v}^\times$ as $\theta_{K_v}(\mathcal{O}_{K_v}^\times)^\times = I_{K_v}^{ab}$ by Theorem 2.1(i). This implies $N_{L/K}(\mathbb{A}_L^\times/L^\times)$ is open, we also know the a product $\prod_v U_v$ of open subgroups from each place $v$, with $U_v \cong \mathcal{O}_{K_v}^\times$ for almost all $v$, has finite index image in $\mathbb{A}_K^\times/K^\times$ (compactness of $(\mathbb{A}_K^\times)^1/K^\times$). Consequently it is the finite index open subgroup of $\mathbb{A}_K^\times/K^\times$ that corresponds to $L$. \qed

Now a generalization of a Dirichlet character for $\mathbb{Q}$ is that

**Definition 2.4.** A Dirichlet character for $K$ is a finite order character on $\mathbb{A}_K^\times/K^\times$, i.e. a continuous homomorphism $\chi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ with finite image.

By Theorem 2.2, a Dirichlet character is exactly a character of $G_K$. A Hecke character is nevertheless more than that:

**Definition 2.5.** A Hecke character (Größencharakter) is a character of $\mathbb{A}_K^\times/K^\times$.

We say a character is unitary if it has image in the circle group $S^1 \subset \mathbb{C}^\times$. Since any compact subgroup of $\mathbb{C}^\times$ is contained in $S^1$ (it must have trivial image in $\mathbb{R}_+ \cong \mathbb{C}^\times/S^1$), any character of a compact group is unitary. Let $\chi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ be a Hecke character. Recall that we have the valuation map $|\cdot|_K : \mathbb{A}_K^\times/K^\times \to \mathbb{R}_+$ given by the product of all normalized valuation; the product formula ensures that it is trivial on $K^\times$. We know that the kernel $\ker |\cdot|_K = (\mathbb{A}_K^\times)^1/K^\times$ is compact, and thus $\chi|_{(\mathbb{A}_K^\times)^1/K^\times}$ is unitary. One then easily sees that any character $\chi$ is of the form $\chi_0 \cdot |s|_K$ for some $\chi_0$ unitary and $s \in \mathbb{C}$, and $\Re(s)$ depends only on $\chi$. We call $\Re(s)$ the exponent of $\chi$.\[4\]
Since the natural embedding $\mathbb{R}_+ \times \hat{\mathbb{Z}}^\times \hookrightarrow \mathbb{A}_Q^\times$ induces an isomorphism $\mathbb{R}_+ \times \hat{\mathbb{Z}}^\times \cong \mathbb{A}_Q^\times / \mathbb{Q}^\times$, a Hecke character for $\mathbb{Q}$ is really a product of a character on $\mathbb{R}_+$ and one on $\hat{\mathbb{Z}}^\times$. Every character on $\hat{\mathbb{Z}}^\times$ has finite order and every character on $\mathbb{R}_+$ of is of the form $x \mapsto |x|^s$. This implies that $\chi \cdot |\cdot|_{\mathbb{Q}}^s$ is a character of finite order; in other words every Hecke character for $\mathbb{Q}$ arises from a Dirichlet character by a twist $K$. Nevertheless, for any number field $K$ that is larger than $\mathbb{Q}$, there exists some Hecke character that is not a twist of a Dirichlet character. We give two examples:

Example 2.6. (i) $K = \mathbb{Q}(i)$. Let $\chi' : \mathbb{A}_K^\times \to \mathbb{C}^\times$ be defined by $\chi'|_{\mathbb{C}^\times} = (z \mapsto z^4)$, and $\chi'|_{K_p^\times} = (x \mapsto z^{-4 \text{val}(x)})$ whenever $p = (z)$ is a prime ideal. One checks that $\chi'$ is trivial on $K^\times$ so that it gives a Hecke character on $\mathbb{A}_K^\times / K^\times$. It is not a twist of a Dirichlet character since $\chi'|_{\mathbb{C}^\times}(S^1)$ has infinite image.

(ii) $K = \mathbb{Q}(\sqrt{2})$. Denote by $\infty$ and $\infty'$ its two real places and $\iota_{\infty} : K \to \mathbb{R}$ the embedding corresponding to the first place. The unit group $\mathcal{O}_K^\times$ is generated by $-1$ and $\sqrt{2} + 1$. Let $\chi' : \mathbb{A}_K^\times \to \mathbb{C}^\times$ be defined by $\chi'|_{\mathbb{C}^\times} = (x \mapsto x^{\frac{-2 \sqrt{2}}{\log(\sqrt{2} + 1)}})$, $\chi'|_{\mathbb{C}^\times} \equiv 1$, and define $\chi'|_{K_p^\times} = (x \mapsto |\iota_{\infty}(z)|^{\frac{-2 \sqrt{2}}{\log(\sqrt{2} + 1)}})$ whenever $p = (z)$ similar to above. Then we get a character $\chi$ as above such that $\chi'|_{K_{\infty}^\times / K^\times}$ has infinite image, and thus is not a twist of Dirichlet characters.

3 A backward interpretation

Recall that the functional equation of the zeta function goes as follows: let $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then $\Lambda(s)$ has a meromorphic continuation to the whole complex plane, and $\Lambda(s) = \Lambda(1-s)$.

How do we prove it? One begins with $g(x) = e^{-\pi x^2}$, for which we have its Fourier transform $\hat{g}(x) = g(x)$. We apply the Poisson summation formula as

$$\sum_{n \in \mathbb{Z}} g(an) = \frac{1}{a} \sum_{n \in \mathbb{Z}} \hat{g}(n/a), \ a \in \mathbb{R}^\times$$

The first equality applies to a general class of functions and its Fourier transform, and so far $a$ is just an innocent scaling. But then for $\text{Re}(s) > 1$ one looks at the integral

$$\int_0^\infty a^{s-1} \left( \sum_{n \in \mathbb{Z}\setminus\{0\}} g(an) \right) da = \sum_{n \in \mathbb{Z}\setminus\{0\}} \int_0^\infty \frac{a^{s-1}}{(n\sqrt{\pi})^s} e^{-\pi a^2} da = \zeta(s)\pi^{-s/2}\Gamma(s/2) = \Lambda(s). \quad (1)$$

On the other hand, one also has

$$\int_0^\infty a^{s-1} \left( \sum_{n \in \mathbb{Z}\setminus\{0\}} g(an) \right) da = \int_1^\infty a^{s-1} \left( \sum_{n \in \mathbb{Z}\setminus\{0\}} g(an) \right) da + \int_0^1 a^{s-1} \left( \sum_{n \in \mathbb{Z}\setminus\{0\}} g(an) \right) da$$
\[ = \int_{1}^{\infty} \frac{a^{s-1}}{a} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} g(an) \right) \, \text{da} + \int_{1}^{1} \frac{a^{s-2}}{a} \left( 1 - \frac{1}{a} + \sum_{n \in \mathbb{Z} \setminus \{0\}} g(n/a) \right) \, \text{da} \]

\[ = \int_{1}^{\infty} \frac{a^{s-1}}{a} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} g(an) \right) \, \text{da} + \int_{1}^{\infty} \frac{a^{s-2}}{a} \left( a^{-s} - a^{-s+1} \right) \, \text{da} + \int_{1}^{\infty} \frac{a^{-s}}{a} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} g(an) \right) \, \text{da} \]

\[ = \int_{1}^{\infty} \frac{a^{s-1}}{a} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} g(an) \right) \, \text{da} + \frac{1}{s} + \frac{1}{1-s} + \int_{1}^{\infty} \frac{a^{-s}}{a} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} g(an) \right) \, \text{da} \]

where the last expression converges for any \( s \), and thus is a meromorphic continuation for others. It is moreover symmetric for \( s \leftrightarrow 1 - s \), and thus \( \Lambda(s) = \Lambda(1 - s) \).

Now how do we make it adelic? From an adelic point of view, instead of a function \( f(x) \) on \( \mathbb{R} \) we should consider a function on \( \mathbb{A}_{\mathbb{Q}} \). The naive (but not too naive) way to produce a function on \( \mathbb{A}_{\mathbb{Q}} \) is to pull back \( f \) from \( \mathbb{R} \) to \( \mathbb{R} \times \prod_{p<\infty} \mathbb{Z}/p; \) let \( f \) be the function on \( \mathbb{A}_{\mathbb{Q}} \) defined by

\[ f(x_\infty x^\infty) = \begin{cases} \frac{g(x_\infty)}{0}, & \text{for } x_\infty \in \mathbb{R}, x^\infty \in \prod_{p<\infty} \mathbb{Z}/p; \\
0, & \text{for } x^\infty \in \prod_{p<\infty} \mathbb{Q}/p \setminus \prod_{p<\infty} \mathbb{Z}/p. \end{cases} \]

Then there is a really nice way to write the sum over integers:

\[ \sum_{n \in \mathbb{Z}} g(n) = \sum_{x \in \mathbb{Q}} f(x), \]

and thus the Poisson summation formula get translated to

\[ \sum_{x \in \mathbb{Q}} f(a_\infty x) = \frac{1}{a_\infty} \sum_{x \in \mathbb{Q}} \hat{f}(a^{-1}_\infty x), \text{ for } a_\infty \in \mathbb{R}^\times \subset \mathbb{A}_{\mathbb{Q}}^\times. \] (2)

Here \( \hat{f} = f \) as \( \hat{g} = g \). Likewise the LHS of equation (1) is equal to

\[ \int_{1}^{\infty} \frac{a^{s-1}}{a} \left( \sum_{x \in \mathbb{Q}^\times} f(a_\infty x) \right) \, \text{da}_\infty. \] (3)

There are things we can complain about (2) and (3). For example, having \( a_\infty \in \mathbb{R}^\times \) (or \( \mathbb{R}^+_\infty \)) only for the archimedean place seems not beautiful enough. So we observe that for (3) using \( \mathbb{A}_{\mathbb{Q}}^\times = \mathbb{R}^+_\infty \times \hat{\mathbb{Z}}^\times \times \mathbb{Q}^\times \) we can rewrite (2) as

\[ \sum_{x \in \mathbb{Q}} f(ax) = |a|^{-1} \sum_{x \in \mathbb{Q}} \hat{f}(a^{-1}x), \text{ for } a \in \mathbb{A}_{\mathbb{Q}}^\times. \] (4)
and (3) as (for some measure $da$)

$$\int_{0}^{\infty} a_{\infty}^{s-1} \left( \sum_{x \in \mathbb{Q}^\times} f(a_{\infty} x) \right) da_{\infty} = \int_{\mathbb{A}} |a|^{s-1} f(a) da.$$  \hspace{1cm} (5)

And thus the functional equation $\Lambda(s) = \Lambda(1 - s)$ becomes

$$\int_{\mathbb{A}} |a|^s f(a) \frac{da}{|a|} = \int_{\mathbb{A}} |a|^{1-s} \hat{f}(a) \frac{da}{|a|}.$$  \hspace{1cm} (6)

Now you bet that if we have (4) for general global fields, we can derive (4) for general global field, and also allow the Hecke characters to enter!