14 From modular forms to automorphic representations

We fix an even integer $k$ and $N > 0$ as before. Let $f \in \mathcal{M}_k(N)$ be a modular form. We would like to produce a function on $GL_2(\mathbb{A}_\mathbb{Q})$ out of it. Recall that we have the action $(\gamma \cdot_k f)(\tau) := j(\gamma^{-1}, \tau)^{-k}f(\gamma^{-1}\tau)$ for $\gamma \in GL_2(\mathbb{R})^+$ for which $f$ is invariant by $\Gamma = \Gamma_0(N)$ under this action. We now define $\phi_\infty : GL_2(\mathbb{R})^+ \to \mathbb{C}$ by $\phi_\infty(g) = f(gi)j(g, i)^{-k}$ for any $g \in GL_2(\mathbb{R})^+$. We have

$$\phi_\infty(\gamma g) = \phi_\infty(g)$$

for any $\gamma \in \Gamma$ and any $g \in GL_2(\mathbb{R})^+$.

The stabilizer of $i \in \mathcal{H}$ under $GL_2(\mathbb{R})^+$ is $\mathbb{R}_+^* \times SO_2$; here $\mathbb{R}_+^* \subset GL_2(\mathbb{R})^+$ is in the center and we write $K_\infty = SO_2$ be the group of orthogonal matrices with determinant 1 on the standard real inner product space $\mathbb{R}^2$. Every element in $K_\infty$ may be expressed as $k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$. We have

$$\phi_\infty(z k_\theta) = e^{i\theta k} \phi_\infty(e)$$

where $e \in GL_2(\mathbb{R})^+$ is the identity. For any $z \in \mathbb{R}_+^*$ and $k_\theta$ as above. This implies for any $g \in GL_2(\mathbb{R})^+$

$$\phi_\infty(g z k_\theta) = e^{i\theta k} \phi_\infty(g). \quad (1)$$

In particular, (1) allows us to easily recover the weight of $f$. Lastly, we would like to interpret the holomorphy of $f$ in terms of $\phi_\infty$. As holomorphy of functions is described by a first order partial equation (i.e. Cauchy-Riemann equation), it is natural to consider the Lie algebra of $GL_2(\mathbb{R})$ acting $\phi_\infty$. More precisely, for any $2 \times 2$ matrix $X$ in $\mathbb{R}$ and any $g \in GL_2(\mathbb{R})^+$, we put

$$R_X(\phi_\infty)(g) := \lim_{t \to 0} \frac{\phi_\infty(g e^{tX}) - \phi_\infty(g)}{t}.$$ 

This is evidently $\mathbb{R}$-linear in $X$, and we shall extend it to $X \in M_2(\mathbb{C})$ by $\mathbb{C}$-linearity. We now put

$$H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X_+ = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad X_- = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$ 

They satisfy $X_+ X_- - X_- X_+ = H$, $HX_+ - X_+ H = 2X_+$ and $HX_- - X_- H = -2X_-$. They are also a basis for $\mathfrak{sl}_2(\mathbb{C})$, the space of $2 \times 2$ matrices in $\mathbb{C}$ with trace zero.

**Proposition 14.1.** The holomorphy of $f$ is equivalent to $R_{X_-}(\phi_\infty) = 0$.

$^1$That is to say, they form a so-called $\mathfrak{sl}_2$-triple.
Proof. We will only show that \( R_{X_-}(\phi_\infty) = 0 \) when \( f \) is holomorphic. The differential of the map \( g \mapsto \tau i \) is an \( \mathbb{R} \)-linear map \( \delta : M_2(\mathbb{R}) \to T_1(\mathfrak{H}) \cong \mathbb{R}^2 \). By extending the map \( \mathbb{C} \)-linearly we have \( \delta_\mathbb{C} : M_2(\mathbb{C}) \to \mathbb{C}^2 \). But we may instead use the complex structure on \( T_1(\mathfrak{H}) \cong \mathbb{C} \) to have \( \delta' : M_2(\mathbb{C}) \to \mathbb{C} \). One computes directly that \( \delta_\mathbb{C}(X_-) = 0 \). This means that \( \delta_\mathbb{C}(X_-) \) lies in the anti-holomorphic tangent space of \( T_1(\mathfrak{H}) \).

Write \( \delta_0(g) := f(gi). \) Then the above implies \( R_{X_-}(\delta_0)(\text{Id}) = 0. \) Moreover, as \( GL_2(\mathbb{R})^+ \) acts on \( \mathfrak{H} \) by conformal maps, we have in fact \( R_{X_-}(\delta_0)(g) \equiv 0 \) for any \( g \in GL_2(\mathbb{R}). \) This leaves the \( j(g,i) \)-term to worry. In other words, with \( j'(g) := j(g,i) \) we would like to have also \( R_{X_-}(j')(g) \equiv 0. \) One notes that \( j(ge^{tX_-},i) = j(g,e^{tX_-}i,j(e^{tX_-},i). \) The derivative of the first term is also zero as \( X_- \) is in the anti-holomorphic tangent space. For the second term we would like \( R_{X_-}(j')(\text{Id}) = 0. \). This is a property of the function \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ci + d) \) that one may check.

Let us collect a bit about what we have

**Proposition 14.2.** The association \( f \mapsto \phi_\infty \) is an isomorphism between \( \mathcal{M}_k(N) \) to the space of smooth functions on \( GL_2(\mathbb{R})^+ \) satisfying
\[
(i) \quad \phi_\infty(\gamma gzk_\theta) = e^{i\theta k_\theta} \phi_\infty(g) \quad \text{for any} \quad \gamma \in \Gamma, \quad g \in GL_2(\mathbb{R})^+ \quad \text{arbitrary, and} \quad z, \ k_\theta \quad \text{as in (1)}.
\]
\[
(ii) \quad \phi_\infty \text{ has polynomial growth in the entries of} \quad g \quad \text{and} \quad g^{-1}.
\]
\[
(iii) \quad R_{X_-}(\phi_\infty) \equiv 0.
\]

It suffices to explain (ii). As \( j(g,i)^{-1} \) has polynomial growth in the coefficients of \( g \) and \( g^{-1} \), if \( \phi_\infty \) comes from a modular form it has polynomial growth. Conversely, a Laurent series in \( q \) that has logarithmic growth in \( q \) is automatically holomorphic, thus (ii) is enough to ensure holomorphy at cusps.

We will write \( \mathbb{A} = \mathbb{A}_\mathbb{Q}, \) and \( \mathbb{A}^\infty := \prod_{p<\infty} \mathbb{Q}_p \) be the restrict product of the finite completions. To translate functions on \( GL_2(\mathbb{R})^+ \) to \( GL_2(\mathbb{A}) \) we rely on the following results

**Theorem 14.3.** (Strong approximation for the additive group) Let \( v \) be any place of \( \mathbb{Q} \) and \( \mathbb{Q}_v \) be the corresponding completion. Then \( \mathbb{Q}_v \) is dense in \( \mathbb{Q} \setminus \mathbb{A}. \)

**Theorem 14.4.** (Strong approximation for \( SL_2 \)) Let \( v \) be any place of \( \mathbb{Q} \) and \( \mathbb{Q}_v \) be the corresponding completion. Then \( SL_2(\mathbb{Q}_v) \) is dense in \( SL_2(\mathbb{Q}) \setminus SL_2(\mathbb{A}). \)

Proof. We want to prove that \( SL_2(\mathbb{Q}) SL_2(\mathbb{Q}_v) \) is dense in \( SL_2(\mathbb{A}). \) This follows from Theorem 14.3 and the observation that elements of the form \( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \) with any \( x \in \mathbb{A} \) generate \( SL_2(\mathbb{A}). \)
In particular, \( SL_2(\mathbb{Q})SL_2(\mathbb{R}) \) is dense in \( SL_2(\mathbb{A}) \). Hence for any open subgroup \( K' \subset SL_2(\mathbb{A}^\infty) \) we have \( SL_2(\mathbb{A}) = SL_2(\mathbb{Q})SL_2(\mathbb{R})K' \). Now consider in \( GL_2 \):

\[
K^\infty = K_0(N) := \{ g \in GL_2(\hat{\mathbb{Z}}) \mid g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } N|c \} \subset GL_2(\mathbb{A}^\infty).
\]

In particular \( K' := K^\infty \cap SL_2(\mathbb{A}^\infty) \) is open. As \( \mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}^\times \hat{\mathbb{Z}}^\times \) and determinants of elements in \( K \) can take all values in \( \hat{\mathbb{Z}}^\times \), we obtain

\[
GL_2(\mathbb{A}) = GL_2(\mathbb{Q})GL_2(\mathbb{R})^+K^\infty.
\]

One also easily see that \( GL_2(\mathbb{Q}) \cap GL_2(\mathbb{R})^+K^\infty = \Gamma = \Gamma_0(N) \). Thus we may define \( \phi : GL_2(\mathbb{A}) \rightarrow \mathbb{C} \) by

\[
\phi(\gamma g_\infty k) := \phi_\infty(g_\infty), \text{ for any } \gamma \in GL_2(\mathbb{Q}), g_\infty \in GL_2(\mathbb{R})^+, k \in K^\infty.
\]

As a summary, we now have

**Theorem 14.5.** The association \( f \mapsto \phi \) is an isomorphism between \( \mathcal{M}_k(N) \) and the space of smooth functions on \( GL_2(\mathbb{A}) \) satisfying

(i) For any \( \gamma \in GL_2(\mathbb{Q}) \), and \( k \in K^\infty \), we have \( \phi(\gamma gk) = \phi(g) \) for any \( g \in GL_2(\mathbb{A}) \).

(ii) For any \( k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in GL_2(\mathbb{R})^+ \) as in (1), we have \( \phi(gk_\theta) = e^{i\theta k}\phi(g) \) for any \( g \in GL_2(\mathbb{A}) \).

(iii) \( \phi \) is translation invariant by the center \( \mathbb{A}^\times \subset GL_2(\mathbb{A}) \).

(iv) \( \phi(g) \) is of polynomial growth in the entries of \( g \) and \( g^{-1} \).

(v) \( R_{X_-}(\phi) \equiv 0 \).

Moreover the subspace \( \mathcal{S}_k(N) \) is mapped to cuspidal function on \( GL_2(\mathbb{A}) \), namely those with

(iii) If any \( g \in GL_2(\mathbb{A}) \),

\[
\int_{\mathbb{Q} \setminus \mathbb{A}} \phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) dx = 0
\]

**Proof.** It remains to prove the assertion with (vi). Write \( g = \gamma g_\infty k \) as any decomposition with \( \gamma \in GL_2(\mathbb{Q}) \), \( g_\infty \in GL_2(\mathbb{R})^+ \) and \( k \in K^\infty \). By continuity, there exists some open compact subgroup \( U_g \subset \mathbb{A}^\infty \) such that \( \phi(\begin{pmatrix} 1 & x+u \\ 0 & 1 \end{pmatrix} g) = \phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) \) for any \( u \in U_g \). By Theorem 14.3 we have \( \mathbb{Q} \setminus \mathbb{A} \) is covered by \( \mathbb{R} \times U_g \), and thus integration over \( \mathbb{Q} \setminus \mathbb{A} \) becomes (up to a constant) integration over \( \tilde{U}_g \setminus \mathbb{R} \) where \( \tilde{U}_g = \mathbb{Q} \cap (\mathbb{R} \times U_g) \). In fact we may assume

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\( U_g \) is of the form \( M \mathbb{Z} \), and by shrinking \( U_g \) we may assume \( M \) is as divisible as we like. In any case we arrive at the integral \( \int_{M \mathbb{Z} \setminus \mathbb{R}} \phi \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) g dx \).

Choose any decomposition \( g = \gamma g_\infty k \) with \( \gamma \in GL_2(\mathbb{Q}) \), \( g_\infty \in GL_2(\mathbb{R})^+ \) and \( k \in K^\infty \) as before. By condition (i) in the theorem we then have, for \( x \in \mathbb{R} \),

\[
\phi \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) g = \phi \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) \gamma g_\infty k = \phi(\gamma^{-1} \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) \gamma g_\infty) = \phi_\infty(\gamma^{-1} \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) \gamma g_\infty)
\]

where \( \gamma_\infty \) is the \( GL_2(\mathbb{R}) \)-component of \( \gamma \in GL_2(\mathbb{Q}) \subset GL_2(\mathbb{A}) \), i.e. the image of \( \gamma \) in \( GL_2(\mathbb{R}) \). Now by Euclid’s algorithm one may find \( \gamma' \in SL_2(\mathbb{Z}) \subset GL_2(\mathbb{R}) \) such that \( \gamma_\infty = b \gamma' \) where \( b \in GL_2(\mathbb{R}) \) is upper triangular. This gives

\[
\phi \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) g = \phi_\infty(\gamma^{-1} \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) \gamma g_\infty) = \phi_\infty(\gamma^{-1} b^{-1} \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) b \gamma' g_\infty).
\]

But conjugation by \( b \) simply scales \( x \) by some non-zero rational number! We thus get, up to a constant,

\[
\int_{\mathbb{Q} \setminus \mathbb{A}} \phi \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) g dx = \int_{M' \mathbb{Z} \setminus \mathbb{A}} \phi_\infty(\gamma^{-1} \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) \gamma g_\infty) dx.
\]

If \( \gamma' \in \Gamma \), then \( \phi_\infty(\gamma^{-1} \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) \gamma g_\infty) = \phi_\infty(\left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) \gamma g_\infty) \) and the vanishing of the integral says \( f \) has a zero at \( q = 0 \). For general \( \gamma' \in SL_2(\mathbb{Z}) \), this says \( f \) is a cusp form. \( \square \)

Nevertheless, this function on \( GL_2(\mathbb{A}) \) should live inside a particular representation of \( GL_2(\mathbb{A}) \)! On a first thought, we would like to have \( GL_2(\mathbb{A}) \) acting on a subspace (containing \( \phi \)) of the space of functions on \( GL_2(\mathbb{A}) \) by right translation. Our experience at the real place however suggest something less. We have seen the action of \( \mathfrak{gl}_2(\mathbb{C}) := M_2(\mathbb{C}) \) by differentiation (of the right translation), and the action of \( K^\infty = SO_2 \) by right translation. Let us also recall that \( \mathfrak{gl}_2(\mathbb{C}) \) can be generated by

\[
H = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \ X_+ = \frac{1}{2} \left( \begin{array}{cc} 1 & i \\ i & -1 \end{array} \right), \ X_- = \frac{1}{2} \left( \begin{array}{cc} 1 & -i \\ -i & -1 \end{array} \right), \ \text{and} \ Z = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).
\]

Let us still denote by \( R_X \) the action of \( X \in \mathfrak{gl}_2(\mathbb{C}) \). From basic differential geometry we have \( R_{[X,Y]-YX} = [R_X, R_Y] \) for \( X, Y \in \mathfrak{gl}_2(\mathbb{C}) \), where \( [X, Y] := XY - YX \) and \( [R_X, R_Y] := R_X \circ R_Y - R_Y \circ R_X \). This suggests to consider the universal enveloping algebra \( U(\mathfrak{gl}_2) \) of \( \mathfrak{gl}_2(\mathbb{C}) \), defined to be the quotient, of the free associative algebra generated by \( \mathfrak{gl}_2(\mathbb{C}) \) (as a \( \mathbb{C} \)-vector space, i.e. generated by any basis of \( \mathfrak{gl}_2(\mathbb{C}) \), by the ideal generated by \( XY - YX - [X,Y] \). The algebra \( U(\mathfrak{gl}_2) \) act on smooth functions on \( GL_2(\mathbb{R}) \) and thus those on \( GL_2(\mathbb{A}) \). By direct computation one may prove
Lemma 14.6. The center $Z(\mathfrak{gl}_2)$ of the algebra $U(\mathfrak{gl}_2)$ is generated by $H^2+2X_+X_-+2X_-X_+$ and $Z$.

We can now state what kind of representations we are studying. First we state the ambient space. Let $\chi : \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ be a (continuous) unitary character.

**Definition 14.7.** An automorphic form of $GL_2$ over $\mathbb{Q}$ with central character $\chi$ is a function $\phi : GL_2(\mathbb{A}) \to \mathbb{C}$ satisfying

(i) For any $\gamma \in GL_2(\mathbb{Q})$, we have $\phi(\gamma g) = \phi(g)$.

(ii) For any $a \in \mathbb{A}_\mathbb{Q}$, $\phi(g \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}) = \chi(a)\phi(g)$.

(iii) There exists an open subgroup $K^\infty \subset GL_2(\mathbb{A}_\mathbb{A})$ such that $\phi$ is right $K^\infty$-invariant.

(iv) $\phi$ is right $K^\infty$-finite, i.e. right $K^\infty$-translates of $\phi$ generate a finite-dimensional vector space.

(v) $\phi$ is smooth at the real place, and right $Z(\mathfrak{gl}_2)$-finite.

(vi) $\phi$ has polynomial growth, and $\int_{GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{A})/\mathbb{A}^\times} |\phi(g)|^2 dg < \infty$ where $\mathbb{A}^\times \to GL_2(\mathbb{A})$ is the center.

Moreover, the automorphic form is **cuspidal** if it satisfies

(vii) $\int_{\mathbb{Q} \backslash \mathbb{A}} \phi\left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0$ for any $g \in GL_2(\mathbb{A})$.

We denote by $\mathcal{A}(GL_2, \chi)$ the space of automorphic forms and $\mathcal{A}_o(GL_2, \chi)$ the subspace of cuspidal automorphic forms. They carry actions of $K^\infty$, $U(\mathfrak{gl}_2)$ and $GL_2(\mathbb{A}_\mathbb{A})$ by right translation. We remark that the action of $K^\infty$ and $U(\mathfrak{gl}_2)$ have to satisfy certain compatibility property. Such a representation of $K^\infty$ and $U(\mathfrak{gl}_2)$ is typically called a ($\mathfrak{gl}_2(\mathbb{C}), K^\infty$)-module (or just ($\mathfrak{g}, K$)-module).

**Definition 14.8.** An automorphic representation (with central character $\chi$) is an irreducible subquotient representation of $\mathcal{A}(GL_2, \chi)$. A cuspidal automorphic representation is an irreducible subrepresentation of $\mathcal{A}_o(GL_2, \chi)$.

A fundamental feature about automorphic representations is that they factor into local components. Let $(\pi, V^\sharp)$ be an automorphic representation. To avoid difficulties at the real place, we focus on those vectors for which $R_{X_-} = 0$ and $k_\theta \in K^\infty$ acts by $e^{ik\theta}$ for some fixed positive integer $k$. In other words, let $V \subset V^\sharp$ be the space of elements satisfying the above archimedean condition, and we assume $V \neq 0$. The following is a result of representation theory of $GL_2(\mathbb{R})$ that we will not prove.

**Lemma 14.9.** $V$ is an irreducible representation of $GL_2(\mathbb{A}^\infty)$. 

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By (iii) of Definition 14.7, every vector in $V$ is invariant under some open subgroup of $GL_2(\mathbb{A}_\infty)$. We may assume this open subgroup is of the form $K^\infty = \prod_{p<\infty} K_p$ with $K_p \subset GL_2(\mathbb{Q}_p)$ and $K_p = GL_2(\mathbb{Z}_p)$ for almost all $p$. We may consider the space $\overline{V} := V^{K^\infty}$ of $K^\infty$-fixed vectors in $V$. If $K^\infty = K_0(N)$, then we have seen in Theorem 14.5 that $\overline{V}$ is a subspace of $S_k(N)$ and hence finite-dimensional by complex function theory. It’s possible to show this finite-dimensional property in general\(^2\), and we now assume $\dim_{\mathbb{C}} \overline{V} < \infty$. We have the Hecke algebra

$$\mathcal{H}^\infty := \bigotimes \mathcal{H}(GL_2(\mathbb{Q}_p), K_p)$$

acting on $\overline{V}$. As $V$ is irreducible, we may prove as in local theory (Lemma 12.3) that $\overline{V}$ is an irreducible $\mathcal{H}^\infty$-module. Let $S$ be a finite set of primes so that $K_p = GL_2(\mathbb{Z}_p)$ for $p \not\in S$. For any $p \not\in S$, as $\overline{V}$ is finite-dimensional, the algebra $\mathcal{H}_p := \mathcal{H}(GL_2(\mathbb{Q}_p), GL_2(\mathbb{Z}_p))$ has to act through a maximal ideal $I_p \subset \mathcal{H}_p$ for any element in $\mathcal{H}_p$ doesn’t act by a constant, its eigenspace will give a proper subrepresentation. We write $\overline{V}_p := \mathcal{H}_p/I_p$ as a $\mathcal{H}_p$-module. For $p \in S$, as an $\mathcal{H}_p := \mathcal{H}(GL_2(\mathbb{Q}_p), GL_2(\mathbb{Z}_p))$-module $\overline{V}$ has to decompose into a direct sum of isomorphic irreducible submodule for the same reason. We call any such irreducible submodule $\overline{V}_p$. We then have

$$\overline{V} \cong \bigotimes \overline{V}_p$$

as an $\mathcal{H}^\infty$-module. Note that the RHS is finite-dimensional exactly because almost all $\overline{V}_p$ are 1-dimensional. For every $p \not\in S$, we know that $\overline{V}_p$ arises as $\overline{V}_p = (V^p)^{K_p}$ for some $GL_2(\mathbb{Q}_p)$-representation $V_p$. In general, $V_p$ may be explicitly found within $V$ as follows: with a fixed $p$, consider a basis of open compact subgroups $K_{p,1} \supset K_{p,2} \supset \ldots$ of $GL_2(\mathbb{Q}_p)$. Let $K_i^\infty$ be as $K^\infty$ with $K_p$ replaced by $K_{p,i}$, and $\overline{V}_i := V^{K_i^\infty}$. Then $\overline{V}_1 \subset \overline{V}_2 \subset \ldots$ and for each we have $\overline{V}_i \cong \overline{V}_{p,i} \otimes \bigotimes_{p' \not\in p} \overline{V}_{p'}$, where $\overline{V}_{p,i}$ is an irreducible $\mathcal{H}_{p,i} := \mathcal{H}(GL_2(\mathbb{Q}_p), K_{p,i})$-module. By choosing embeddings $\overline{V}_{p,i} \rightarrow \overline{V}_{p,i+1}$ (unique up to constant) we have $\overline{V}_p := \bigcup \overline{V}_{p,i}$ is a representation of $GL_2(\mathbb{Q}_p)$ such that $\overline{V}_{p,i} = (V_p)^{K_{p,i}}$. In particular $\overline{V}_p = (V_p)^{K_p}$.

Moreover, we may enlarge shrink finitely many $K_p$ (and there enlarge $\overline{V}_p$ and $\mathcal{H}_p$) at the same time. Since all vectors in $V$ are invariant under some open subgroup of $GL_2(\mathbb{A}_\infty)$, we then have

$$V \cong \bigotimes V_p$$

where the restricted tensor product means the following: a vector in $\bigotimes V_p$ is a finite linear combination of $\otimes v_p$ for $v_p \in V_p$ where almost all $v_p \in \overline{V}_p$. On the other hand, suppose we begin with a sequence of irreducible smooth representations $W_p$ of $GL_2(\mathbb{Q}_p)$ and form their restricted tensor product $W = \bigotimes W_p$ consisting of linear combinations of $\otimes v_p$ with $v_p \in W_p^{GL_2(\mathbb{Z}_p)}$ for almost all $p$, then $W$ is also an irreducible smooth representation of $GL_2(\mathbb{A}_\infty)$. Here smoothness means every vector in $W$ is invariant by an open subgroup of $GL_2(\mathbb{A}_\infty)$. In fact, with a bit of the archimedean theory, we can also take $V_\infty$ to be any irreducible $(\mathfrak{g}_2(\mathbb{C}), K_\infty)$-submodule of $V^\perp$ and have

\(^2\)Without the holomorphic condition, it will be a result of elliptic partial differential equations.
Theorem 14.10. The automorphic representation $V^\sharp$ is a restricted tensor product of all $V_v$ for all places $v$ of $\mathbb{Q}$.

Note that we have only used property (iii), (iv) and (v) in Definition 14.7, but not the most important (i). In other words, there is no need in the theorem that $V^\sharp$ is automorphic. It is then natural to wonder that what condition (i) in Definition 14.7 means. That is, what it means to be automorphic? Let us look at instead the $GL_1$-example. An automorphic representation of $GL_1$ can be readily checked to be a Hecke character $\chi : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$. Instead, a character of $\mathbb{A}_\mathbb{Q}^\times$ is a tensor product of characters of $\mathbb{Q}^\times_v$ that are unramified for almost all $v$. Let us furthermore suppose that $\chi|_{\mathbb{R}_+^\times} \equiv 1$. Then class field theory tells us that this character is trivial on $\mathbb{Q}^\times$ if and only if the Galois characters $\text{rec}(\chi|_{\mathbb{Q}^\times_p}) : G_{\mathbb{Q}_p} \to \mathbb{C}^\times$ obtained via class field theory combines to a 1-dimensional Galois representation from $\rho : G_\mathbb{Q} \to \bar{\mathbb{Q}}_{\ell}^\times$ such that $\rho|_{G_{\mathbb{Q}_{\ell}}}$ is $\chi_{\text{cyc},\ell}^{-n}$ twisted by a finite order character. We are now ready to state the global Langlands conjecture.

Conjecture 14.11. There is a bijection $\pi \mapsto \text{rec}(\pi)$ between the set of isomorphic classes of automorphic representations $(\pi, V)$ whose real part $V_\infty$ is algebraic, and the set of isomorphic classes of semisimple Galois representations $\rho : G_\mathbb{Q} \to GL_n(\bar{\mathbb{Q}}_{\ell})$ that are unramified almost everywhere and de Rham at $\ell$. This bijection satisfies:

(i) For any $p \neq \ell$, $\text{rec}(V_p)$ is isomorphic to the Frobenius-semisimplification of $\rho|_{G_{\mathbb{Q}_p}}$.

(ii) The real part $V_\infty$ is given by certain ($p$-adic Hodge theoretic) invariant of $\rho|_{G_{\mathbb{Q}_{\ell}}}$, called Hodge-Tate weights.

(iii) Cuspidal automorphic representations correspond to irreducible Galois representations. If $\rho = \bigoplus \rho_i$ and $\rho_i = \text{rec} \pi_i$ with $\pi$ being cuspidal automorphic representations of $GL_{n_i}(\mathbb{A}_\mathbb{Q})$ (where $n_i = \dim \rho_i$ and $n = \sum n_i$), then $\pi$ can be formed from $\pi_i$ by a formation generalizing Eisenstein series.

We need some more precise statement about how cusp forms enter the picture. To see this, we need the following global analytic result that we will not prove:

Theorem 14.12. (Multiplicity one) Let $V, V' \subset A_0(GL_2, \chi)$ be two cuspidal automorphic representations. Suppose $V_p \cong V'_p$ for almost all $p$. Then $V = V'$.