# SPECTRAL SEQUENCE CALCULATIONS ARE NP-COMPLETE

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Abstract. Many problems in algebraic topology require making computations in spectral sequences which have the structure of a module over an algebra. We will show that the problem of determining whether a given differential can occur as part of a self-consistent collection of differentials is NP-complete. This verifies the folk theorem that "spectral sequence computations can be hard".

The computation of the stable homotopy groups of spheres via the Adams spectral sequence is quickly approaching the point where it is not feasible for a human to compute all the differentials by hand. This barrier is not one of mathematical difficulty, but rather one of the size of dataset one must consider, which grows quasi-polynomially in the stem [Bur21]. While it is infeasible for humans to compute differentials on millions of classes on the $E_2$-page, such computations remain feasible through a much larger range with the aid of computers.

Considerations such as these have led to a renewed effort by Hood Chatham, Dexter Chua, Guozhen Wang and others to produce a software suite specifically taylored to the problem of computing stable stems. Two components which this software suite will likely contain are propogation of differentials using the Liebniz rule and speculative evalutation of potential differentials. What we mean by "speculative evaluation" is roughly the following,

- In some situations one can show, using the Liebniz rule, that a certain differential does not occur as part of any self-consistent collection of possible differentials. On this basis, such a differential can be excluded.

The most basic instance in which a differential may be excluded by speculative evaluation is given in fig. 1. What we show in this note is that the problem which the computer must solve as part of "speculative evaluation" is NP-complete. As in many situations where NP-complete problems arise, in practice we find it unlikely that real-world instances will approach worst-case runtimes. Instead we have written this note as a justification of the humorous meta-theorem that, "spectral sequence computations can be hard".

Recall that a decision problem is in NP if "yes" solutions can be verified in polynomial time and a decision problem $A$ is NP-hard if every problem in NP can be reduced to an instance of $A$ in polynomial time [Coo71]. We will now give a formalize the decision problem which arises in "speculative evaluation". This begins with setting up a toy model of a spectral sequence.

**Definition 1.** For the purpose of this note a spectral sequence of modules over a fixed bigraded ring $R$ will consist of [1]

- a sequence of bigraded $R$-module $E_r^{*,*}$ for $r \geq 1$,
- $R$-linear differentials $d_r : E_r^{x,y} \to E_r^{x-1,y+r}$,
- $R$-linear isomorphisms $H^{*,*}(E_r, d_r) \cong E_{r+1}^{*,*}$.

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[1]In this definition we have chosen the indexing that makes the most sense for the way we display charts.

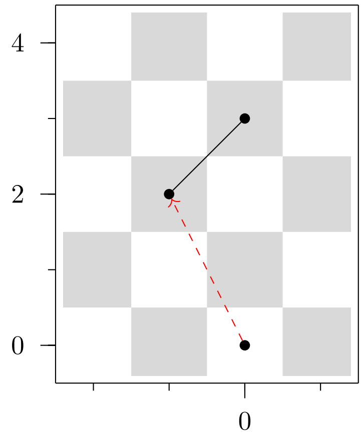


FIGURE 1. The line of slope 1 indicates multiplication by some permanent cycle. The dashed red differntial can be excluded on the basis of the Liebniz rule.

With this we define *spectral sequence speculative evaluation* decision problem. Informally, the problem we will define asks whether a given $d_2$-differential occurs as part of a consistent spectral sequence.

**Definition 2.** An instance of the problem $\mathrm{SSeq}_n$ will consist of

- a bigraded ring $R \coloneqq \mathbb{F}_2[x_1, \cdots, x_n]$ (this is just a choice of bidegrees in which the $x_i$ live) [2],
- a bigraded $\mathbb{F}_2$-vector space $E^{*,*}$ equipped with an $R$-module structure,
- a potential differential $d_2(x) = y$ we wish to evaluate.

An accepting configuration for this problem will consist of a spectral sequence of $R$-modules such that $E = E_1$, $d_1(x) = 0$ and $d_2(x) = y$.

Note that the size of an accepting configuration is at most polynomial in the dimension of the vector space $E$ we started with.

**Proposition 3.** *The decision problem $SSeq_4$ is NP-complete.*

We will prove this proposition in two steps. First we will show that $\mathrm{SSeq}_n$ is in NP. Then we will show that $\mathrm{SSeq}_4$ is NP-hard. Since $\mathrm{SSeq}_n$ for $n \geq 4$ is at least as hard as $\mathrm{SSeq}_4$, this will in turn imply that $\mathrm{SSeq}_n$ is NP-complete for larger values of $n$ as well.

In order to show that $\mathrm{SSeq}_n$ is in NP we observe that given a collection of differentials and pages, verifying that the differentials are $R$-linear and the pages are the homology of the previous differential uses only linear algebra operations which can be accomplished in polynomial time (polynomial in the total dimension of $E$).

Now we turn to the more difficult task of showing that $\mathrm{SSeq}_4$ is NP-hard. We will accomplish this via a two-step reduction to CNFSAT.

**Definition 4.** An instance of the *subspace union* decision problem (which we abbreviate to SSU) will consist of

- an $\mathbb{F}_2$-vector space $V$,
- a collection of subspaces $W_1, \ldots, W_n \subset V$.

An accepting configuration for this problem will consist of a vector $v \in V$ such that $v \notin W_i$ for all $i$.

[2]We have chosen to work with $\mathbb{F}_2$ because this makes reductions to boolean logic simplest. This restriction is not necessary, though we will not prove this.

Another way of phrasing this question is that it asks whether $W_1 \cup \cdots \cup W_n$ is equal to $V$ [3].

**Lemma 5.** *SSU is NP-complete.*

*Proof.* In order to conclude that SSU is in NP, we note that checking membership $v \in W_i$ can be done in polynomial time (polynomial in the dimension of $V$) by row reducing a basis for $W_i$.

Now we must show that SSU is NP-hard. We will do this by embedding instances of CNFSAT into SSU. Since CNFSAT is NP-hard [Coo71] this will allow us to conclude that SSU is NP-hard. Replacing the subspaces $W_i$ by quotient maps out of $V$ with kernel $W_i$, we can rephrase SSU as follows:

- We're given an $\mathbb{F}_2$-vector space $V$,
- a collection of quotients $M_i : V \to V_i$ for $i = 1, \ldots, n$,
- and an accepting configuration consists of a vector $v \in V$ such that $M_i v \neq 0$ for all $i$.

After fixing a basis for $V$ and each of the $V_i$ let $M_i^j$ denote the $j^{\text{th}}$ row of $M_i$. Then, the condition for $v$ to be an accepting configuration can be expressed by the predicate

$$((M_1^1 \cdot v \neq 0) \vee \cdots \vee (M_1^{a_1} \cdot v \neq 0)) \wedge \cdots \wedge ((M_n^1 \cdot v \neq 0) \vee \cdots \vee (M_n^{a_n} \cdot v \neq 0)).$$

Each of the dot-products can then be interpretted as xor'ing the appropriate terms of the vector $v$.

From this perspective we can embed an instance of CNFSAT into SSU as follows. Use the variable $x_1, \ldots, x_k$ appearing in the instance of CNFSAT as the components of $v$. Then, if we use matrices $M$ whose rows have at most a single non-zero entry we can construct any instance of CNFSAT which doesn't use not. In order to simulate "not" we will add a new variable $x_0$ and a matrix $M_0$ that forces $x_0 = 1$. Then, we can make the entry of the $M_i^j$ that aligns with $x_0$ non-zero when necessary in order to build a "not" (xor with 1). □

**Example 6.** Under the embedding given in the proof of Lemma 5 the instance of CNFSAT given by

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_1) \wedge (\neg x_3 \vee x_1)$$

is encoded by $V = (\mathbb{F}_2)^4$ and matrices

$$M_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Two example of accepting configurations are $v = (1, 1, 1, 1)$ and $v_2 = (1, 0, 0, 0)$.

We will now finish the proof of Proposition 3 by embedding SSU in $\mathrm{SSeq}_4$.

**Construction 7.** Given an instance of SSU we associate to it an instance of $\mathrm{SSeq}_4$ according to the following procedure.

- The operators $x_1, x_2, x_3$ and $x_4$ have degrees
$$|x_1| = (2, 0), \qquad |x_2| = (2, 1), \qquad |x_3| = (-2, 0), \qquad |x_4| = (-2, 1).$$
- The bigraded $R$-module $E$ is given by the sum of $R$-modules $X \oplus B \oplus D$ where these summands are defined below.
- The module $D$ has a single genator $d$ in bidegree $(4n - 1, 2)$ which is subject to relations,
$$x_1 d = 0, \quad x_2 d = 0, \quad (x_3, x_4)^{n+1} d = 0.$$

[3] This question makes sense over any field $k$, but it is only hard over finite fields.

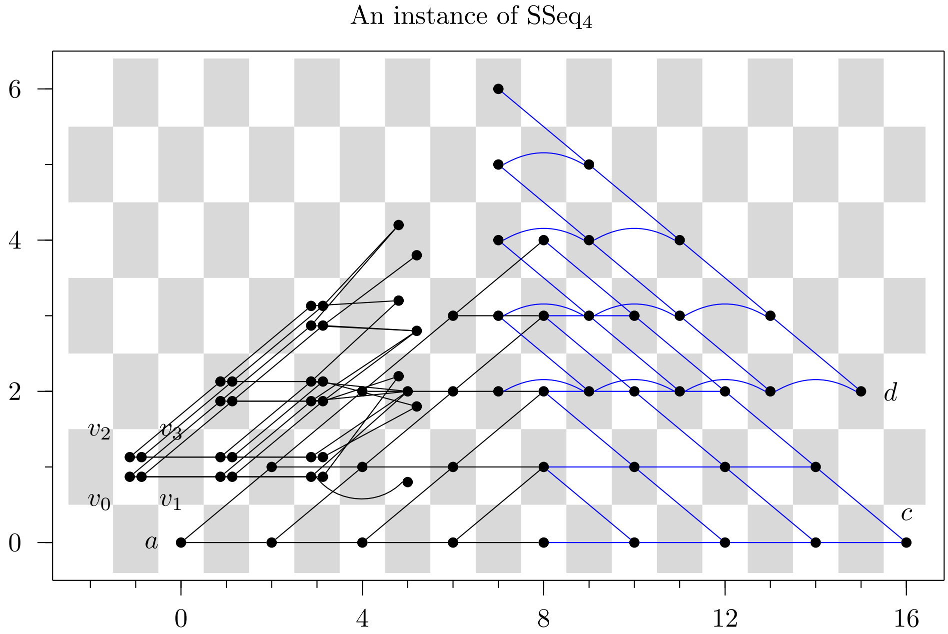


FIGURE 2. This chart illustrates the $E_1$ page of the instance of $\mathrm{SSeq}_4$ which Construction 7 associates to the instance of SSU from Example 6. The $x_1$ and $x_2$ multiplications are indicated by black lines while the $x_3$ and $x_4$ multiplications are indicated by blue lines.

- The module $X$ has two generators $a$ and $c$ in degrees $(0, 0)$ and $(4n, 0)$ respectively. The generators $a$ and $c$ are subject to relations,

$$x_3 a = 0, \quad x_4 a = 0, \quad x_1 c = 0, \quad x_2 c = 0$$

$$x_1^i x_2^{n-i} a = x_3^i x_4^{n-i} c \quad \text{for } i = 0, \dots, n$$

- Let $W$ denote the bigraded $R$-submodule of $V \otimes R/(x_3, x_4)$ generated by $x_1^{i-1} x_2^{n-i} W_i$ for $i = 1, \dots, n$ together with $(x_1, x_2)^n$. Then, we set $B = (V \otimes R/(x_3, x_4))/W$ where the copy of $V$ sits in degree $(-1, 1)$.
- The differential we would like to speculatively evaluate is $d_2(c) = d$.

Since the size of the instance of $\mathrm{SSeq}_4$ given by Construction 7 is polynomial in $n$ and the dimension of $V$, this is a polynomial reduction.

We now analyze the instance of $\mathrm{SSeq}_4$ associated to an instance of SSU by Construction 7. By considering the degrees of the various nontrivial elements and using the Liebniz rule we can conclude the following:

- All differentials on the elements of $B$ are zero.
- All differentials on the elements of $D$ are zero.
- $d_1(a) = v$ for some vector $v \in V \subset B$.
- $d_2(c) = d$ (by hypothesis).
- $d_2(x_3^a x_4^b c) = x_3^a x_4^b d$ for all $a + b \leq n$.

Suppose that $d_1(x_1^{i-1} x_2^{n-i} a) = 0$. Then, on the $E_2$ page we would have

$$0 = x_1 \cdot 0 = x_1 d_2(x_1^{i-1} x_2^{n-i} a) = d_2(x_1^i x_2^{n-i} a) = d_2(x_3^i x_4^{n-i} c) = x_3^i x_4^{n-i} d \neq 0.$$

Thus, any accepting configuration has the property that $d_1(x_1^{i-1} x_2^{n-i} a) \neq 0$ for all $i$. Moreover, it is not difficult to see that this is the only necessary condition for the given differentials to form an accepting configuration. Finally, we observe that our

An accepting configuration of an instance of $\mathrm{SSeq}_4$

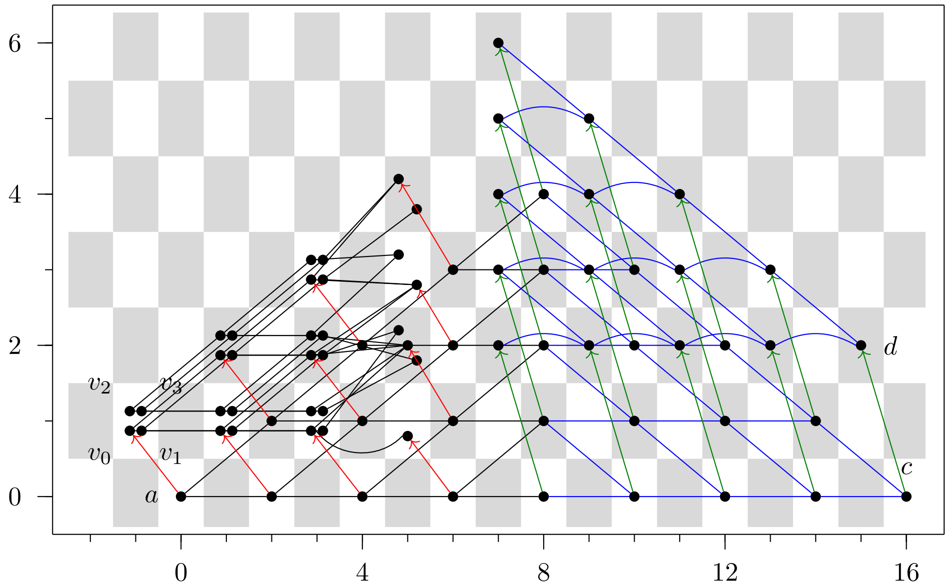


FIGURE 3. This chart illustrates an accepting configuration of the instance of $\mathrm{SSeq}_4$ associated to the instance of SSU from Example 6. This configuration is associated to the vector $v = (1, 0, 0, 0)$.

construction of the module $B$ was tailored so that $d_1(x_1^{i-1}x_2^{n-i}a) = x_1^{i-1}x_2^{n-i}v = 0$ if and only if $v \in W_i$.

## REFERENCES

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