$\mathbb{F}_p$-LOCAL DOES NOT IMPLY $\mathbb{F}_p$-NILPOTENT COMPLETE

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Abstract. We give an example of a non-connective spectrum which is $\mathbb{F}_p$-local, but not $\mathbb{F}_p$-nilpotent complete.

Given a spectrum E we can define the Bousfield localization at E,

Definition 1 ([Bou79, Section 1])**.**

- A spectrum X is E-acyclic if $X \otimes E \cong 0$.
- A morphism $f : X \to Y$ is an E-equivalence if $f \otimes E$ is an equivalence.
- A spectrum X is E-local if every map into X from an E-acyclic spectrum is nullhomotopic.

Proposition 2 (Bousfield)**.** *Every spectrum X sits in an essentially unique fiber sequence*

$$F_E X \to X \to L_E X$$

where $L_E X$ is E-local, $F_E X$ is E-acyclic and the second map is an E-equivalence.

If E was equipped with an $\mathbb{E}_0$-ring structure we can also define the nilpotent completion functor $(-)^{\wedge}_E : \mathrm{Sp} \to \mathrm{Sp}$.

Definition 3. Let I denote the fiber of the unit map on E, then we define[1]

$$X^{\wedge}_E := \varprojlim_n \mathrm{cof}(I^{\otimes n} \otimes X \to X).$$

While L_E exists for very general reasons and has many nice properties, computing its value on an object X can be a pain unless $L_E(X) \cong X^{\wedge}_E$. The essential reason for this is that nilpotent-completion involved only a countable inverse limit, whereas the construction of L_E may involve an involve an inverse limit over a rather large set.

In this note we will show that even in the (usually well behaved) situation where $E = \mathbb{F}_p$, nilpotent completion and localization can differ. Before giving our counter-example we recall without proof the following result.

Proposition 4 ([Bou79, Theorem 6.6])**.** *If X is bounded below, then $L_{\mathbb{F}_p} X \cong X^{\wedge}_{\mathbb{F}_p}$.*

We now proceed with constructing a counter-example when X is not connective. For any X we have a map $X \to X^{\wedge}_{\mathbb{F}_p}$. In order to show that the associated map $L_{\mathbb{F}_p} X \to X^{\wedge}_{\mathbb{F}_p}$ is not an equivalence it will suffice to show that $X \to X^{\wedge}_{\mathbb{F}_p}$ is not an equivalence on $\mathbb{F}_p$-homology. Stated another way, we would like to find an X such that $\mathrm{cof}(X \to X^{\wedge}_{\mathbb{F}_p})$ has non-zero $\mathbb{F}_p$-homology.

We will construct our counter-example by building an X such that $\mathrm{cof}(X \to X^{\wedge}_{\mathbb{F}_p})$ is connective with bottom homotopy group an $\mathbb{F}_p$-module. This allows us to avoid computing the homology of an inverse limit (something which is apriori difficult).

Let $k(n)$ denote the connective Morava K-theory, define X as follows,

$$X := \oplus_{i>0} \Sigma^{-|v_{n_i}|i} k(n_i)$$

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[1]If E is a ring object then this definition will agree with the more standard definition in terms of a choice of Adams tower.

Where $\{n_i\}$ is an increasing sequence of natural numbers chosen to have the following property:

$$|v_{n_i}| > |v_{n_{i-1}}|(i-1) \tag{1}$$

This condition guarantees the following:

(1) $\pi_0 X$ is the direct sum of $\mathbb{F}_p\{v_{n_i}^i\}$ over $i \in \mathbb{N}$.
(2) In negative degrees the homotopy groups of X are either 0 or $\mathbb{F}_p$.
(3) In positive degrees the homotopy groups of X are finite dimensional $\mathbb{F}_p$-vector spaces.

Now we need to examine an Adams resolution for X. Since a sum of Adams towers is an Adams tower itself it will suffice to pick one for each of the $k(n)$. For each $k(n)$ we will use the Adams resolution given by the v_n-adic filtration. In particular, the homotopy groups of the tower are just homotopy groups of X equipped with a filtration. Then, by construction, the associated graded of $\pi_0 X$ consists of one copy of $\mathbb{F}_p$ in each degree.

We can now compute the homotopy groups of $X^\wedge_{\mathbb{F}_p}$ and what the map $X \to X^\wedge_{\mathbb{F}_p}$ does on homotopy groups. In non-zero degrees the inverse limit of the homotopy groups of the Adams tower stabilize at a finite stage so we learn that the map above is an isomorphism. In degree zero, the connecting maps in the inverse system are surjective so there is no $\lim^1$ term. Passing to the inverse limit has the effect of completing π_0 with respect to the Adams filtration. Altogether, we learn that the homotopy groups of $\mathrm{cof}(X \to X^\wedge_{\mathbb{F}_p})$ are zero for $n \neq 0$ and

$$\pi_0 \mathrm{cof}(X \to X^\wedge_{\mathbb{F}_p}) = \mathrm{coker}\left(\bigoplus_{\mathbb{N}} \mathbb{F}_p \to \prod_{\mathbb{N}} \mathbb{F}_p\right).$$

This implies that $\mathrm{cof}(X \to X^\wedge_{\mathbb{F}_p})$ is an uncountable sum of copies $\mathbb{F}_p$, an object with nonzero homology.

References

[Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

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