SOME LIMITS IN MOTIVIC SPECTRA I

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Abstract. This is the first of a pair of posts where we will examine how limits in the category of motivic spectra can be poorly behaved. In this post we show that the p-completion of the unit in $SH(k)$ is not cellular over an algebraically closed field k of characteristic zero.

The stable motivic homotopy category $\mathcal{SH}(k)$ as introduced by Morel and Voevodsky in [MV99] can be a rather wild place. It is full to the brim with objects recording an essentially unknowably complicated tangle of arithmetic, geometric and topological information. In practice this means that it is common place to work in a simpler variant of this category. The following are three of the most common variations:

- Working in the full subcategory of *cellular* objects i.e. the smallest subcategory which includes the bigraded spheres and is closed under colimits.
- Working in the full subcategory of p-complete objects.
- Working in the category of modules over $\mathrm{M}\mathbb{Z}$ (the object representing motivic cohomology).

The purpose of this post is to provide an example which demonstrates that the first two simplifications are not particularly compatible and that the third does not alleviate the essential issue. The core computation is an analysis of the motivic cohomology of an elliptic curve and how it changes upon p-completion. Before we can get to this we make a detour to set up some computational techniques.

1. Weight structures

In the final section a key role will be played by a certain weight structure on the motivic category, to which we provide a lightning introduction. Weight structures were developed by Bondarko [Bon10] and provide a generalization of the stupid filtrations in the derived category of a ring.

Definition 1.1. A *weight structure* on a stable category $\mathcal{C}$ consists of pair of full subcategories $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ such that

(1) $\mathcal{C}_{\geq 0}$ is closed under finite colimits, retracts and extensions while $\mathcal{C}_{\leq 0}$ is closed under finite limits, retracts and extensions.
(2) For any $X \in \mathcal{C}_{\leq 0}$ and $Y \in \mathcal{C}_{\geq 0}$ the set of homotopy classes of maps $[X, \Sigma Y]$ contains only the trivial map.
(3) Every object $X \in \mathcal{C}$ can be put in a cofiber sequence,

$$\Sigma^{-1} A \to X \to B$$

where $A \in C_{\leq 0}$ and $B \in C_{\geq 0}$.

The intersection $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ is referred to as the *heart* of the weight structure. We will call the weight structure compactly generated if (suspensions of) its heart contain a family of compact generators for $\mathcal{C}$. Extending $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ to be integer graded families of subcategories we can associate to each object X in $\mathcal{C}$ a weight complex,

$$\cdots \to X_{\leq -1} \to X_{\leq 0} \to X_{\leq 1} \to \cdots .$$

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In the compactly generated case the weight complex recovers X when viewed as a filtered colimit.

Definition 1.2. Given a weight structure on a category $\mathcal{C}$ we say that an object X is locally compact (dualizable) if we can choose a weight complex

$$\cdots \to X_{\leq -1} \to X_{\leq 0} \to X_{\leq 1} \to \cdots$$

which is bounded on the left and such that each X_i is compact (dualizable).

Our reason for introducing the notion of locally compact is the following lemma.

Lemma 1.3. *Suppose $\mathcal{C}$ is equipped with a compactly generated weight structure. Then, for bounded below objects tensoring with a locally dualizable object commutes with p-completion.*

Proof. Suppose that $X \in \mathcal{C}_{\geq 0}$ and Y is locally dualizable, then for any $Z \in \mathcal{C}_{=0}$ and N sufficiently large we have isomorphisms

$$[\Sigma^k Z, X \otimes Y] \cong [\Sigma^k Z, X \otimes Y_{\leq N}] \cong [\Sigma^k Z \otimes (Y_{\leq N})^\vee, X]$$

which allow us to move the completion around freely. □

We now turn to our example of interest.

Example 1.4. As a consequence of Morel's connectivity theorem [Mor05] we can construct a weight structure on $\mathcal{SH}(k)^{\mathrm{cell}}$ with $\mathcal{SH}(k)^{\mathrm{cell}}_{\geq 0}$ given by the objects X with $\pi_{p,q} X = 0$ for $p < q$. The heart of this weight structure is given by filtered colimits of sums of copies of $\mathbb{S}^{n,n}$.

Lemma 1.5. *The object* $\mathrm{M}\mathbb{Z}$ *representing motivic cohomology is cellular, locally compact and locally dualizable.*

Proof. From [DI05, Theorem 6.4] we know that MGL is cellular and using the Hopkins–Morel theorem (as in [DI05, Remark 6.6]) we may conclude that $\mathrm{M}\mathbb{Z}$ is cellular as well. We also have an explicit description of the cells used to build these objects which allows us to conclude. □

At this point we find that

$$\mathrm{M}\mathbb{Z} \otimes \mathrm{Cof}(\mathbb{S} \to \mathbb{S}_p) \simeq \mathrm{Cof}(\mathrm{M}\mathbb{Z} \to \mathrm{M}\mathbb{Z} \otimes \mathbb{S}_p) \simeq \mathrm{Cof}(\mathrm{M}\mathbb{Z} \to \mathrm{M}\mathbb{Z}_p)$$

which allows us to provide our counterexample while working at the level of motivic cohomology.

2. Completing an elliptic curve

Let E be an elliptic curve over k. The basepoint provides a splitting in the stable motivic category into a reduced suspension motive $\Sigma^\infty E$ and a copy of the unit. We will let $\bar{E}$ denote this reduced suspension.

Theorem 2.1. *Let k be an algebraically closed field of characteristic zero, then the p-completion of the unit in $\mathcal{SH}(k)$ is not cellular.*

Proof. We begin by recalling some information about the motivic cohomology of an elliptic curve. We display this information in the diagrams below where the horizontal axis contains the pure groups (i.e. degrees $2n, n$ in which the Chow groups appear) and the vertical axis contains the Tate twists $\mathbb{S}^{0,n}$.

$\mathbb{S}$				$\mathbb{S}_p$		
0		0		0		0
	$\mathbb{Z}$				$\mathbb{Z}_p$	
$k^\times$		0		0		0

$\bar{E}$						$\bar{E}_p$				
0		0		0		0		0		0
	$E(k)$		$\mathbb{Z}$				0		$\mathbb{Z}_p$	
?		$k^\times$		0		?		$T_p(E)$		0

The identifications of the uncompleted groups can be extracted from [MVW06]. The completed groups are obtained from these by p-completion. The appearance of the Tate module of the elliptic curve can be traced to the inclusion of the p-divisible group of E into $E(k)$, the k-points of E.

We will now argue that the way p-completion affects the homotopy groups of $\mathbb{S}$ and E differently precludes the possibility that $\mathbb{S}_p$ is cellular. Assume that $\mathbb{S}_p$ were cellular. Now we let X denote $\mathrm{Cof}(\mathbb{S} \to \mathbb{S}_p) \otimes \mathrm{M}\mathbb{Z}$ (which would also be cellular). From the above we know that $\pi_{p,q}X$ is zero when $p < 2q$ or $p < q$ and $\pi_{0,0}X$ is rational vector space of uncountable dimension while $\pi_{0,0}(\bar{E} \otimes X)$ is zero.

Using the weight structure of Example 1.4 we conclude that X must have a weight complex whose associated graded consisting of spheres $\mathbb{S}^{p,q}$ with $p \geq 2q$ and $p \geq q$. Since X is rational we can then tensor this with the rationals without changing the result. Comparing the associated spectral sequences for X and $\bar{E} \otimes X$ we obtain an isomorphism

$$\pi_{0,0}(\bar{E} \otimes X) \cong E(k) \otimes \pi_{0,0}(\bar{E} \otimes X)$$

which means this group is in particular non-zero since $E(k)$ contains many copies of $\mathbb{Q}$[1]. However, this is in contradiction with our earlier computation that $\pi_{0,0}(\bar{E} \otimes X) = 0$. $\square$

Remark 2.2. Based on various finiteness conjectures for algebraic K-theory in arithmetic settings we speculate that this (counter)example has more to do with the fact that an algebraically closed field is "too large of a base" than anything else.

Remark 2.3. In practice this result doesn't actually cause many issues since one may simply use the notion of p-completion internal to the category of cellular objects.

References

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[1]Here we have used our characteristic zero hypothesis to conclude that E has a non-torsion point and our algebraic closure condition to extend this to a copy of $\mathbb{Q}$.