Math 103, Spring 2007 Extra Problem
Due: Friday, March 9th

The third of Littlewood's three principles says:

"Every convergent sequence of measurable functions is nearly uniformly convergent."

One mathematically rigorous incarnation of this statement is known as Egoroff's theorem.

1. Prove the following weak form of Egoroff's theorem:

   Let E be a measurable set (finite measure), and f_n a sequence of measurable functions defined on E such that, for each $x \in E$, $f_n(x) \to f(x)$, where f is a real-valued function. Then show that given any $\epsilon, \delta > 0$, there exists a measurable set $A \subset E$ with $\mu(A) < \delta$ and an integer N such that, for all $x \notin A$, and all $n \geq N$,

   $$|f_n(x) - f(x)| < \epsilon.$$

   (**Hint:** Consider the sets G_n consisting of points where f_n and f differ by at least ϵ.)

2. Give an example that shows the assumption $\mu(E) < \infty$ is necessary in the above result.

3. Prove Egoroff's theorem:

   Let E be a measurable set (finite measure), and f_n a sequence of measurable functions defined on E such that, for each $x \in E$, $f_n(x) \to f(x)$, where f is a real-valued function. Then given any $\eta > 0$, there exists a measurable set $A \subset E$ with $\mu(A) < \eta$ such that f_n converges uniformly to f on $E - A$.

   (**Hint:** Use the weak form of Egoroff's theorem repeatedly, with clever choice of ϵ's and δ's.)