Analytic Continuation for Cubic Multiple Dirichlet Series

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Contents

Chapter 1

Introduction

1.1 History and Background

Over the past twenty years, my advisor, Jeffrey Hoffstein of Brown University, in collaboration with Daniel Bump, Solomon Friedberg, Dorian Goldfeld and others, has made a careful study of objects we now refer to as "multiple Dirichlet series." These are Dirichlet series of the form

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \frac{a(m_1, \ldots, m_k)}{m_1^{s_1} \cdots m_k^{s_k}}.$$

We often think of this object as a Dirichlet series in one of the variables m_i whose coefficients are Dirichlet series in several other variables. The first example of such a series was originally investigated by C.L. Siegel in 1956 (cf. [25]). He notes that the Fourier coefficients of a half-integral weight Eisenstein series are roughly quadratic L-functions $L(s, \chi_d)$ where χ_d is the quadratic character associated to $\mathbb{Q}(\sqrt{d})$. Upon taking the Mellin transform of the Fourier coefficients, he obtains a series of approximate form

$$\sum_{d} \frac{L(s, \chi_d)}{d^w}, \tag{1.1}$$

and suggests that this series should be viewed as a function of two complex variables.

Somewhat more precisely, if d_0 is a square-free integer, we can associate a primitive quadratic character of conductor d_0. Then extend the definition of the L-series to all integers d by writing $d = d_0 d_1^2$ with d_0 square-free and define

$$L(s, \chi_d) = L(s, \chi_{d_0}) P(s, d)$$

where the $P(s, d)$ are Dirichlet polynomials which complete the primitive L-series in order to preserve functional equations for $L(s, \chi_d)$ as $s \to 1-s$. These polynomials are precisely the additional arithmetic terms in the Fourier coefficients of Siegel's Eisenstein series and its Mellin transform yields the appropriate definition of (1.1).

Now properly defined, the above multiple Dirichlet series takes the form

$$Z(s, w) = \sum_{d=1}^{\infty} \frac{L(s, \chi_d)}{d^w} = \sum_{d=1}^{\infty} \frac{L(s, \chi_{d_0}) P(s, d)}{d^w},$$

where $d = d_0 d_1^2$. This series also obeys a rather surprising additional functional equation when the order of summation in the series is reversed: $Z(s, w) = Z(w, s)$. Some care does need to be taken over bad primes to formulate this precisely. We can now write down two exact functional equations for the object $Z(s,w)$. They are

$$(s,w) \to (1-s, w+s-1/2) \quad \text{and} \quad (s,w) \to (w,s)$$

These two transformations generate a finite group of functional equations isomorphic to the dihedral group of order 12. Interestingly, we can take the above properties as axioms and solve for the correction factors $P(s,d)$. One can show that these conditions uniquely determine the Dirichlet polynomials $P(s,d)$. This is truly miraculous as the correction factors seem to be satisfying so many desirable conditions at once.

Classical growth estimates for L-series give a region of absolute convergence for this multiple Dirichlet series. Applying functional equations to $Z(s,w)$ transforms this domain of convergence into a new domain, which has a non-empty intersection with the original. This provides an analytic continuation to the union of the original domain and its translates. The resulting region of convergence has convex hull equal to the entire space $\mathbb{C}^2$, so by Hartog's theorem, we obtain a continuation to all of $\mathbb{C}^2$. The process of continuation described above was first observed by Bump, Friedberg, and Hoffstein. This raises the natural question: which multiple Dirichlet series admit such an analytic continuation to the entire complex space?

In response to this question, generalizations of the above were done in [3] and [4]. There, the numerators are quadratic twists of L-series associated to automorphic forms on $GL(2)$ or $GL(3)$. In particular, they observed that in these cases (i.e., $GL(m)$ with $m = 1,2,3$), the completed multiple Dirichlet series possesses a finite group of functional equations which permit the continuation to all of $\mathbb{C}^2$ and this set of functional equations uniquely determines the form of the correction factor $P(s,d)$. However, for $m \geq 4$, the group of functional equations is infinite and the uniqueness principle is lost, corresponding to an inability to analytically continue beyond a line of essential singularities.

These methods have a number of applications. Standard Tauberian techniques (cf. [12]) may be applied to these series to obtain information about the L-series in the numerator at the center of the critical strip, which give distribution information about important arithmetic quantities. Non-vanishing results for quadratic twists of $L(1/2, f, \chi_d)$ were also obtained for f, an arbitrary automorphic form on $GL(m)$ for $m = 2,3$. In the cases $m \geq 4$, the prospect of continuing other multiple Dirichlet series past this line of singularities leads to conjectures about mean values of products of L-series and in particular, would imply the moment conjectures of Conrey, Ghosh, Keating, and Snaith (cf. [7] and [5]). Lastly, continuation in the $GL(3)$ case gives a new proof of the holomorphy of the symmetric square L-function of an automorphic form on $GL(3)$, as it is the residue of this multiple Dirichlet series at $w = 1$ (up to particular zeta factors).

This doctoral thesis addresses the analytic continuation of the multiple Dirichlet series

$$\sum_d \frac{L(s_1, \chi_{d_0}) L(s_2, \chi_{d_0}) P(s_1, s_2, d)}{\mathbb{N}d^w} \tag{1.2}$$

where the sum is over a restricted set of algebraic integers d in K, a number field containing the cubic roots of unity, and χ_{d_0} is a cubic character associated to the primitive (cube-free) part d_0 of d. (The precise statement of the result is included in the next section.) The major obstacle to continuation is the determination of the appropriate correction factor $P(s_1, s_2, d)$. This is complicated by the fact that such a numerator has not yet been realized as the Fourier coefficient of a metaplectic Eisenstein series, so its determination rests solely on an appropriately defined axiomatic approach. Moreover, these Dirichlet polynomials don't have a functional equation in the variables s_i, as in the quadratic case. The functional equations for the cubic L-series as $s_i \to 1-s_i$ for $i = 1, 2$ introduce a cubic Gauss sum (whereas, for quadratic characters, the Gauss sum was just the positive root of the associated conductor), so the transformed object is of an essentially different form. As such, the determination of these factors $P(s_1, s_2, d)$ requires all the information able to be gleaned from the functional equations and previously known one-variable cases. This is achieved in the thesis and the finite group of functional equations corresponding to this multiple Dirichlet series is exhibited. Previous attempts to determine this correction factor had proven too unwieldy to exhibit exact functional equations; the new methods of the thesis greatly reduce such computations.

Many of the same applications carry over to this case. Using a Tauberian theorem, we can obtain the first known asymptotics for the second moment of cubic L-series. Previously, these had not even been conjectured. We include a proof of these mean-value estimates at the end of the thesis. The same correction factors with variables specialized so that $s_1 = s_2$ can also be used to complete the multiple Dirichlet series whose numerator is a $GL(2)$ automorphic form twisted by a cubic character. Upon taking the residue at $w = 1$, we obtain the symmetric cube L-function of a $GL(2)$ automorphic form, and a new proof of the holomorphy of this object can be derived from the analytic continuation of the multiple Dirichlet series [1]. The first such proof was obtained by Shahidi and Kim [16] in 1999.

1.2 Statement of Results

The main result of the thesis is the following.

Theorem 1.1. *Let $K = \mathbb{Q}(\sqrt{-3})$ with ring of integers $\mathcal{O}_K$. Given an integer $d \in \mathcal{O}_K$, write $d = d_1 d_2^2 d_3^3$ with d_1 and d_2 cube-free. Let $\chi_{d_0} = \chi_{d_1}\bar{\chi}_{d_2}$ denote the product of cubic residue characters with conductor $d_1 d_2$. Let ψ_1 and ψ_2 be primitive cubic Hecke characters of a fixed conductor $N|9$. Define the function*

$$Z_1(s_1, s_2, w; \psi_1, \psi_2) = \sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\,(3)}} \frac{L(s_1, \chi_{d_0}\psi_1) L(s_2, \chi_{d_0}\psi_1)\psi_2(d) P(s_1, s_2; d, \psi_1)}{\mathbb{N}d^w}$$

where $P(s_1, s_2; d, \psi_1)$ is a certain finite, Eulerian Dirichlet polynomial in two variables s_1 and s_2 depending only on the indicated quantities. Then, letting $\Lambda(s) = (2\pi)^{-s}\Gamma(s)\zeta_K(s)$, the function

$$\begin{aligned} Z_1^*(s_1, s_2, w; \psi_1, \psi_2) \stackrel{def}{=} & \, 4(2\pi)^{-s_1-s_2}\Gamma(s_1)\Gamma(s_2)\Gamma(w)\Lambda(3w + 3s_1 - 2)\Lambda(3w + 3s_2 - 2) \\ & \Lambda(3s_1 + 3s_2 + 6w - 5)\Lambda(6w + 6s_1 + 6s_2 - 8) Z_1(s_1, s_2, w; \psi_1, \psi_2) \end{aligned}$$

has a meromorphic continuation to a region of $\mathbb{C}^3$ containing the point $(1/2, 1/2, 1/2)$.

Moreover, the function $Z_1(s_1, s_2, w; 1, 1)$ is analytic in this region except for the following 18 polar planes:

$$\begin{array}{llll} s_1 = 1, & s_1 = 0, & s_1 + 2s_2 + 2w - 3 = 0, & s_1 + 2s_2 + 2w - 2 = 0, \\ s_2 = 1, & s_2 = 0, & 2s_1 + s_2 + 2w - 3 = 0, & 2s_1 + s_2 + 2w - 2 = 0, \\ w = 1, & w = 0, & 2s_1 + 2s_2 + 3w - 3 = 0, & 2s_1 + 2s_2 + 3w - 4 = 0, \end{array}$$

$$\begin{aligned} w+s_1+s_2-5/3=0, &\quad w+s_1+s_2-4/3=0,\\ w+s_1-7/6=0, &\quad w+s_1-5/6=0,\\ w+s_2-7/6=0, &\quad w+s_2-5/6=0. \end{aligned}$$

The determination of both the Dirichlet polynomial $P(s_1, s_2; d, \psi_1)$ and the polar planes are intimately related to the method of analytic continuation. The characters ψ_1 and ψ_2 are used to correct the theory at bad primes (namely, those dividing 9 in $\mathcal{O}_K$). Then ignoring a finite number of bad primes, the reader may safely assume that these characters are trivial upon the first reading and still retain all of the important analytic number theory content.

1.3 Outline of Methods: The Square-Free Heuristic

We will take great care in formulating this result precisely over the following chapters. In particular, Chapter 2 contains all of the important definitions of the objects under consideration. Here, we sketch the basic outline of the argument, postponing all of the technical details for the main body of the work. As such, we will neglect notating all congruence conditions in our sums and refrain from writing down Gamma factors associated to L-series with perfect functional equations.

The method of multiple Dirichlet series is used to accomplish the continuation. Roughly, the method uses a group of functional equations among Dirichlet series to extend the region of absolute convergence for these series. As a first step, we need to understand the collection of possible functional equations.

Our initial object $Z_1(s_1, s_2, w; \psi_1, \psi_2)$ inherits functional equations from the L-series in the numerator. These functional equations behave well for primitive characters χ_{d_0}. In particular, if we restrict our attention to square-free integers and make further simplifying assumptions that our sums are over relatively prime integers and there are no bad primes, then all functional equations should be as nice as possible and we should be able to get a simplified view of the essential picture from them. We call this collection of assumptions, and the simplified picture they provide, the "square-free heuristic." We use this extensively throughout the thesis to motivate our constructions. (Of course, most of the work will go into eventually removing these assumptions.) Using the "square-free heuristic" where all our simplifying assumptions hold, then $Z_1(s_1, s_2, w)$ can be manipulated as follows.

$$\sum_d \frac{L(s_1,\chi_d\psi_1)L(s_2,\chi_d\psi_1)\psi_2(d)}{\mathbb{N}d^w} =$$

$$\begin{aligned}
&= \sum_d \frac{L(1-s_1,\overline{\chi_d\psi_1})L(1-s_2,\overline{\chi_d\psi_1})\psi_2(d)\mathbb{N}d^{1-s_1-s_2}G_3^2(1,d)}{\mathbb{N}d^w}\\
&= \sum_{d,m,n} \frac{\bar\chi_d(m)\bar\psi_1(m)\bar\chi_d(n)\bar\psi_1(n)\psi_2(d)G_3^2(1,d)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^{w+s_1+s_2-1}}\\
\text{(by Davenport-Hasse relation)} &= \sum_{d,m,n} \frac{\overline{G_6(1,d)}\bar\chi_d(mn)\bar\psi_1(m)\bar\psi_1(n)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^{w+s_1+s_2-1}}\\
&= \sum_{m,n,d} \frac{\overline{G_6(m^2n^2,d)}\;\overline{\psi}_1(m)\overline{\psi}_1(n)\psi_2(d)}{\mathbb{N}m^{1-s_1}\mathbb{N}n^{1-s_2}\mathbb{N}d^{w+s_1+s_2-1}}\\
&\stackrel{\text{def}}{=} Z_6(1-s_1,1-s_2,w+s_1+s_2-1;\overline{\psi}_1,\psi_2)
\end{aligned}$$

where $G_n(m,d)$ denotes the normalized n^{th} order Gauss sum:

$$G_n(m,d) = \frac{1}{\sqrt{\mathbb{N}d}} \sum_{r\ (d)} \left(\frac{r}{d}\right)_n e\left(\frac{mr}{d}\right).$$

We have decided to call this new series Z_6 according to the sixth order Gauss sum. Removing the characters ψ_i for clarity of notation, we may write

$$Z_6(s_1, s_2, w) = \sum_{m,n,d} \frac{\overline{G_6(m^2n^2, d)}}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^w}$$

Reversing the order of summation so that the inner sum is over integers d, then for fixed m and n, the inner sum is seen to be the $(m^2n^2)^{th}$ Fourier coefficient of a metaplectic Eisenstein series defined on the six-fold cover of an appropriate subgroup of $GL(2)$ (cf. [18] or [13] for a detailed survey of metaplectic Eisenstein series). Owing to the automorphy of the Eisenstein series, the inner sum inherits a functional equation whose properties were detailed by Kazhdan and Patterson in [17]. Applying this in the case of sixth order series,

$$\sum_d \frac{\overline{G_6(m,d)}}{d^w} \stackrel{\text{def}}{=} D_6(w;m) = D_6(1-w;m)\mathbb{N}m^{1/2-w}$$

so we expect a functional equation for Z_6 into itself of form

$$Z_6(s_1, s_2, w) = Z_6(s_1 + 2w - 1, s_2 + 2w - 1, 1 - w).$$

Additionally, we can interchange the order of summation in the original object Z_1 and we have, under the square-free heuristic, that

$$\begin{aligned}
\sum_d \frac{L(s_1, \chi_d\psi_1)L(s_2, \chi_d\psi_1)\psi_2(d)}{\mathbb{N}d^w} &= \sum_{d,m,n} \frac{\chi_d(mn)\psi_1(m)\psi_1(n)\psi_2(d)}{\mathbb{N}d^w\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} \\
\text{(by cubic recip.)} &= \sum_{d,m,n} \frac{\chi_{mn}(d)\psi_1(m)\psi_1(n)\psi_2(d)}{\mathbb{N}d^w\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} \\
\text{(summing over } d) &= \sum_{m,n} \frac{L(w, \chi_{mn}\psi_2)\psi_1(m)\psi_1(n)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}}.
\end{aligned}$$

From this altered form, we see that $Z_1(s_1, s_2, w; \psi_1, \psi_2)$ has an additional functional equation as $w \to 1 - w$. Performing this, we roughly obtain

$$\begin{aligned}
\sum_{m,n} \frac{L(w, \chi_{mn}\psi_2)\psi_1(m)\psi_1(n)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} &= \sum_{m,n} \frac{L(1-w, \overline{\chi_{mn}\psi_2})\mathbb{N}(mn)^{1/2-w}G_3(1,mn)\psi_1(m)\psi_1(n)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} \\
&= \sum_{m,n,d} \frac{G_3(d, mn)\psi_1(m)\psi_1(n)\overline{\psi}_2(d)}{\mathbb{N}m^{w+s_1-1/2}\mathbb{N}n^{w+s_2-1/2}\mathbb{N}d^{1-w}} \\
&\stackrel{\text{def}}{=} Z_3(w + s_1 - 1/2, w + s_2 - 1/2, 1 - w; \psi_1, \psi_2)
\end{aligned}$$

The resulting function is labeled Z_3 according to the cubic Gauss sum in the numerator.

Again dismissing the additional characters ψ_i for the moment, we may rewrite Z_3 as

$$\begin{aligned} Z_3(s_1, s_2, w) &= \sum_{d,m,n} \frac{G_3(d, mn)}{\mathbb{N}d^w \mathbb{N}m^{s_1} \mathbb{N}n^{s_2}} \\ \text{(writing } M = mn\text{)} &= \sum_{\substack{d,M \\ M=mn}} \frac{G_3(d, M)\sigma_{s_1-s_2}(M)}{\mathbb{N}d^w \mathbb{N}M^{s_1}} \\ \text{(more symmetrically)} &= \sum_{\substack{d,M \\ M=mn}} \frac{\chi_d(M) G_3(1, M)\sigma_{s_1-s_2}(M)\mathbb{N}M^{(s_1-s_2)/2}}{\mathbb{N}M^{(s_1+s_2)/2}} \end{aligned}$$

where $\sigma_k(M)$ is the usual divisor function $\sum_{n|M} n^k$. Recall that the cubic theta function, realized as a residue of a metaplectic Eisenstein series on the three-fold cover of $GL(2)$ (cf. [23]), has as its m^{th} Fourier coefficient

$$\tau(m) = \begin{cases} \overline{G(1, m_1)} m_3^{1/2} & \text{if } m = m_1 m_3^3,\ m_1 \text{ square-free} \\ 0 & \text{otherwise.} \end{cases}$$

Then substituting this back into the series $Z_3(s_1, s_2, w)$ using the square-free heuristic, we obtain

$$Z_3(s_1, s_2, w) = \sum_{\substack{d,M \\ M=mn}} \frac{\chi_d(M)\overline{\tau(M)}\sigma_{s_1-s_2}(M)\mathbb{N}M^{(s_1-s_2)/2}}{\mathbb{N}M^{(s_1+s_2)/2}}$$

which can be regarded as the twisted Rankin-Selberg convolution of a cubic theta function and an ordinary (non-metaplectic) Eisenstein series $E(z, (s_1 - s_2 + 1)/2)$ (recalling that the Fourier coefficients of Eisenstein series are just divisor sums). As such, we expect a functional equation for Z_3 into itself of form

$$Z_3(s_1, s_2, w) = Z_3(1 - s_1, 1 - s_2, w + 2s_1 + 2s_2 - 2)$$

since the convolution of two $GL(2)$ automorphic forms is a $GL(4)$ automorphic form so we have the resulting shift in the w variable.

The set of these transformations just discussed can be summarized in the following diagram:

$$\begin{array}{rcll} Z_1(s_1, s_2, w) & \xleftrightarrow{(1-s_1,1-s_2,w+s_1+s_2-1)} & Z_6(s_1, s_2, w) \circlearrowleft & (s_1+2w-1, s_2+2w-1, 1-w) \\ \text{interchange} \ \Big\| & & & \\ Z_1(s_1, s_2, w) & \xleftrightarrow{(s_1+w-1/2, s_2+w-1/2, 1-w)} & Z_3(s_1, s_2, w) \circlearrowleft & (1-s_1, 1-s_2, w+2s_1+2s_2-2) \end{array}$$

These transformations are both involutions and so we may read two equalities from each arrow of the diagram. For example, the top arrow gives both $Z_1(s_1, s_2, w) = Z_6(1 - s_1, 1 - s_2, w + s_1 + s_2 - 1)$ and $Z_6(s_1, s_2, w) = Z_1(1 - s_1, 1 - s_2, w + s_1 + s_2 - 1)$.

The eventual goal is to use these functional equations to obtain an analytic continuation. We can determine regions of absolute convergence for each of these objects in the diagram using classical estimates. Then the functional equations transform these regions of convergence for Z_3 and Z_6 into regions of convergence for Z_1. The collection of these transformed regions not only have non-empty intersection, but also have a convex hull equal to all of $\mathbb{C}^3$. By the convexity principle for analytic functions of several complex variables, this is enough to show that our function Z_1 (and in fact Z_3 and Z_6 as well) is meromorphic over all of $\mathbb{C}^3$. Further, we know enough about the naturally occurring poles of each of the objects Z_1, Z_3, and Z_6 so that we can keep track of poles by applying these same functional equations to the equations of the polar planes coming from each object. This is how we arrive at the list of polar planes in the statement of the theorem.

The major obstacle to implementing the above strategy is the lack of exact functional equations. We have relied heavily on the assumptions of the square-free heuristic in all of the above arguments. For example, we would like our sum for Z_1 to be defined over all integers d, but the L-series transform according to the conductor of the character χ, which must be a cube-free integer. In interchanging the order of summation above, reciprocity led to perfect objects only after assuming that all integers were relatively prime and dismissing necessary congruence conditions. It is the Dirichlet polynomials added to our object which turn our rough outline of transformations into bona fide functional equations. In fact, the coefficients of these correction polynomials are determined by requiring that such functional equations and interchanging summation are exact. We will spend great effort in detailing this process in the following pages. Upon completion, we will find that the exact objects take form:

$$Z_1(s_1, s_2, w; \psi_1, \psi_2) = \sum_d \frac{L(s_1, \chi_{d_0}\psi_1)L(s_2, \chi_{d_0}\psi_1)\psi_2(d)P(s_1, s_2; d, \psi_1)}{\mathbb{N}d^w}$$

where the correction factor P begins

$$P(s_1, s_2; d, \psi_1) = \prod_{p^\alpha || d_3} [1 - \chi_{d_0}(p)(\psi_1(p)\mathbb{N}p^{-s_1} + \psi_1(p)\mathbb{N}p^{-s_2}) + \chi^2(p)\psi_1^2(p)\mathbb{N}p^{-s_1-s_2} + \cdots$$
$$\cdots + a_{i,j}(d_0, p^\alpha) + \cdots]$$

Similarly, turning to the other form of the object Z_1 after interchange of summation. We will write the product $mn = \underline{mn}_0\ \underline{mn}_3^3$ where $\underline{mn}_0$ denotes the cube-free part of the product. Then

$$Z_1(s_1, s_2, w; \psi_1, \psi_2) = \sum_{m,n} \frac{L(w, \chi_{\underline{mn}_0}\psi_2)\psi_1(m)\psi_1(n)Q(w; m, n, \psi_2)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}}$$

where the correction factor Q begins

$$Q(w; m, n, \psi_2) = \prod_{p^\beta || \underline{mn}_3} [1 - \chi_{\underline{mn}_0}(p)\psi_2(p)\mathbb{N}p^{-w} + \mathbb{N}p^{2-3w} + \delta_3(m, n)\mathbb{N}p^{3-3w} + \cdots$$
$$\cdots + b_k(m, n)\mathbb{N}p^{-kw} + \cdots]$$

and (writing $(m, n) < (a, b)$ if either $m < a$ or $n < b$),

$$\delta_3(m, n) = \begin{cases} 2 & \text{if } (m, n) \geq (3, 3) \\ 1 & \text{if } (2, 3) \leq (m, n) < (3, 3) \text{ or } (3, 2) \leq (m, n) < (3, 3) \\ 0 & \text{otherwise} \end{cases}$$

The existence of this δ_3 function in the above expression already attests to the combinatorial complexity of these correction polynomials. We give these explicit first terms to orient the reader to our set-up. However, we will not be able to give a closed expression for their form. Instead we resort to other methods that will be described at the end of the introduction.

1.4 Additional Functional Equations and the Determination of Polar Planes

Because Z_1, using either order of summation, contains L-series in the numerator with arguments s_1 and s_2 or w, then it has poles whenever any of these arguments take the value 1. Now we need to determine all of the functional equations of Z_1 into itself. If we reflect these polar planes at $s_i = 1$ and $w = 1$ according to these transformations, then we will determine all the poles associated to the L-series in the numerator. Recall that we presented the diagram earlier:

$$\begin{array}{rcccl} & Z_1(s_1,s_2,w) & \xleftrightarrow{(1-s_1,1-s_2,w+s_1+s_2-1)} & Z_6(s_1,s_2,w) & \circlearrowleft (s_1+2w-1,s_2+2w-1,1-w) \\ \text{interchange} & \Big\| & & & \\ & Z_1(s_1,s_2,w) & \xleftrightarrow{(s_1+w-1/2,s_2+w-1/2,1-w)} & Z_3(s_1,s_2,w) & \circlearrowright (1-s_1,1-s_2,w+2s_1+2s_2-2) \end{array}$$

But Z_6 contains a sixth order Gauss sum in the numerator, so its cube-free part is essentially the Dirichlet series associated to an Eisenstein series on the 6-fold cover of $GL(2)$. That is, the numerator took the form $G(m^2n^2, d)$. Thus according to the Selberg theory, as a sum over integers d, it has a functional equation as $w \to 1 - w$ and poles at $w = 1/2 + 1/6$. Similarly, Z_3 contains a cubic Gauss sum in the numerator and its cube-free part is essentially the Mellin transform of a Rankin-Selberg convolution of a twisted cubic theta function and a non-metaplectic Eisenstein series $E(z, (s_1 - s_2 + 1)/2)$. As mentioned previously, this object has a $GL(4)$ functional equation as $s_i \to 1 - s_i$ so that $w \to w + 2s_1 + 2s_2 - 2$ and poles at $2s_i - 1/2 = 1/2 + 1/3$ according to the usual Fourier analysis which transforms the argument of the Eisenstein series. We will address these functional equations in detail later, but for now, we'd like to just list them and show how they lead to poles. In total, our functional equations are (up to natural zeta factors to be discussed later and up to finite correction polynomials which do not affect convergence):

$$\begin{aligned} A &: Z_1(s_1, s_2, w) = Z_6(1 - s_1, 1 - s_2, w + s_1 + s_2 - 1), \\ B &: Z_6(s_1, s_2, w) = Z_6(s_1 + 2w - 1, s_2 + 2w - 1, 1 - w), \\ C &: Z_1(s_1, s_2, w) = Z_3(s_1 + w - 1/2, s_2 + w - 1/2, 1 - w), \\ D &: Z_3(s_1, s_2, w) = Z_3(1 - s_1, 1 - s_2, w + 2s_1 + 2s_2 - 2). \end{aligned}$$

Each of these transformations is an involution. This implies that the total collection of functional equations of Z_1 into itself is described by the set of transformations $\{ABA, ABACDC, ABACDCABA, \ldots\}$ and $\{CDC, CDCABA, CDCABACDC, \ldots\}$. This produces the following list of functional equations from Z_1 into Z_1.

$$\begin{aligned} (s_1, s_2, w) &\longrightarrow (s_1 + 2s_2 + 2w - 2, 2s_1 + s_2 + 2w - 2, -2s_1 - 2s_2 - 3w + 4) \\ (s_1, s_2, w) &\longrightarrow (1 - s_1, 1 - s_2, 1 - w) \\ (s_1, s_2, w) &\longrightarrow (3 - s_1 - 2s_2 - 2w, 3 - 2s_1 - s_2 - 2w, -3 + 2s_1 + 2s_2 + 3w) \end{aligned}$$

Applying this set of functional equations to the polar planes $s_i = 1$ and $w = 1$ (the poles associated to Z_1 in its original form with L-series in the numerator) produces the first twelve poles in our list in theorem 1. That is, we obtain the polar planes

$$\begin{array}{llll} s_1 = 1, & s_1 = 0, & s_1 + 2s_2 + 2w - 3 = 0, & s_1 + 2s_2 + 2w - 2 = 0, \\ s_2 = 1, & s_2 = 0, & 2s_1 + s_2 + 2w - 3 = 0, & 2s_1 + s_2 + 2w - 2 = 0, \\ w = 1, & w = 0, & 2s_1 + 2s_2 + 3w - 3 = 0, & 2s_1 + 2s_2 + 3w - 4 = 0, \end{array}$$

We can similarly generate all of the functional equations of Z_6. They are given by the sets of transformations $\{B, BACDCA, \ldots\}$ and $\{ACDCA, ACDCAB, \ldots\}$. This yields transformations which all take $w \to 1 - w$. Z_6 now has both the original polar plane $w = 2/3$ and $1 - w = 2/3$. These polar planes translate to polar planes of Z_1 via the involution A taking $w \to w + s_1 + s_2 - 1$, producing the pair of planes:

$$w + s_1 + s_2 - 5/3 = 0, \quad w + s_1 + s_2 - 4/3 = 0.$$

Lastly, Z_3 functional equations into itself are given by the sets $\{C, CDABAD, \ldots\}$ and $\{DABAD, DABADC, \ldots\}$ and all take $s_i \to 1 - s_i$. Then the polar planes are given by $s_i = 2/3$ and $1 - s_i = 2/3$ and translated to Z_1 by the involution C taking $s_i \to 1 - s_i$. This produces the final four planes in our original list in Theorem 1. That is, we obtain the polar planes:

$$\begin{gathered} w + s_1 - 7/6 = 0, \quad w + s_1 - 5/6 = 0, \\ w + s_2 - 7/6 = 0, \quad w + s_2 - 5/6 = 0. \end{gathered}$$

1.5 Mean-Value Estimates for Cubic L-Series

We want to determine the second moment of cubic L-series at the center of the critical strip. To do this, we can set $s_1 = s_2 = 1/2$ and then determine the number of polar planes of $Z_1(s_1, s_2, w; 1, 1)$ which contain a point of form $(1/2, 1/2, w)$. The values of w which occur from these distinct planes will determine the asymptotic behavior of the second moment according to standard Tauberian techniques. Revisiting the list of polar planes listed in Theorem 1, the planes $s_1 = 0, s_1 = 1, s_2 = 0$, and $s_2 = 1$ are the only planes which do not include such a point $(1/2, 1/2, w)$. This leaves 14 remaining planes and one can check that they have w values

$$\{1, 3/4 \text{ (2 times)}, 2/3 \text{ (4 times)}, 1/3 \text{ (4 times)}, 1/4 \text{ (2 times)}, 0\}$$

Values of w which occur more than once indicate the possibility of a non-simple pole at $(1/2, 1/2, w)$. According to a Tauberian theorem (i.e., the method of contour integration using an appropriate smoothing function), this can increase the associated growth term at $X^{Re(w)}$ by a factor of $\log(x)^{k-1}$ where k is the multiplicity of the polar contribution at w. Hence, using a sieving argument similar to that of Diaconu, Goldfeld, and Hoffstein in [7], one can obtain estimates of the following form:

Theorem 1.2. *For d corresponding to primitive cubic character χ_{d_0},*

$$\sum_{|d| \leq X} (L(1/2, \chi_{d_0}))^2 P(1/2, 1/2, d) e^{-\mathbb{N}d/X} = c_1 X + c_2 X^{3/4} + c_3 X^{2/3} F(\log X) + O(X^{1/2+\epsilon})$$

where c_1, c_2, and c_3 are explicit constants, with c_1 and c_2 non-zero and F is a polynomial with $deg(F) \leq 3$.

Note that while the value $(1/2, 1/2, 3/4)$ occurs with multiplicity two in the above list of planes containing poles at $(1/2, 1/2, w)$, the residue at $w = 3/4$ of our object does not contain a singularity, so the pole is in fact simple. We also remark that since the continuation of our initial object was proved for any characters ψ_i, not just the trivial ones, we may recompute the poles of the initial object and transform them according to functional equations to form a different collection of polar planes. Then repeating the above procedure produces asymptotics for $L(1/2, \chi_d \psi)^2$ for some fixed character ψ.

1.6 Outline of Method for Determining the Correction Factor

As discussed earlier, we begin by determining all the natural functional equations of our initial object,

$$Z_1(s_1, s_2, w) = \sum_{d} \frac{L(s_1, \chi_{d_0})L(s_2, \chi_{d_0})P(s_1, s_2, d)}{\mathbb{N}d^w} = \sum_{m,n} \frac{L(w, \chi_{\underline{mn}_0})Q(w, m, n)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}}, \tag{1.3}$$

Note that the original object comes in two forms according to the order in which summation is performed and inherits natural functional equations from the L-series which occur in the numerator of each. Note that there are even series arising from transforming just one of the L-series according to the functional equation in either s_1 or s_2. The resulting series will be given the names

$$Z_4(s_1, s_2, w) \stackrel{\text{def}}{=} Z_1(1 - s_1, s_2, w + s_1 - 1/2)$$

and

$$Z_5(s_1, s_2, w) \stackrel{\text{def}}{=} Z_1(s_1, 1 - s_2, w + s_2 - 1/2),$$

respectively. They have similar square-free heuristics which lead to additional functional equations. We have refrained from mentioning them up to this point because their functional equations are not essential to the continuation.

While the picture we outlined earlier works beautifully under the assumptions of the square-free heuristic, it is unclear how we might go about solving for correction factors P and Q to make these functional equations exact for series summed over all integers. However, the series Z_3 and Z_6 obtained using the square-free heuristic and defined by

$$Z_3(s_1, s_2, w) = \sum_{\substack{m,n,d \\ \text{sq. free}}} \frac{G_3(d, mn)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^w} \quad \text{and} \quad Z_6(s_1, s_2, w) = \sum_{\substack{m,n,d \\ \text{sq. free}}} \frac{\overline{G_6(m^2n^2, d)}}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^w}$$

have very natural generalizations to sums over all integers (taking the sums to be unrestricted over integers). This presents a tantalizingly simple and elegant possible solution. We can begin with the series Z_6 summed over all integers and determine the correction factor $P(s_1, s_2, d)$ according to the transformation

$$Z_1(s_1, s_2, w) = Z_6(1 - s_1, 1 - s_2, w + s_1 + s_2 - 1).$$

Note that such a definition immediately gets us half of the functional equations we need since we automatically get the additional functional equation for Z_6 into itself. Moreover, such an approach can be shown to work in the slightly simpler case of a two variable Dirichlet series whose numerator consists of a single cubic Dirichlet L-series. Unfortunately, the correction factor determined by this definition does not behave well under interchange; the polynomial $Q(w; m, n)$ it determines does not satisfy the required properties so that it inherits additional functional equations.

However, we can make more modest gains by reducing from the three-variable situation to series containing two variables by taking limits as variables go to infinity. In Chapter 3, we use these two-variable series to solve for a certain subset of coefficients of the correction factors P and Q. Chapter 4 contains a detailed discussion of the restrictions forced on P and Q by the interchange equality (1.3).

In Chapter 5, we show that a natural definition exists for the series $Z_4(s_1, s_2, w)$ exists by building up from two-variable series. The definition we arrive at is similar enough to the square-free heuristics to guarantee the predicted additional functional equation of Z_4 into itself and also defines the correction factor $P(s_1, s_2, d)$ via the involution $Z_1(s_1, s_2, w) = Z_4(1 - s_1, s_2, w + s_1 - 1/2)$ with the appropriate properties to guarantee a well-behaved interchange for Z_1.

In Chapter 6, we show how this definition begets the additional anticipated functional equations for Z_4 and Z_6, and an asymmetric functional equation for Z_3 . The first, as we just mentioned, is almost automatic while the latter two will take considerable work. We then determine regions of absolute convergence for these series and finish the analytic continuation by mapping these regions according to our functional equations.

We conclude in Chapter 7 by describing one way in which we can apply this result to mean-value estimates using a Tauberian theorem by methods sketched earlier in the introduction.

Chapter 2

Algebraic Preliminaries and Definitions

2.1 Cubic Reciprocity and Gauss Sums

We begin by recalling some basic constructions of algebraic number theory which provide the appropriate setting to define cubic characters. Let $K = \mathbb{Q}(\sqrt{-3})$ with ring of integers $\mathcal{O}_{\mathcal{K}} = \mathbb{Z}[\omega]$ where $\omega = \frac{-1+\sqrt{3}}{2}$ is a primitive cubic root of unity. Recall that if $\mathfrak{a}$ is an ideal of $\mathcal{O}_K$ relatively prime to (3), then there is a unique generator $a \equiv 1$ (3) such that a can be decomposed as a product of "primary" primes (i.e. primes congruent to 1 mod 3).

Given any integer r and primary prime p in $\mathcal{O}_K$, let $\left(\frac{r}{p}\right)_3$ denote the cubic residue symbol in $\mathcal{O}_K$.

Theorem 2.1 (Cubic Reciprocity). *If p_1 and p_2 are primary in $\mathcal{O}_K$ with $\mathbb{N}p_1, \mathbb{N}p_2 \neq 3$, and $\mathbb{N}p_1 \neq \mathbb{N}p_2$, then*

$$\left(\frac{p_2}{p_1}\right)_3 \stackrel{def}{=} \chi_{p_1}(p_2) = \chi_{p_2}(p_1) \stackrel{def}{=} \left(\frac{p_1}{p_2}\right)_3$$

This result can be extended multiplicatively to all pairs of integers r, d such that $r, d \equiv 1$ (3). One can further check that $3 = -\omega^2(1-\omega)^2$ and $\mathcal{O}_K$ has a group of units $\mathcal{O}_K^\times$ with six elements $\{\pm 1, \pm\omega, \pm\omega^2\}$. By definition, we have $\chi_p(1) = \chi_p(-1) = 1$ since $(-1)^3 = -1$. Hence, the following result completely determines the character χ_d for $d \equiv 1$ (3).

Theorem 2.2 (Supplement to Cubic Reciprocity). *Suppose that p is a primary prime. Then*

$$\chi_p(\omega) = \begin{cases} 1 & \text{if } \mathbb{N}p \equiv 1 \ (9) \\ \omega & \text{if } \mathbb{N}p \equiv 4 \ (9) \\ \omega^2 & \text{if } \mathbb{N}p \equiv 7 \ (9) \end{cases}$$

$$\chi_p(1-\omega) = \omega^m \quad \text{where } m = \begin{cases} \frac{p-1}{3} & \text{if } p \in \mathbb{Q} \\ \frac{a-1}{3} & \text{if } p = a + b\omega \end{cases}$$

For a proof of these results, we refer the reader to section 9.3 of Ireland and Rosen's book [15]. Note that their discussion defines a primary prime to be congruent to 2 mod 3 for historical reasons. Their Proposition 9.3.5 shows that our definition is equivalent.

Definition 2.1. *For m, d integers in $\mathcal{O}_K$ with $(d,3) = 1$, define the cubic Gauss sum $g(m,d)$ by*

$$g(m,d) = \sum_{r\ (d)} \left(\frac{r}{d}\right)_3 e\left(\frac{mr}{d}\right)$$

where, for $z \in \mathbb{C}$, $e(z) = \exp(2\pi i(z + \bar{z}))$ denotes the usual additive character.

Gauss sums will play an important role in the following chapters. As our essential method is the interchanging of the order of summation, we will often decompose and recompose such sums. In particular, we will make extensive use of the following proposition.

Proposition 2.3. *Given integers m and d in $\mathcal{O}_K$, the cubic Gauss sum $g(m,d)$ possesses the following properties.*

- $g(m_1m_2, d) = \overline{\left(\frac{m_1}{d}\right)_3} g(m_2, d) \quad$ *if* $(m_1, m_2) = 1$.
- $g(m, d_1d_2) = \overline{\left(\frac{d_2}{d_1}\right)_3} \overline{\left(\frac{d_1}{d_2}\right)_3} g(m, d_1)g(m, d_2) \quad$ *if* $(d_1, d_2) = 1$.
- *For any* $\alpha, \beta \geq 0$,

$$g(p^\alpha, p^\beta) = \begin{cases} \phi(p^\beta) & \text{if } \alpha \geq \beta, \beta \equiv 0\ (3) \\ -\mathbb{N}p^{\beta-1} & \text{if } \alpha = \beta - 1, \beta \equiv 0\ (3) \\ \mathbb{N}p^{\beta-1} g(1,p) & \text{if } \alpha = \beta - 1, \beta \equiv 1\ (3) \\ \mathbb{N}p^{\beta-1} \overline{g(1,p)} & \text{if } \alpha = \beta - 1, \beta \equiv 2\ (3) \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{g(1,p)} = g(\bar{\chi}_p)$, the usual cubic Gauss sum with character $\bar{\chi}_p$

Proof. The first is immediate. The second follows from a Chinese Remainder Theorem argument. The third, also elementary, can be determined by a slight generalization of the discussion in Chapter 2 of Davenport [6]. □

We will find it convenient to work with a slightly adjusted Gauss sum from the one defined above.

Definition 2.2. *The normalized cubic Gauss sum $G(m,d)$ is defined by*

$$G(m,d) = \frac{1}{\sqrt{\mathbb{N}d}} g(m,d)$$

so that, in particular, $|G(1,p)| = 1$.

Note that Proposition 3 allows us to take any Gauss sum $g(m,d)$ and compute its value up to a unit $G(1, d_0) = G(\chi_{d_0})$. The distribution of these values was originally wrongly conjectured by Kummer and has a long history (see the "Notes" section of Chapter 9 in [15]). A precise expression for $G(\chi_{d_0})$ is now known (cf. [21]), though it will not be important for us here.

Throughout the following chapters, an unindexed Gauss sum will contain a cubic character, though we will sometimes write $g_3(m,d)$ or $G_3(m,d)$ to emphasize this. We will also have occasion to study the sixth-order Gauss sum $g_6(m,d)$, which is similarly defined as

$$g_6(m,d) = \sum_{r\ (d)} \left(\frac{r}{d}\right)_6 e\left(\frac{mr}{d}\right) \quad \text{with} \quad \left(\frac{r}{d}\right)_6 \stackrel{\text{def}}{=} \left(\frac{r}{d}\right)_3 \left(\frac{r}{d}\right)_2.$$

The Hasse-Davenport Relates these Gauss sums of various residue characters of different order.

Proposition 2.4 (Special Case of the Davenport-Hasse Relation). *Given an integer $d \equiv 1$ (3), let $G_3(1,d)$ and $G_6(1,d)$ denote cubic and sixth-order normalized Gauss sums, respectively. Then*

$$\overline{G_6(1,d)} = \chi_d^{(3)}(2)^2 G_3(1,d)^2$$

where $\chi_d^{(3)}$ denotes the cubic residue symbol with modulus d.

Proof. We refer the reader to p. 61 of Lang's book [20] for a proof in full generality. To obtain the above, we use an alternate formulation given in [8]. The relation we need is given in the paper by $(**)$ with their $\tau(\chi)$ and $\tau(\psi)$ defined in our notation by $G_3(1,d)$ with cubic residue character $\chi_d^{(3)}$ and $G_2(1,d)$ with quadratic residue character $\chi_d^{(2)}$. $\square$

2.2 Defining Cubic Triple Dirichlet Series

We can now state the definitions of the basic multiple Dirichlet series to be studied in the subsequent chapters. Because the objects we consider are simultaneously packaging so much information, we will make several approximations to our ultimate Dirichlet series before offering a precise definition.

Definition 2.3. *Let $d \equiv 1$ (3) be an integer in $\mathcal{O}_K$ such that $(d,6)=1$. Writing $d = d_1 d_2^2 d_3^3$ with $d_1 d_2^2$ cube-free, we can associate to d the primitive character $\chi_{d_1}\bar{\chi}_{d_2} \overset{def}{=} \chi_{d_0}$. Further, define the cubic L-series associated to d by*

$$L_6(s,\chi_{d_0}) = \sum_{\substack{m \equiv 1\ (3) \\ (m,6)=1}} \frac{\chi_{d_0}(m)}{\mathbb{N}m^s} = \prod_{\substack{p \text{ primary} \\ p \nmid 6}} \left[1 - \chi_{d_0}(p)\mathbb{N}p^{-s}\right]^{-1}.$$

As we have seen in the last section, cubic reciprocity is properly formulated for pairs of integers m, d with m and d composed of primary primes. Then we can apply cubic reciprocity to characters occurring in the following restricted sum over integers:

$$Z(s_1,s_2,w) = \sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\ (3) \\ (d,6)=1}} \frac{L_6(s_1,\chi_{d_0})L_6(s_2,\chi_{d_0})}{\mathbb{N}d^w}$$

We will refer to this and all other similar series as "multiple" Dirichlet series owing to their dependence on several complex variables. All such Dirichlet series in this thesis should be summed over integers subject to the above three conditions, though for notational clarity we may sometimes omit several of the conditions. The condition that $(d,3)=1$ is clear from cubic reciprocity. The condition that $(d,2)=1$ results from considerations associated to the Davenport-Hasse relation. These will be alluded to in the next section and considered formally in Chapter 6.

While reciprocity works well on such a series, we pay a price for the restricted sums when trying to write down the functional equation for the imprimitive L-series $L_6(s,\chi_{d_0})$.

That is, we must reinsert missing Euler factors to obtain a perfect functional equation. Let $L^*(s,\chi) = (2\pi)^{-s}\Gamma(s)L(s,\chi)$ where $\Gamma(s)$ denotes the usual Gamma function. Then we have

$$\begin{aligned} L_6^*(s,\chi_{d_0}) &= L^*(s,\chi_{d_0})(1-\chi_{d_0}(1-\omega)3^{-s})(1-\chi_{d_0}(-2)4^{-s}) \\ &= L^*(1-s,\bar\chi_{d_0})G(1,d_1)\overline{G(1,d_2)}\ \cdot \\ &\qquad \cdot (1-\chi_{d_0}(1-\omega)3^{-s})(1-\chi_{d_0}(-2)4^{-s})(d_1d_2)^{1/2-s} \\ &= L_6^*(1-s,\bar\chi_{d_0})G(1,d_1)\overline{G(1,d_2)}(1-\chi_{d_0}(1-\omega)3^{-s})(1-\chi_{d_0}(-2)4^{-s})\ \cdot \\ &\qquad \cdot (1-\bar\chi_{d_0}(1-\omega)3^{-(1-s)})^{-1}(1-\chi_{d_0}(-2)4^{-(1-s)})^{-1}(d_1d_2)^{1/2-s} \end{aligned}$$

where $\chi_{d_0}(1-\omega)$ is defined according to the supplement to the law of cubic reciprocity. The above can also be re-expressed in the form

$$\begin{aligned} L_6^*(s,\chi_{d_0})(1-\chi_{d_0}(1-\omega)3^{-s})^{-1}(1-\chi_{d_0}(-2)4^{-s}) &= L_6^*(1-s,\bar\chi_{d_0})G(1,d_1)\overline{G(1,d_2)}\ \cdot \\ &\cdot\ (1-\bar\chi_{d_0}(1-\omega)3^{-(1-s)})^{-1}(1-\chi_{d_0}(-2)4^{-(1-s)})^{-1}(d_1d_2)^{1/2-s} \end{aligned}$$

But

$$\begin{aligned} (1-\chi_{d_0}(1-\omega)3^{-s})^{-1} &= 1+\chi_{d_0}(1-\omega)3^{-s}+\chi_{d_0}^2(1-\omega)3^{-2s}+\chi_{d_0}^3(1-\omega)3^{-3s}+\cdots \\ &= \frac{1}{1-3^{-3s}}\left[1+\chi_{d_0}(1-\omega)3^{-s}+\chi_{d_0}^2(1-\omega)3^{-2s}\right] \end{aligned}$$

since $\chi_{d_0}^3(1-\omega)=1$. We can do a similar rewriting for the Euler factor $(1-\chi_{d_0}(-2)4^{-s})^{-1}$. This suggests that the natural objects to begin with are finite linear combinations of L-series with Euler factor at 3 removed together with cubic characters as above.

Let

$$\psi_{1-\omega}(d) \stackrel{\text{def}}{=} \left(\frac{1-\omega}{d_0}\right) = \chi_{d_0}(1-\omega), \quad \psi_{-2}(d) \stackrel{\text{def}}{=} \left(\frac{-2}{d_0}\right) = \chi_{d_0}(-2)$$

According to the supplement to cubic reciprocity, $\psi_{1-\omega}$ can be extended to a cubic character in $(\mathcal{O}_K/9)^\times$. We will write Ψ' to denote the finite group of characters generated by the following set

$$\Psi' = \langle \psi_{1-\omega}, \psi_{-2}\rangle.$$

Then given $\psi \in \Psi'$, we want to study the adjusted multiple Dirichlet series

$$Z'(s_1,s_2,w) = \sum_{\substack{d\in\mathcal{O}_K \\ d\equiv 1\ (3) \\ (d,6)=1}} \frac{L_6(s_1,\chi_{d_0})L_6(s_2,\chi_{d_0})\psi(d)}{\mathbb{N}d^w}$$

Using this new notation, we may write the functional equation for each of the L-series in the numerator as

$$L_6^*(s,\chi_{d_0}) = \sum_{\psi\in\Psi'} c(s,\psi)L_6^*(1-s,\bar\chi_{d_0})G(1,d_1)\overline{G(1,d_2)}\psi(d)(d_1d_2)^{1/2-s}$$
$$(1-3^{-3(1-s)})^{-1}(1-4^{-3(1-s)})^{-1}$$

where the $c(s,\psi)$ are constants depending on the indicated quantities with $c(s,\psi) \ll 1$ on compact sets. We now make use of the reciprocity we have so carefully preserved for $Z'(s_1,s_2,w)$. Assume, for the following calculation, that our sums are taken over a collection of relatively prime, square-free integers so that reciprocity works perfectly for all pairs of integers. Then

$$\begin{aligned}
Z'(s_1, s_2, w) &= \sum_{\substack{d\in\mathcal{O}_K \\ d\equiv 1\ (3) \\ (d,6)=1}} \frac{L_3(s_1,\chi_d)L_3(s_2,\chi_d)\psi(d)}{\mathbb{N}d^w} \\
&= \sum_{\substack{d,m,n\in\mathcal{O}_K \\ d,m,n\equiv 1\ (3) \\ (dmn,6)=1}} \frac{\chi_d(mn)\psi(d)}{\mathbb{N}d^w\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} \\
&= \sum_{\substack{d,m,n\in\mathcal{O}_K \\ d,m,n\equiv 1\ (3) \\ (dmn,6)=1}} \frac{\chi_{mn}(d)\psi(d)}{\mathbb{N}d^w\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} \quad \text{using cubic reciprocity} \\
&= \sum_{\substack{m,n\in\mathcal{O}_K \\ m,n\equiv 1\ (3) \\ (mn,6)=1}} \frac{L_3(w,\chi_{mn}\psi)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}}
\end{aligned}$$

Now that we have interchanged the order of summation, we see that $Z'(s_1, s_2, w)$ should possess a functional equation as $w \mapsto 1-w$. Just as we saw with the functional equation in s_i, we will need to add characters from Ψ'. Moreover, to make the preceding set of equalities true over all integers, we add certain finite Dirichlet polynomials (or "correction factors" as we will often call them) to our series. The nature of these Dirichlet polynomials will occupy most of the discussion in what remains, so we postpone a rigorous explanation until the next chapter.

With these considerations, our first fundamental multiple Dirichlet series is defined as follows.

Definition 2.4. *Let ψ_1 and ψ_2 be cubic characters with fixed conductor N. Then a cubic triple Dirichlet series $Z_1(s_1, s_2, w)$ is a series of the form*

$$\begin{aligned}
Z_1(s_1, s_2, w) &= \sum_{\substack{d\in\mathcal{O}_K \\ d\equiv 1\ (3) \\ (d,N)=1}} \frac{L_3(s_1,\chi_d\psi_1)L_3(s_2,\chi_d\psi_1)\psi_2(d)P(s_1,s_2;d)}{\mathbb{N}d^w} \\
&= \sum_{\substack{m,n\in\mathcal{O}_K \\ m,n\equiv 1\ (3) \\ (mn,N)=1}} \frac{L_3(w,\chi_{mn}\psi_2)\psi_1(m)\psi_1(n)Q(w;m,n)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}}
\end{aligned}$$

for finite Dirichlet polynomials P and Q depending on the indicated variables and parameters, resp., as well as the choice of ψ_i.

2.3 Fourier Coefficients of Metaplectic Eisenstein Series

We have one final object to define. As presented in the introduction and revisited in the next chapter more formally, the functional equations $s_i \mapsto 1 - s_i$ lead to Dirichlet series of essential form

$$\sum_{m,n,d} \frac{G(m^i n^j, d)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^w} = \sum_{m,n} \frac{1}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} \sum_d \frac{G(m^i n^j, d)}{\mathbb{N}d^w} \quad \text{for } i, j \in \{1, 2\}. \tag{2.1}$$

The inner sum is realized as the Fourier coefficient (of index $m^i n^j$) of a metaplectic Eisenstein series (cf. [13]). The important consequence for us is that the above multiple Dirichlet series, as the Fourier coefficient of an automorphic form, inherits an additional functional equation as $w \mapsto 1-w$. This has been studied in [23] and [17]. According to the law of cubic reciprocity, this twisted Eisenstein series is properly defined on the triple cover of $\Gamma(3)$, the principal congruence subgroup of $SL_2(\mathbb{Z})$ consisting of matrices congruent to the identity matrix mod 3. The complex upper half-plane $\mathcal{H}$ modulo the usual action of $\Gamma(3)$ by linear fractional transformations can be regarded as a compact Riemann surface after adding a finite number of points called "cusps" (to evoke the singular behavior of the surface there). Each cusp has an associated Eisenstein series and the collection of series are essentially translates of each other by fractional linear transformations. Functional equations for these Eisenstein series permute the associated cusps and this behavior is similarly reflected in their Fourier coefficients. (For a careful presentation, see [18].)

For our purposes, we only need to use the resulting form of the functional equation for the inner sum of (2.1) to a set of translates of similar series. This functional equation from [23] is translated to our situation in Section 2 of [11] and will be discussed precisely in our Chapter 5. Borrowing the notation of [11], the translated Fourier coefficients now take the following form.

Definition 2.5.

$$D(w, \mu m^i n^j) \stackrel{def}{=} \zeta_K(3w - 1/2) \sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\ (3) \\ (d,6)=1}} \frac{G(\mu m^i n^j, d)}{\mathbb{N}d^w}$$

where $\mu = \mu(\psi) = \omega^a(1-\omega)^b$ according to the parametrized list of primitive cubic characters whose conductor is divisible by $(1-\omega)$ given by

$$\psi(d) = \overline{\left(\frac{\omega^a(1-\omega)^b}{d}\right)} = \bar{\chi}_d(\omega^a(1-\omega)^b) \quad \textit{with } a \in \{0, 1, 2\} \textit{ and } b \in \{0, 3, 4\}$$

The fact that these characters ψ, listed for certain a and b, give a complete list follows from a simple exercise using [15]. The zeta function in the above definition has been introduced to cancel poles of the Dirichlet series. At times we will add a subscript to $D(w, \mu m^i n^j)$ to emphasize the associated functional equations from $Z_1(s_1, s_2, w)$; this will be clear from the context.

In addition, functional equations performed simultaneously in s_1 and s_2, together with the Davenport-Hasse relation, lead to Dirichlet series of the following similar form.

Definition 2.6.

$$D_6(w, \nu m^2 n^2) \stackrel{def}{=} \zeta_K(6w - 2) \sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\ (3) \\ (d,6)=1}} \frac{\overline{G(\nu m^2 n^2, d)}}{\mathbb{N}d^w}$$

where $\nu = \nu(\psi) = \omega^a(1-\omega)^b(-2)^c$ *according to the parametrized list of primitive cubic characters whose conductor divides by* 18 *given by*

$$\psi(d) = \overline{\left(\frac{\omega^a(1-\omega)^b(-2)^c}{d}\right)} = \bar{\chi}_d(\omega^a(1-\omega)^b(-2)^c) \quad \textit{with } a, c \in \{0,1,2\} \textit{ and } b \in \{0,3,4\}$$

Finally, we need to add these additional characters to the generating set of our finite group of characters Ψ'.

Definition 2.7.

$$\Psi = \Psi' \cup \{\psi \mid \mu(\psi) = \omega^a(1-\omega)^b, a \in \{0,1,2\}, b \in \{0,3,4\}\} \cup \\ \{\psi \mid \nu(\psi) = \omega^a(1-\omega)^b(-2)^c, a, c \in \{0,1,2\}, b \in \{0,3,4\}\}$$

It is this set Ψ that we take to be our finite set of characters from which we choose ψ_1 and ψ_2 in our definition of $Z_1(s_1, s_2, w)$. As we will see in later chapters, this will guarantee that linear combinations of such series taken over all possible choices of characters will be mapped by functional equations to other linear combinations taken over all such choices.

Chapter 3

Taking Limits of Dirichlet Series

3.1 A Naïve Conjecture

Our eventual goal is an analytic continuation for the multiple Dirichlet series $Z_1(s_1, s_2, w)$ which comes in two guises according to the order in which the summation is performed. For cubic characters ψ_1, ψ_1, ψ_2 as defined in the previous chapter, the two forms are

$$Z_1(s_1, s_2, w; \psi_1, \psi_1, \psi_2) = \sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\ (3) \\ (d,6)=1}} \frac{L_6(s_1, \chi_{d_0}\psi_1) L_6(s_2, \chi_{d_0}\psi_1)\psi_2(d) P(s_1, s_2, d)}{\mathbb{N}d^w} \quad (3.1)$$

$$= \sum_{\substack{m,n \in \mathcal{O}_K \\ m,n \equiv 1\ (3) \\ (mn,6)=1}} \frac{L_6(w, \chi_{\underline{mn}_0}\psi_2)\psi_1(m)\psi_1(n) Q(w, m, n)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} \quad (3.2)$$

where $d = d_1 d_2^2 d_3^3$ with $\chi_{d_0} = \chi_{d_1}\bar{\chi}_{d_2}$, $mn = \underline{mn}_1 \underline{mn}_2^2 \underline{mn}_3^3$ with $\chi_{\underline{mn}_0} = \chi_{\underline{mn}_1}\bar{\chi}_{\underline{mn}_2}$, and $P(s_1, s_2, d)$ and $Q(w, m, n)$ are finite Eulerian Dirichlet polynomials. The dependence of the correction factors on the choice of the ψ_i has been suppressed in the notation. Heuristically, the ψ_i may be completely ignored as they have been introduced in order to adjust the reciprocity characters χ at bad primes and provide no additional difficulty in the method we will introduce.

As outlined in the introduction, we will obtain the analytic continuation for $Z_1(s_1, s_2, w)$ by proving the existence of enough functional equations between Z_1 and other Dirichlet series in three variables whose analytic behavior is known to us. We want to show that Dirichlet polynomials can be introduced as weighting factors in the series Z_1 in order to make our conjectural functional equations exact. The Dirichlet polynomials are natural choice for two reasons. First, they are the simplest possible fix to our problem; the finiteness condition implies that they do not influence the convergence of the series and the Euler product preserves the multiplicativity of the numerator of Z_1 in both incarnations. But more importantly, simpler two variable series which are similar to (3.1) arise naturally as the Fourier coefficients of metaplectic Eisenstein series (cf. [13]).

In Sections 1.3 and 1.6, we noted that there should be natural functional equations from the series $Z_1(s_1, s_2, w)$ to other Dirichlet series using the functional equations of the L-series in the numerator. Under the square-free heuristic, we found that

$$Z_1(s_1+w-1/2, s_2+w-1/2, 1-w) \stackrel{\text{def}}{=} Z_3(s_1,s_2,w) \stackrel{\text{sq. free}}{=} \sum_{m,n,d} \frac{\chi_{mn}(d)G(1,mn)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^{w}}, \quad (3.3)$$

$$Z_1(1-s_1, s_2, w+s_1-1/2) \stackrel{\text{def}}{=} Z_4(s_1,s_2,w) \stackrel{\text{sq. free}}{=} \sum_{m,n,d} \frac{G(mn^2,d)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^{w}}, \quad (3.4)$$

$$Z_1(s_1, 1-s_2, w+s_2-1/2) \stackrel{\text{def}}{=} Z_5(s_1,s_2,w) \stackrel{\text{sq. free}}{=} \sum_{m,n,d} \frac{G(m^2n,d)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^{w}}, \quad (3.5)$$

$$Z_1(1-s_1, 1-s_2, w+s_1+s_2-1) \stackrel{\text{def}}{=} Z_6(s_1,s_2,w) \stackrel{\text{sq. free}}{=} \sum_{m,n,d} \frac{\overline{G_6(m^2n^2,d)}}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^{w}}. \quad (3.6)$$

As we noted before, each of these series on the right-hand side have natural generalizations to series taken over all integers m, n, d which suggest additional functional equations of all four Dirichlet series Z_3, Z_4, Z_5, and Z_6 into themselves. Moreover, (3.3), (3.6), and the pair of equations (3.4) and (3.5) are symmetric in s_1 and s_2. This strongly suggests that the correction factor $P(s_1, s_2, d)$ should be symmetric in s_1 and s_2 and the correction factor $Q(w, m, n)$ should be symmetric in m and n. Further, the existence of perfect objects under the assumption that integers are square-free suggests that the correction factor $P(s_1, s_2, d) = 1$ when d is square-free and $Q(w, m, n) = 1$ when mn is square-free. Because we intend to eventually show that correction factors with such desirable properties exist, we need to keep a running total of these assumptions.

Assumption 1. *The Dirichlet polynomials $P(s_1, s_2, d; \psi_1, \psi_1)$ and $Q(w, m, n; \psi_2)$ should satisfy the following properties:*

- *They are finite, Eulerian Dirichlet polynomials depending only on the indicated quantities.*
- *$P(s_1, s_2, d) = P(s_2, s_1, d)$ and $Q(w, m, n) = Q(w, n, m)$.*
- *$P(s_1, s_2, d) = 1$ if d is square-free.*
- *$Q(w, m, n) = 1$ if mn is square-free.*
- *They can be chosen so that the interchange equality (3.1)=(3.2) is satisfied.*
- *They can be chosen so that Z_3, Z_4, Z_5 and Z_6, defined according to (3.3)-(3.6), satisfy additional functional equations into themselves.*

The obvious question arises: how can we find such correction factors? The square-free heuristics in (3.4)-(3.6) above offer a tantalizing possibility. Unlike the Dirichlet series Z_1, the series Z_4, Z_5 and Z_6 restricted to square-free integers have natural generalizations as sums over all integers–simply take the sums to be over all integers. We might naïvely conjecture that we could begin with such a definition of either Z_4, Z_5, or Z_6 as the initial object (with sums over all integers) and define $Z_1(s_1, s_2, w)$ (and hence implicitly define $P(s_1, s_2, d)$ as well) according to the associated involution.

This idea has merit. For example, taking the definition of $Z_4(s_1, s_2, w)$ to be

$$Z_4(s_1, s_2, w) \stackrel{\text{def}}{=} \sum_{\substack{m,n,d \in \mathcal{O}_K \\ m,n,d \equiv 1\ (3) \\ (mnd,6)=1}} \frac{G(mn^2, d)\bar{\psi}_1(m)\psi_1(n)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^{w}} \tag{3.7}$$

is consistent with the initial definition $Z_4(s_1, s_2, w) = Z_1(1 - s_1, s_2, w + s_1 - 1/2)$. Indeed, transforming the Dirichlet series Z_4 in (3.7) by

$$(s_1, s_2, w) \mapsto (1 - s_1, s_2, w + s_1 - 1/2)$$

and interchanging the order of summation does result in a series summed over d whose numerator is a product of the correct L-series multiplied by a finite, Eulerian Dirichlet polynomial. Moreover, for double Dirichlet series (i.e. two-variable series) with one Dirichlet L-series in the numerator, one can show that similarly completing a series suggested by the square-free heuristic does lead to the right definition for the Dirichlet polynomials.

Before getting too excited, we immediately note that Z_4 and Z_5 are not symmetric in s_1 and s_2, so they fail to meet one of the desired properties of Assumption 1 above. A more careful inspection of Z_6 shows that it fails to provide an adequate definition of $Q(w, m, n)$ upon interchange. (Such a Q is not trivial at square-free mn.) Moreover, Z_4 and Z_6 lead to different definitions of a correction factor $P(s_1, s_2, d)$, and hence, an inconsistent set of functional equations. However, the simple fact that these series, upon interchanging the order of summation and transforming, do produce L-series multiplied by additional factors is an indication that we are close to the correct interpretation of this picture.

3.2 A First Step

In this section, we take a first step toward a more refined conjecture. While it was too ambitious to try to infer the form of an ideal three-variable object from the restricted sum over square-free integers, we can make more modest claims upon careful study of the form of these Dirichlet polynomials.

Having assumed that $P(s_1, s_2, d)$ and $Q(w, m, n)$ are Eulerian Dirichlet polynomials, we may write

$$P(s_1, s_2, d) = \prod_{p^\alpha || d_3} \sum_{i,j} a_{i,j}(d_0, p^\alpha)\mathbb{N}p^{-is_1 - js_2} = \sum_{e_1 e_2 | d_3^\infty} a_{e_1,e_2}(d_0, d_3)\mathbb{N}e_1^{-s_1}\mathbb{N}e_2^{-s_2}$$

where $d = d_0 d_3^3$ with d_0 cube-free. Similarly,

$$Q(w, m, n) = \prod_{p^\beta || M_3} \sum_{l} b_l(m, n, p^\beta)\mathbb{N}p^{-lw} = \sum_{f | M_3^\infty} b_f(m, n, M_3)\mathbb{N}f^{-w}$$

where $mn = M_0 M_3^3$ with M_0 square-free.

From this assumption, we see that if we take the limit of $Z_1(s_1, s_2, w)$ (written as a sum over d) as $Re(s_2) \to \infty$ (which is valid since our sum is absolutely convergent for all values of s_1, s_2 with $Re(s_i) \geq 1$), we can obtain a simpler object which we understand. That is,

$$\sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\ (3) \\ (d,6)=1}} \frac{L_6(s_1, \chi_{d_0}\psi_1)L_6(s_2, \chi_{d_0}\psi_1)\psi_2(d)P(s_1, s_2, d)}{\mathbb{N}d^{w}} =$$

$$= \sum_{\substack{d,m,n \in \mathcal{O}_K \\ d,m,n \equiv 1\ (3)}} \frac{\chi_{d_0}(m)\psi_1(m)\chi_{d_0}(n)\psi_1(n)\psi_2(d)P(s_1,s_2,d)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^{w}}$$

so in the limit as $Re(s_2) \to \infty$, the only non-zero terms are those associated to $n = 1$. Hence, we have

$$\lim_{Re(s_2)\to\infty} Z_1(s_1,s_2,w) = \sum_{\substack{d,m \in \mathcal{O}_K \\ d,m \equiv 1\ (3) \\ (m,6)=1}} \frac{\chi_{d_0}(m)\psi_1(m)\psi_2(d)P(s_1,\infty,d)}{\mathbb{N}m^{s_1}\mathbb{N}d^{w}}$$

where $P(s_1,\infty,d)$ is defined according to the Euler product form of P. That is,

$$P(s_1,\infty,d) \stackrel{\text{def}}{=} \lim_{Re(s_2)\to\infty} \prod_{p^\alpha || d_3} \sum_{i,j} a_{i,j}(d_0,p^\alpha)\mathbb{N}p^{-is_1-js_2} = \prod_{p^\alpha || d_3} \sum_{i} a_{i,0}(d_0,p^\alpha)\mathbb{N}p^{-is_1}$$

Performing a similar limit calculation for Z_1 as a sum over m and n gives:

$$\lim_{Re(s_2)\to\infty} Z_1(s_1,s_2,w) = \sum_{\substack{m,d \in \mathcal{O}_K \\ m,d \equiv 1\ (3) \\ (d,6)=1}} \frac{\chi_{m_0}(d)\psi_1(m)\psi_2(d)Q(w,m,1)}{\mathbb{N}m^{s_1}\mathbb{N}d^{w}}$$

This limit reduces the interchange equality to

$$\sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\ (3) \\ (d,6)=1}} \frac{L_6(s_1,\chi_{d_0}\psi_1)\psi_2(d)P(s_1,\infty,d)}{\mathbb{N}d^{w}} = \sum_{\substack{m \in \mathcal{O}_K \\ m \equiv 1\ (3) \\ (m,6)=1}} \frac{L_6(w,\chi_{m_0}\psi_2)\psi_1(m)Q(w,m,1)}{\mathbb{N}m^{s_1}}.$$

This is precisely the form of the one-variable interchange equality; the Dirichlet polynomials $P(s_1,\infty,d)$ and $Q(w,m,1)$ are serving the same roles as the one-variable correction factors to complete double Dirichlet series with cubic characters. However, we know from the one-variable cubic case that there is a unique pair of correction factors P and Q which satisfy the desired properties of assumption 1. If $P_{d_0,d_3}(s)$ denotes the one-variable correction factor, then $P(s_1,\infty,d) = P_{d_0,d_3}(s)$ and $Q(w,m,1) = P_{m_0,m_3}(w)$ where it can be shown that

$$P_{d_0,d_3}(s) = \prod_{p|d_3} \left[1 - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{-s}\right] \tag{3.8}$$

(This was done in general for nth order twists of Hecke L-series by Friedberg, Hoffstein, and Lieman in [11]. There they write that the Dirichlet correction factor has form

$$P_{d_0,d_3}(s) = \sum_{r_1 r_2 r_3 = d_3} \frac{\mu(r_3)\chi_{d_0}(r_3)\psi_1(r_3)}{\mathbb{N}r_3^{s}\mathbb{N}r_1^{ns-(n-1)}}.$$

To translate into our case, set $n = 3$ and remove the zeta factor $\zeta(3w + 3s - 2)$ from their original Dirichlet series $Z(s,w)$ defined with one cubic L-series in the numerator. The result is (3.8).)

The form of these Dirichlet polynomials in one variable imply the form for our correction coefficients $a_{i,j}(d_0, p^\alpha)$ of the two-variable $P(s_1, s_2, d)$ when $j = 0$.

$$a_{i,0}(d_0, p^\alpha) = \begin{cases} 1 & \text{if } i = 0, \alpha \geq 0 \\ -\chi_{d_0}(p)\psi_1(p) & \text{if } i = 1, \alpha > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

for any value of d where $d = d_1 d_2^2 d_3^3$ and $ord_p(d_3) = \alpha$. Similarly, this implies the form of correction coefficients $b_l(m, n, p^\beta)$ of $Q(w, m, n)$ when $n = 1$.

$$b_l(m, n, p^\beta) = \begin{cases} 1 & \text{if } l = 0, \beta \geq 0 \\ -\chi_{d_0}(p)\psi_2(p) & \text{if } l = 1, \beta \geq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.10}$$

for any value of m where $m = m_1 m_2^2 m_3^3$ and $ord_p(m_3) = \beta$.

We can repeat our limiting argument by taking the limit of Z_1 as $Re(s_1) \to \infty$. However, our object is completely symmetric in s_1 and s_2, so this process gives identical information to the above for the coefficients $a_{0,j}(d_0, p^\alpha)$ of P and $b_i(1, n)$ of Q.

3.3 Taking Variables to Infinity

In the previous section, we determined information about the Dirichlet polynomials P and Q by taking limits of the original series $Z_1(s_1, s_2, w)$. This reduced Z_1 to a simpler object in two complex variables which was much easier to study. Here we repeat this idea with a subtle twist. First, we apply a transformation of variables as predicted in (3.3)-(3.6). Then we take limits of variables to infinity to determine new information about the polynomials P and Q and their coefficients. It is not immediately clear why the resulting object from this process should be something familiar. This will be further explained in what follows.

3.3.1 A Careful Transformation of $Z_1(s_1, s_2, w)$

We begin with a change of variables suggested by (3.4).

$$Z_1(s_1, s_2, w) = \sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\ (3) \\ (d,6)=1}} \frac{L_6(s_1, \chi_{d_0}\psi_1) L_6(s_2, \chi_{d_0}\psi_1)\psi_2(d) P(s_1, s_2, d)}{\mathbb{N}d^w}, \text{ so}$$

$$Z_1(1-s_1, s_2, w+s_1-1/2) = \sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\ (3) \\ (d,6)=1}} \frac{L_6(1-s_1, \chi_{d_0}\psi_1) L_6(s_2, \chi_{d_0}\psi_1)\psi_2(d) P(1-s_1, s_2, d)}{\mathbb{N}d^{w+s_1-1/2}}$$

$$= \sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\ (3) \\ (d,6)=1}} \frac{L(1-s_1, \chi_{d_0}\psi_1) L(s_2, \chi_{d_0}\psi_1)\psi_2(d) P(1-s_1, s_2, d)}{\mathbb{N}d^{w+s_1-1/2}}$$

$$\cdot \left[1 - \chi_{d_0}(1-\omega)\psi_1(1-\omega) 3^{-(1-s_1)}\right]$$

$$= \sum_{\substack{d\in\mathcal{O}_K\\ d\equiv 1\ (3)\\ (d,6)=1}} \frac{L(s_1,\bar\chi_{d_0}\bar\psi_1)L(s_2,\chi_{d_0}\psi_1)\psi_2(d)G(\chi_{d_1})G(\bar\chi_{d_2})G(\psi_1)\bar\psi_1(d_2)(\mathbb{N}d_1d_2)^{s_1-1/2}}{\mathbb{N}d^{w+s_1-1/2}}$$

$$\cdot P(1-s_1,s_2,d)\left[1-\chi_{d_0}(1-\omega)\psi_1(1-\omega)3^{-(1-s_1)}\right]$$

$$= \sum_{\substack{d\in\mathcal{O}_K\\ d\equiv 1\ (3)\\ (d,6)=1}} \frac{L_6(s_1,\bar\chi_{d_0}\bar\psi_1)L_6(s_2,\chi_{d_0}\psi_1)\psi_2(d)G(\chi_{d_1})G(\bar\chi_{d_2})G(\psi_1)\bar\psi_1(d_2)(\mathbb{N}d_1d_2)^{s_1-1/2}}{\mathbb{N}d^{w+s_1-1/2}}$$

$$\cdot P(1-s_1,s_2,d)\left[1-\chi_{d_0}(1-\omega)\psi_1(1-\omega)3^{-(1-s_1)}\right]\cdot\left[1-\bar\chi_{d_0}(1-\omega)\bar\psi_1(1-\omega)3^{-s_1}\right]^{-1}$$

$$= \sum_{\substack{d\in\mathcal{O}_K\\ d\equiv 1\ (3)\\ (d,6)=1}} \frac{\bar\chi_{d_0}(mn^2)\bar\psi_1(m)\psi_1(n)\psi_2(d)G(\chi_{d_1})G(\bar\chi_{d_2})G(\psi_1)\bar\psi_1(d_2)P(1-s_1,s_2,d)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^{w}(\mathbb{N}d_2d_3^3)^{s_1-1/2}}$$

$$\cdot\left[1-\chi_{d_0}(1-\omega)\psi_1(1-\omega)3^{-(1-s_1)}\right]\cdot\left[1-\bar\chi_{d_0}(1-\omega)\bar\psi_1(1-\omega)3^{-s_1}\right]^{-1}$$

Since our characters are cubic, we can rewrite the final Euler factor above as

$$\left[1-\bar\chi_{d_0}(1-\omega)\bar\psi_1(1-\omega)3^{-s_1}\right]^{-1} =$$

$$= \left[1+\bar\chi_{d_0}(1-\omega)\bar\psi_1(1-\omega)3^{-s_1}+\chi_{d_0}(1-\omega)\psi_1(1-\omega)3^{-2s_1}\right]\left[1-3^{-3s_1}\right]^{-1}$$

so we have shown that

$$\left[1-3^{-3s_1}\right]Z_1(1-s_1,s_2,w+s_1-1/2) = G(\psi_1)\left[1-\chi_{d_0}(1-\omega)\psi_1(1-\omega)3^{-(1-s_1)}\right]$$
$$\left[1+\bar\chi_{d_0}(1-\omega)\bar\psi_1(1-\omega)3^{-s_1}+\chi_{d_0}(1-\omega)\psi_1(1-\omega)3^{-2s_1}\right]Z_4(s_1,s_2,w)$$

where

$$Z_4(s_1,s_2,w) = \sum_{\substack{d\in\mathcal{O}_K\\ d\equiv 1\ (3)\\ (d,6)=1}} \frac{\bar\chi_{d_0}(mn^2)\bar\psi_1(m)\psi_1(n)\psi_2(d)G(\chi_{d_1})G(\bar\chi_{d_2})\bar\psi_1(d_2)P(1-s_1,s_2,d)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^{w}(\mathbb{N}d_2d_3^3)^{s_1-1/2}}.$$

Now for primes $p\nmid d_2$, let $d_3 = p^{3\alpha}d_3'$ with $(p,d_3')=1$ as usual. Then we have correction terms of form

$$P^{(p)}(s_1,s_2,d) = \left[1+a_{1,0}^{(\alpha)}\mathbb{N}p^{-s_1}+\cdots+a_{3\alpha-1,3\alpha}^{(\alpha)}\mathbb{N}p^{-(3\alpha-1)s_1-3\alpha s_2}+a_{3\alpha,3\alpha}^{(\alpha)}\mathbb{N}p^{-3\alpha s_1-3\alpha s_2}\right]$$

So

$$P(1-s_1,s_2,d) = \prod_{p^\alpha||d_3}\left[1+a_{1,0}^{(\alpha)}\mathbb{N}p^{s_1-1}+\cdots+a_{3\alpha-1,3\alpha}^{(\alpha)}\mathbb{N}p^{-3\alpha+1+(3\alpha-1)s_1-3\alpha s_2}+\right.$$
$$\left.a_{3\alpha,3\alpha}^{(\alpha)}\mathbb{N}p^{-3\alpha+3\alpha s_1-3\alpha s_2}\right]$$

Incorporating the $\mathbb{N}(d_3^3)^{s_1-1/2}$ from the above denominator into the correction factor, we have:

$$\begin{aligned} P(1-s_1,s_2,d) &= \prod_{p^\alpha\|d_3} \mathbb{N}p^{3\alpha/2-3\alpha s_1}\left[1+\cdots\right] \\ &= \prod_{p^\alpha\|d_3} \Big[a^{(\alpha)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2} + a^{(\alpha)}_{3\alpha-1,0}\mathbb{N}p^{-3\alpha/2+1-s_1} + \cdots \\ &\qquad \cdots + a^{(\alpha)}_{0,3\alpha}\mathbb{N}p^{3\alpha/2-3\alpha s_1-3\alpha s_2}\Big] \end{aligned}$$

For primes $p|d_2$, again letting $d_3 = p^{3\alpha}d_3'$ with $(p,d_3')=1$, we have

$$P^{(p)}(s_1,s_2,d) = \Big[1 + a^{(\alpha)}_{1,0}\mathbb{N}p^{-s_1} + \cdots + a^{(\alpha)}_{3\alpha,3\alpha+1}\mathbb{N}p^{-3\alpha s_1-(3\alpha+1)s_2} + a^{(\alpha)}_{3\alpha+1,3\alpha+1}\mathbb{N}p^{-(3\alpha+1)s_1-(3\alpha+1)s_2}\Big]$$

So

$$P(1-s_1,s_2,d) = \prod_{\substack{p^\alpha\|d_3\\ p|d_2}} \Big[1 + a^{(\alpha)}_{1,0}\mathbb{N}p^{s_1-1} + \cdots \cdots + a^{(\alpha)}_{3\alpha,3\alpha+1}\mathbb{N}p^{-3\alpha+3\alpha s_1-(3\alpha+1)s_2} + a^{(\alpha)}_{3\alpha+1,3\alpha+1}\mathbb{N}p^{-(3\alpha+1)+(3\alpha+1)s_1-(3\alpha+1)s_2}\Big]$$

Incorporating the $(\mathbb{N}d_2d_3^3)^{s_1-1/2}$ from the above denominator into the correction factor, we have:

$$\begin{aligned} P(1-s_1,s_2,d) &= \prod_{\substack{p^\alpha\|d_3\\ p|d_2}} \mathbb{N}p^{3\alpha/2+1/2-(3\alpha+1)s_1}\left[1+\cdots\right] \\ &= \prod_{\substack{p^\alpha\|d_3\\ p|d_2}} \Big[a^{(\alpha)}_{3\alpha+1,0}\mathbb{N}p^{-3\alpha/2-1/2} + a^{(\alpha)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2+1/2-s_1} + \cdots \\ &\qquad \cdots + a^{(\alpha)}_{0,3\alpha+1}\mathbb{N}p^{3\alpha/2+1/2-(3\alpha+1)s_1-(3\alpha+1)s_2}\Big] \end{aligned}$$

Rewriting $Z_4(s_1,s_2,w)$ to include this altered form of the correction factor, we have

$$\sum_{\substack{d,m,n\in\mathcal{O}_K\\ d,m,n\equiv 1\ (3)\\ (dmn,6)=1}} \frac{\bar{\chi}_{d_0}(mn^2)\bar{\psi}_1(m)\psi_1(n)\psi_2(d)G(\chi_{d_1})G(\bar{\chi}_{d_2})\bar{\psi}_1(d_2)\prod_{\substack{p^\alpha\|d_3\\ p\nmid d_2}}\Big[a^{(\alpha)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2}+\cdots\Big]}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^w}\cdot$$
$$\cdot \prod_{\substack{p^\alpha\|d_3\\ p|d_2}} \Big[a^{(\alpha)}_{3\alpha+1,0}\mathbb{N}p^{-3\alpha/2-1/2} + a^{(\alpha)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2+1/2-s_1} + \cdots\Big] \quad (3.11)$$

3.3.2 The Limit as $\Re(s_2) \to \infty$

If we take the limit of the above series as $\Re(s_2) \to \infty$, then the only non-zero terms are those with $n = 1$ and no contribution to terms with exponent s_2 from the correction factor. Hence, we are left with $Z_4(s_1, \infty, w) =$

$$\sum_{\substack{d,m \in \mathcal{O}_K \\ d,m \equiv 1\ (3) \\ (m,6)=1}} \frac{\bar{\chi}_{d_0}(m)\bar{\psi}_1(m)\psi_2(d)G(\chi_{d_1})G(\bar{\chi}_{d_2})\bar{\psi}_1(d_2)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} \prod_{\substack{p^\alpha || d_3 \\ p \nmid d_2}} \left[a^{(\alpha)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2} + \cdots \right.$$

$$\left. \cdots + a^{(\alpha)}_{0,0}\mathbb{N}p^{3\alpha/2-3\alpha s_1}\right] \prod_{\substack{p^\alpha || d_3 \\ p | d_2}} \left[a^{(\alpha)}_{3\alpha+1,0}\mathbb{N}p^{-3\alpha/2-1/2} + \cdots + a^{(\alpha)}_{0,0}\mathbb{N}p^{3\alpha/2+1/2-3\alpha s_1}\right] \quad (3.12)$$

Note that the result (3.12) above is the same series as $Z_1(1-s_1, \infty, w+s_1-1/2)$, the series which results by first taking the limit as $\Re(s_2) \to \infty$ of $Z_1(s_1, s_2, w)$ and *then* performing the translation. But, as we noted before,

$$Z_1(s_1, \infty, w) = \sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\ (3) \\ (d,6)=1}} \frac{L_6(s_1, \chi_{d_0}\psi_1)\psi_2(d)P(s_1, \infty, d)}{\mathbb{N}d^w}$$

is just a double Dirichlet series for which we know that, under transformation $(s_1, w) \mapsto (1 - s_1, w + s_1 - 1/2)$,

$$Z_1(1 - s_1, \infty, w + s_1 - 1/2) = \sum_{\substack{d,m \in \mathcal{O}_K \\ d,m \equiv 1\ (3) \\ (m,6)=1}} \frac{G(m, d)\bar{\psi}_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w}.$$

This functional equation is established in [11]. In brief, it exists because the two-variable multiple Dirichlet series $Z_1(1-s_1, \infty, w+s_1-1/2)$ is the double Dirichlet series which occurs naturally as the Fourier coefficient of a metaplectic Eisenstein series on the three-fold cover of $GL(2)$.

Thus we may determine the correction coefficients $a^{(\alpha)}_{i,0}(d_0, p)$ of $P(s_1, s_2, d)$ found in (3.12) via the following known equality

$$\sum_{\substack{d,m \in \mathcal{O}_K \\ d,m \equiv 1\ (3) \\ (m,6)=1}} \frac{G(m, d)\bar{\psi}_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} =$$

$$\sum_{\substack{d,m \in \mathcal{O}_K \\ d,m \equiv 1\ (3) \\ (m,6)=1}} \frac{\bar{\chi}_{d_0}(m)G(\chi_{d_1})G(\bar{\chi}_{d_2})\bar{\psi}_1(d_2)\bar{\psi}_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} \prod_{\substack{p^\alpha || d_3 \\ p \nmid d_2}} \left[a^{(\alpha)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2} + \cdots \right.$$

$$\left. \cdots + a^{(\alpha)}_{0,0}\mathbb{N}p^{3\alpha/2-3\alpha s_1}\right] \cdot \prod_{\substack{p^\alpha || d_3 \\ p | d_2}} \left[a^{(\alpha)}_{3\alpha+1,0}\mathbb{N}p^{-3\alpha/2-1/2} + \cdots + a^{(\alpha)}_{0,0}\mathbb{N}p^{3\alpha/2+1/2-3\alpha s_1}\right] \quad (3.13)$$

To solve for the unknown coefficients, we must decompose the series on the left-hand side of (3.13) in terms of primes dividing d.

Proposition 3.1. *Decomposing the Gauss sum according to primes dividing $d = d_1 d_2^2 d_3^3$, we obtain*

$$\sum_{\substack{d,m\in\mathcal{O}_K \\ d,m\equiv 1\ (3) \\ (m,6)=1}} \frac{G(m,d)\bar{\psi}_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} = \sum_{\substack{d\in\mathcal{O}_K \\ d\equiv 1\ (3) \\ (d,6)=1}} \frac{\psi_2(d)G(1,d_1)\overline{G(1,d_2)}L(s_1,\bar{\chi}_{d_0}\bar{\psi}_1)\bar{\psi}_1(d_2)}{\mathbb{N}d^w}$$

$$\cdot \prod_{\substack{p^\alpha || d_3 \\ p\nmid d_0}} \left(\mathbb{N}p^{3\alpha/2-3\alpha s_1} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha/2-1-(3\alpha-1)s_1}\right)$$

$$\prod_{\substack{p^\alpha || d_3 \\ p| d_1}} \mathbb{N}p^{3\alpha/2-3\alpha s_1} \prod_{\substack{p^\alpha || d_3 \\ p| d_2}} \mathbb{N}p^{3\alpha/2+1/2-(3\alpha+1)s_1}$$

Then, in conjunction with the above equality, we have

Corollary 3.2. *Let $a_{i,j}^{(\alpha)}(d_0,p)$ be the coefficient of $\mathbb{N}p^{-is_1-js_2}$ in the pth Euler factor of the Dirichlet polynomial $P(s_1,s_2,d)$ depending on the indicated variables. Then*

$$a_{i,0}^{(\alpha)}(d_0,p) = a_{0,i}^{(\alpha)}(d_0,p) = \begin{cases} 1 & \text{if } i=0, \alpha\ge 0 \\ -\chi_{d_0}(p)\psi_1(p) & \text{if } i=1, \alpha>0 \\ 0 & \text{otherwise} \end{cases}$$

Proof of Corollary: Simply equating $\sum_{d,m} \frac{G(m,d)\bar{\psi}_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w}$ with $Z_4(s_1,\infty,w)$ and cancelling common factors at any fixed d, we have that

$$\prod_{\substack{p^\alpha || d_3 \\ p\nmid d_0}} \left(\mathbb{N}p^{3\alpha/2-3\alpha s_1} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha/2-1-(3\alpha-1)s_1}\right) \prod_{\substack{p^\alpha || d_3 \\ p| d_1}} \mathbb{N}p^{3\alpha/2-3\alpha s_1} =$$

$$\prod_{\substack{p^\alpha || d_3 \\ p\nmid d_2}} \left[a_{3\alpha,0}^{(\alpha)}\mathbb{N}p^{-3\alpha/2} + \cdots + a_{0,0}^{(\alpha)}\mathbb{N}p^{3\alpha/2-3\alpha s_1}\right]$$

and

$$\prod_{\substack{p^\alpha || d_3 \\ p| d_2}} \mathbb{N}p^{3\alpha/2+1/2-(3\alpha+1)s_1} = \prod_{\substack{p^\alpha || d_3 \\ p| d_2}} \left[a_{3\alpha+1,0}^{(\alpha)}\mathbb{N}p^{-3\alpha/2-1/2} + \cdots + a_{0,0}^{(\alpha)}\mathbb{N}p^{3\alpha/2+1/2-(3\alpha+1)s_1}\right].$$

Equating terms on both sides gives the result for $a_{i,0}$. By symmetry, we know the same must be true for $a_{0,i}$. □

The reader will note that this is precisely the information determined by the limiting process in the previous section. While it seems redundant to reprove this now, in the following chapters, we will need the fact that the decomposition of a Gauss sum results in the determination of correction coefficients. Moreover, it will serve to motivate the assumptions we will make over the rest of this chapter.

Proof of Proposition: We do this by case method. Suppose that $p \nmid d_0$. Write $d = p^{3\alpha}d'$ with $(p,d') = 1$ and $m = p^\gamma m'$ with $(m',p) = 1$. Then

$$G(m,d) = G(p^\gamma m', p^{3\alpha}d') = \frac{g(p^\gamma m', p^{3\alpha}d')}{\sqrt{\mathbb{N}p^{3\alpha}\mathbb{N}d'}} = \frac{g(p^\gamma m', d')}{\sqrt{\mathbb{N}d'}}\frac{g(p^\gamma m', p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}} =$$

$$= \bar{\chi}_{d'}(p^\gamma)\frac{g(m',d')}{\sqrt{\mathbb{N}d'}}\frac{g(p^\gamma,p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}} = \bar{\chi}_{d_0}(p^\gamma)G(m',d')\frac{g(p^\gamma,p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}$$

So we may write

$$\sum_{d,m}\frac{G(m,d)\psi_2(d)\bar{\psi}_1(m')}{\mathbb{N}m^{s_1}\mathbb{N}d^w} = \sum_{\substack{m',d\\ d=p^{3\alpha}d'}}\frac{G(m',d')\psi_2(d)\bar{\psi}_1(m)}{(\mathbb{N}m')^{s_1}\mathbb{N}d^w}\left[\sum_{\gamma\geq 0}\frac{\bar{\chi}_{d_0}(p^\gamma)\bar{\psi}_1(p^\gamma)}{\mathbb{N}p^{\gamma s_1}}\frac{g(p^\gamma,p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}\right]$$

But we know that

$$g(p^\gamma,p^{3\alpha}) = \begin{cases}\phi(p^{3\alpha}), & \text{if } \gamma\geq 3\alpha\\ -\mathbb{N}p^{3\alpha-1}, & \text{if } \gamma = 3\alpha-1, \alpha>0\\ 0, & \text{otherwise.}\end{cases}$$

In particular, if $\alpha = 0$ (i.e. $p\nmid d$), then $g(p^\gamma,p^{3\alpha}) = 1$ and so

$$\sum_{\gamma\geq 0}\frac{\bar{\chi}_{d_0}(p^\gamma)\bar{\psi}_1(p^\gamma)}{\mathbb{N}p^{\gamma s_1}}\frac{g(p^\gamma,p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}} = \sum_{\gamma\geq 0}\frac{\bar{\chi}_{d_0}(p^\gamma)\bar{\psi}_1(p^\gamma)}{\mathbb{N}p^{\gamma s_1}} = L^{(p)}(s_1,\bar{\chi}_{d_0}\bar{\psi}_1),$$

the pth Euler factor of the L-series. Then substituting this information and removing all such prime factors p such that $p\nmid d_0$, our initial series reduces to

$$\sum_{\substack{m',d\\ d=d'd_3^3}}\frac{G(m',d')\psi_2(d)\bar{\psi}_1(m')}{(\mathbb{N}m')^{s_1}\mathbb{N}d^w}\prod_{p\nmid d}L^{(p)}(s_1,\bar{\chi}_{d_0}\bar{\psi}_1)\prod_{\substack{p^\alpha||d_3\\ p\nmid d_0}}\left(\sum_{\gamma\geq 3\alpha}\frac{\bar{\chi}_{d_0}(p^\gamma)\bar{\psi}_1(p^\gamma)}{\mathbb{N}p^{\gamma s_1}}\frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}} - \frac{\mathbb{N}p^{3\alpha-1}}{\sqrt{\mathbb{N}p^{3\alpha}}}\frac{\bar{\chi}_{d_0}(p^{(3\alpha-1)})\bar{\psi}_1(p^{(3\alpha-1)})}{\mathbb{N}p^{(3\alpha-1)s_1})}\right)$$

Examining the Euler factors in the latter product more closely, we see that

$$\sum_{\gamma\geq 3\alpha}\frac{\bar{\chi}_{d_0}(p^\gamma)\bar{\psi}_1(p^\gamma)}{\mathbb{N}p^{\gamma s_1}}\frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}} - \frac{\mathbb{N}p^{3\alpha-1}}{\sqrt{\mathbb{N}p^{3\alpha}}}\frac{\bar{\chi}_{d_0}(p^{(3\alpha-1)})\bar{\psi}_1(p^{(3\alpha-1)})}{\mathbb{N}p^{(3\alpha-1)s_1})} =$$

$$= \frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}\mathbb{N}p^{-3\alpha s_1}L^{(p)}(s_1,\bar{\chi}_{d_0}\bar{\psi}_1) - \frac{\mathbb{N}p^{3\alpha-1}}{\sqrt{\mathbb{N}p^{3\alpha}}}\frac{\chi_{d_0}(p)\psi_1(p)}{\mathbb{N}p^{(3\alpha-1)s_1})}.$$

Factoring the pth Euler factor of the L-series from each of the terms, we have

$$L^{(p)}(s_1,\bar{\chi}_{d_0}\bar{\psi}_1)\left(\frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}\mathbb{N}p^{-3\alpha s_1} - \frac{\mathbb{N}p^{3\alpha-1}}{\sqrt{\mathbb{N}p^{3\alpha}}}\frac{\chi_{d_0}(p)\psi_1(p)}{\mathbb{N}p^{(3\alpha-1)s_1})}(1-\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{-s_1})\right) =$$

$$= L^{(p)}(s_1,\bar{\chi}_{d_0}\psi_1)\left(\mathbb{N}p^{3\alpha/2-3\alpha s_1} - \chi_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{3\alpha/2-1-(3\alpha-1)s_1}\right)$$

Rewriting the above, we now have

$$\sum_{\substack{d,m\in\mathcal{O}_K\\ d,m\equiv 1\ (3)\\ (dm,6)=1}}\frac{G(m,d)\bar{\psi}_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} = \sum_{\substack{m',d\\ d'|d\\ m',d'|d_0^\infty}}\frac{G(m',d')\bar{\psi}_1(m')\psi_2(d)}{(\mathbb{N}m')^{s_1}\mathbb{N}d^w}\prod_{p\nmid d}L^{(p)}(s_1,\bar{\chi}_{d_0}\bar{\psi}_1)$$

$$\prod_{\substack{p^\alpha||d_3\\ p\nmid d_0}}L^{(p)}(s_1,\bar{\chi}_{d_0}\bar{\psi}_1)\ \cdot\ \left(\mathbb{N}p^{3\alpha/2-3\alpha s_1} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha/2-1-(3\alpha-1)s_1}\right)$$

This completes the treatment of the case $p \nmid d_0$. We are now left to consider the remaining sum

$$\sum_{\substack{m,d=d_0d_3^3 \\ d_3,m|d_0^\infty}} \frac{G(m,d)\psi_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w}.$$

Now suppose that $p|d_1$. Then, similar to the previous case, we may write $d'' = p^{3\alpha+1}d'$ with $(d',p) = 1$ and $m = p^\gamma m'$ with $(m',p) = 1$. In this case,

$$\begin{aligned} G(m,d) = G(p^\gamma m', p^{3\alpha+1}d') &= \frac{g(p^\gamma m', p^{3\alpha+1}d')}{\sqrt{\mathbb{N}p^{3\alpha+1}\mathbb{N}d'}} \\ &= \frac{g(p^\gamma m', p^{3\alpha+1})}{\sqrt{\mathbb{N}p^{3\alpha+1}}} \frac{g(p^\gamma m', d')}{\sqrt{\mathbb{N}d'}} \bar{\chi}_{d'}(p^{3\alpha+1})\bar{\chi}_p(d') \end{aligned}$$

But $g(p^\gamma, p^{3\alpha+1}) = 0$ unless $\gamma = 3\alpha$. Hence we may write

$$\frac{g(p^\gamma m', p^{3\alpha+1}d')}{\sqrt{\mathbb{N}p^{3\alpha+1}\mathbb{N}d'}} = \frac{g(p^{3\alpha}m', p^{3\alpha+1}d')}{\sqrt{\mathbb{N}p^{3\alpha+1}\mathbb{N}d'}} = \frac{\mathbb{N}p^{3\alpha}}{\sqrt{\mathbb{N}p^{3\alpha}}} G(m', pd')$$

Removing all such prime factors p such that $p|d_1$, we have

$$\sum_{\substack{m,d=d_0d_3^3 \\ d_3,m|d_0^\infty}} \frac{G(m,d)\bar{\psi}_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} = \sum_{\substack{m,d \\ d_1,d'|d \\ m|d_2^\infty}} \frac{G(m,d_1d')\bar{\psi}_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} \prod_{\substack{p^\alpha||d_3 \\ p|d_1}} \mathbb{N}p^{3\alpha/2-3\alpha s_1}$$

This completes the case where $p|d_1$. Finally, we remove the primes $p|d_2$ from the remaining sum. Write $d' = p^{3\alpha+2}d''$ with $(d'',p) = 1$ and $m = p^\gamma m'$ with $(m',p) = 1$. Then

$$G(m,d_1d') = G(p^\gamma m', p^{3\alpha+2}d_1d'') = \frac{g(p^\gamma m', p^{3\alpha+2}d_1d'')}{\sqrt{\mathbb{N}p^{3\alpha+2}\mathbb{N}d_1\mathbb{N}d''}} = 0$$

unless $\gamma = 3\alpha + 1$. In this case,

$$\frac{g(p^\gamma m', p^{3\alpha+2}d_1d'')}{\sqrt{\mathbb{N}p^{3\alpha+2}\mathbb{N}d_1\mathbb{N}d''}} = \frac{g(p^{3\alpha+1}m', p^{3\alpha+2}d_1d'')}{\sqrt{\mathbb{N}p^{3\alpha+2}\mathbb{N}d_1\mathbb{N}d''}} = \frac{\mathbb{N}p^{3\alpha}}{\sqrt{\mathbb{N}p^{3\alpha}}} \frac{g(pm', p^2d_1d'')}{\sqrt{\mathbb{N}p^2\mathbb{N}d_1\mathbb{N}d''}}$$

Then removing all such primes p such that $p|d_2$ (so that we've exhausted all possible divisors of m) we have

$$\sum_{\substack{m,d \\ d_1d'|d \\ m|d_2^\infty}} \frac{G(m,d_1d')\bar{\psi}_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} = \sum_{d=d_1d_2^2d_3^3} \frac{g(d_2,d_1d_2^2)\bar{\psi}_1(d_2)\psi_2(d)}{\mathbb{N}d^w\sqrt{\mathbb{N}d_1d_2^2}} \prod_{\substack{p^\alpha||d_3 \\ p|d_2}} \mathbb{N}p^{3\alpha/2-(3\alpha+1)s_1} =$$

$$= \sum_d \frac{g(1,d_1)\overline{g(1,d_2)}\mathbb{N}d_2\bar{\psi}_1(d_2)\psi_2(d)}{\mathbb{N}d^w\sqrt{\mathbb{N}d_1d_2^2}} \prod_{\substack{p^\alpha||d_3 \\ p|d_2}} \mathbb{N}p^{3\alpha/2-(3\alpha+1)s_1}$$

$$= \sum_d \frac{G(1,d_1)\overline{G(1,d_2)}\bar{\psi}_1(d_2)\psi_2(d)}{\mathbb{N}d^w} \prod_{\substack{p^\alpha||d_3 \\ p|d_2}} \mathbb{N}p^{3\alpha/2+1/2-(3\alpha+1)s_1}$$

This completes the case $p|d_2$ and we can now reconstruct the initial object.

$$\sum_{d,m} \frac{G(m,d)\bar{\psi}_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} = \sum_{d} \frac{G(1,d_1)\overline{G(1,d_2)}\bar{\psi}_1(d_2)\psi_2(d)}{\mathbb{N}d^w} \prod_{p\nmid d} L^{(p)}(s_1,\bar{\chi}_{d_0}\bar{\psi}_1)$$
$$\prod_{\substack{p^\alpha||d_3\\ p\nmid d_0}} L^{(p)}(s_1,\bar{\chi}_{d_0}\bar{\psi}_1)\Big(\mathbb{N}p^{3\alpha/2-3\alpha s_1}$$
$$-\chi_{d_0}(p)\mathbb{N}p^{3\alpha/2-1-(3\alpha-1)s_1}\Big) \prod_{\substack{p^\alpha||d_3\\ p|d_1}} \mathbb{N}p^{3\alpha/2-3\alpha s_1} \prod_{\substack{p^\alpha||d_3\\ p|d_2}} \mathbb{N}p^{3\alpha/2+1/2-(3\alpha+1)s_1}$$

Noting that, for primes p dividing d_0, the pth Euler factor of $L(s_1,\bar{\chi}_{d_0}\bar{\psi}_1)$ is trivial, this completes the proof of the theorem. □

3.3.3 The Limit as $\Re(s_1)\to\infty$

Recall that from (3.11), we had an expression for $Z_1(1-s_1,s_2,w+s_1-1/2) = Z_4(s_1,s_2,w) =$

$$\sum_{\substack{d,m,n\in\mathcal{O}_K\\ d,m,n\equiv 1\ (3)}} \frac{\bar{\chi}_{d_0}(mn^2)\bar{\psi}_1(m)\psi_1(n)\psi_2(d)G(\chi_{d_1})G(\bar{\chi}_{d_2})\bar{\psi}_1(d_2)\prod_{\substack{p^\alpha||d_3\\ p\nmid d_2}}\left[a^{(\alpha)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2}+\cdots\right]}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^w}\cdot$$
$$\cdot\prod_{\substack{p^\alpha||d_3\\ p|d_2}}\left[a^{(\alpha)}_{3\alpha+1,0}\mathbb{N}p^{-3\alpha/2-1/2}+a^{(\alpha)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2+1/2-s_1}+\cdots\right]$$

If, instead, we take the limit of this series as the real part of the transformed variable $\Re(s_1)\to\infty$, then we are left with

$$\sum_{\substack{n,d\in\mathcal{O}_K\\ n,d\equiv 1\ (3)\\ (d,6)=1}} \frac{\bar{\chi}_{d_0}(n^2)\psi_1(n)\psi_2(d)\bar{\psi}_1(d_2)G(1,d_1)\overline{G(1,d_2)}}{\mathbb{N}n^{s_2}\mathbb{N}d^w} \prod_{\substack{p^\alpha||d_3\\ p\nmid d_2}}\Big[a^{(\alpha)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2}+\cdots$$
$$+a^{(\alpha)}_{3\alpha,3\alpha}\mathbb{N}p^{-3\alpha/2-3\alpha s_2}\Big]\prod_{\substack{p^\alpha||d_3\\ p|d_2}}\left[a^{(\alpha)}_{3\alpha+1,0}\mathbb{N}p^{-3\alpha/2-1/2}+\cdots+a^{(\alpha)}_{3\alpha+1,3\alpha}\mathbb{N}p^{-3\alpha/2+1/2-3\alpha s_2}\right]$$

Note that for square-free d with $(n,d)=1$, after stripping away all of the characters ψ_i, this reduces to $\displaystyle\sum_{\substack{(n,d)=1\\ d\text{ sq. free}}}\frac{G(n^2,d)}{\mathbb{N}n^{s_2}\mathbb{N}d^w}$. Based on the results of the previous section, we might hope to determine the above unknown correction coefficients using the equality

$$\sum_{\substack{n,d\in\mathcal{O}_K\\ n,d\equiv 1\ (3)\\ (d,6)=1}} \frac{G(n^2,d)\psi_1(n)\psi_2(d)}{\mathbb{N}n^{s_2}\mathbb{N}d^w} = Z_4(\infty,s_2,w) =$$

$$= \sum_{n,d} \frac{\bar{\chi}_{d_0}(n^2)\psi_1(n)\psi_2(d)\bar{\psi}_1(d_2)G(1,d_1)\overline{G(1,d_2)}}{\mathbb{N}n^{s_2}\mathbb{N}d^w} \prod_{\substack{p^\alpha || d_3 \\ p \nmid d_2}} \left[a^{(\alpha)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2} + \cdots \right.$$

$$\left. \cdots + a^{(\alpha)}_{3\alpha,3\alpha}\mathbb{N}p^{-3\alpha/2-3\alpha s_2}\right] \prod_{\substack{p^\alpha || d_3 \\ p | d_2}} \left[a^{(\alpha)}_{3\alpha+1,0}\mathbb{N}p^{-3\alpha/2-1/2} + \cdots + a^{(\alpha)}_{3\alpha+1,3\alpha}\mathbb{N}p^{-3\alpha/2+1/2-3\alpha s_2}\right] \tag{3.14}$$

where the left-hand side is now the complete sum over all integers $n, d \equiv 1$ (3) $(d, 6) = 1$. While there is no one-variable situation to confirm this, we will adopt this assumption in order to determine the above correction coefficients and attempt to find a complete description of the $a^{(\alpha)}_{i,j}$ which is consistent with this hypothesis and also satisfies the other proposed axioms.

Proposition 3.3. *Decomposing the Gauss sum according to primes dividing d, we obtain*

$$\sum_{\substack{n,d \in \mathcal{O}_K \\ n,d \equiv 1 \ (3) \\ (nd,6)=1}} \frac{G(n^2,d)\psi_1(n)\psi_2(d)}{\mathbb{N}n^{s_2}\mathbb{N}d^w} = \sum_{\substack{d \\ d=d_1d_2^2d_3^3}} \frac{G(1,d_1)\overline{G(1,d_2)}L(s_2,\chi_{d_0}\psi_1)\psi_2(d)\bar{\psi}_1(d_2)}{\mathbb{N}(d_1d_2)^w}$$

$$\prod_{\substack{p \nmid d_0 \\ p^\alpha || d_3 \\ \alpha \text{ even}}} \frac{\phi(p^{3\alpha})}{\mathbb{N}p^{3\alpha/2+3\alpha s_2/2+3\alpha w}} \cdot \prod_{\substack{p \nmid d_0 \\ p^\alpha || d_3 \\ \alpha \text{ odd}}} \left(\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{3\alpha/2-(3\alpha+1)s_2/2-3\alpha w} - \right.$$

$$\left. \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha/2-1-(3\alpha-1)s_2/2-3\alpha w}\right) \cdot \prod_{\substack{p | d_1 \\ p^\alpha || d_3 \\ \alpha \text{ even}}} \mathbb{N}p^{3\alpha/2-3\alpha s_2/2-3\alpha w}$$

$$\prod_{\substack{p | d_2 \\ p^\alpha || d_3 \\ \alpha \text{ odd}}} \mathbb{N}p^{(3\alpha+1)/2-(3\alpha+1)s_2/2-(3\alpha+1)w} \tag{3.15}$$

Before performing this analysis, we note the resulting implications on the correction coefficients.

Corollary 3.4. *Keeping the same notation as before and supposing that*

$$\sum_{\substack{n,d \in \mathcal{O}_K \\ n,d \equiv 1 \ (3) \\ (nd,6)=1}} \frac{G(n^2,d)\psi_1(n)\psi_2(d)}{\mathbb{N}n^{s_2}\mathbb{N}d^w} = Z_4(\infty, s_2, w) = \lim_{\Re(s_1)\to\infty} Z_4(s_1,s_2,w)$$

we have the following determination of correction coefficients. If α is even, then if $p \nmid d_0$,

$$a^{(\alpha)}_{3\alpha,i}(d_0,p) = \begin{cases} \mathbb{N}p^{3\alpha} - \mathbb{N}p^{3\alpha-1}, & \text{if } p \nmid d_0, i = 3\alpha/2 \\ 0 & \text{if } p \nmid d_0, i \neq 3\alpha/2 \\ \mathbb{N}p^{3\alpha} & \text{if } p | d_1, i = 3\alpha/2, \\ 0 & \text{if } p | d_1, i \neq 3\alpha/2. \end{cases}$$

If α even and $p|d_2$, $a^{(\alpha)}_{3\alpha+1,i}(d_0,p)=0$ for all i. If α is odd, then

$$a^{(\alpha)}_{3\alpha,i}(d_0,p)=\begin{cases}\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{3\alpha} & \text{if } p\nmid d_2, i=(3\alpha+1)/2\\ -\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha-1} & \text{if } p\nmid d_2, i=(3\alpha-1)/2\\ 0 & \text{if } p\nmid d_2, i\neq(3\alpha+1)/2,(3\alpha-1)/2\end{cases}$$

If α is odd and $p|d_2$,

$$a^{(\alpha)}_{3\alpha+1,i}(d_0,p)=\begin{cases}\mathbb{N}p^{3\alpha+1} & \text{if } p|d_2, i=(3\alpha+1)/2\\ 0 & \text{if } p|d_2, i\neq(3\alpha+1)/2.\end{cases}$$

Proof of Corollary. This follows immediately by directly comparing the two sides of the equality using the result of the proposition and the careful transformation in Equation (3.14). □

Proof of Proposition. We proceed as before, doing this case by case according to the divisibility of d by each prime p. First, suppose $p\nmid d_0$. Write $d=p^{3\alpha}d'$ with $(d',p)=1$ and $n=p^{\gamma}n'$ with $(n',p)=1$. Then

$$\sum_{n,d}\frac{G(n^2,d)\psi_1(n)\psi_2(d)}{\mathbb{N}n^{s_2}\mathbb{N}d^w}=\sum_{\substack{n',d'\\(n'd',p)=1}}\frac{G((n')^2,d')\psi_1(n')\psi_2(d)}{(\mathbb{N}n')^{s_2}\mathbb{N}d^w}\sum_{\alpha\geq0}\sum_{\gamma\geq0}\frac{g(p^{2\gamma},p^{3\alpha})\bar{\chi}_{d_0}(p^{2\gamma})\psi_1(p^{\gamma})}{\mathbb{N}p^{3\alpha/2+\gamma s_2+3\alpha w}}$$

since

$$\begin{aligned}G(n^2,d)=\frac{g(p^{2\gamma}(n')^2,p^{3\alpha}d')}{\sqrt{\mathbb{N}p^{3\alpha}\mathbb{N}d'}} &= \frac{g(p^{2\gamma}(n')^2,d')}{\sqrt{\mathbb{N}d'}}\frac{g(p^{2\gamma}(n')^2,p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}\\ &= \bar{\chi}_{d_0}(p^{2\gamma})G((n')^2,d')\frac{g(p^{2\gamma},p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}.\end{aligned}$$

We can evaluate the Gauss sum $g(p^{2\gamma},p^{3\alpha})$ according to the following case analysis.

$$g(p^{2\gamma},p^{3\alpha})=\begin{cases}\phi(p^{3\alpha}) & \text{if } 2\gamma\geq3\alpha,\\ -\mathbb{N}p^{3\alpha-1} & \text{if } 2\gamma=3\alpha-1,\\ 0 & \text{otherwise.}\end{cases}$$

Then if α is even, we have

$$\begin{aligned}\sum_{\gamma\geq0}\frac{g(p^{2\gamma},p^{3\alpha})\bar{\chi}_{d_0}(p^{2\gamma})\psi_1(p^{\gamma})}{\mathbb{N}p^{3\alpha/2+\gamma s_2+3\alpha w}} &= \sum_{\gamma\geq3\alpha/2}\frac{\phi(p^{3\alpha})\bar{\chi}_{d_0}(p^{2\gamma})\psi_1(p^{\gamma})}{\mathbb{N}p^{3\alpha/2+\gamma s_2+3\alpha w}}\\ &= L^{(p)}(s_2,\chi_{d_0}\psi_1)\frac{\phi(p^{3\alpha})}{\mathbb{N}p^{3\alpha/2+3\alpha s_2/2+3\alpha w}}\end{aligned}$$

If α is odd, we have $\sum_{\gamma\geq 0} \frac{g(p^{2\gamma},p^{3\alpha})\bar{\chi}_{d_0}(p^{2\gamma})\psi_1(p^\gamma)}{\mathbb{N}p^{3\alpha/2+\gamma s_2+3\alpha w}} =$

$$\begin{aligned}
&= -\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha/2-1-(3\alpha-1)s_2/2-3\alpha w} + \sum_{\gamma\geq(3\alpha+1)/2} \frac{\phi(p^{3\alpha})\bar{\chi}_{d_0}(p^{2\gamma})\psi_1(p^\gamma)}{\mathbb{N}p^{3\alpha/2+\gamma s_2+3\alpha w}} \\
&= -\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha/2-1-(3\alpha-1)s_2/2-3\alpha w} + L^{(p)}(s_2,\chi_{d_0}\psi_1)\frac{\phi(p^{3\alpha})\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)}{\mathbb{N}p^{3\alpha/2+(3\alpha+1)s_2/2+3\alpha w}} \\
&= L^{(p)}(s_2,\chi_{d_0}\psi_1)\Big[\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{3\alpha/2-(3\alpha+1)s_2/2-3\alpha w} - \\
&\qquad \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha/2-1-(3\alpha-1)s_2/2-3\alpha w}\Big]
\end{aligned}$$

Piecing this back together, we have completed the analysis of the case $p \nmid d_0$. Suppose instead that $p|d_1$. Then we may write $d = p^{3\alpha+1}d'$ with $(d',p) = 1$ and $n = p^\gamma n'$ with $(n',p) = 1$. In this case,

$$G(n^2,d) = \frac{g(p^{2\gamma}(n')^2,p^{3\alpha+1}d')}{\sqrt{p^{3\alpha+1}d'}} = 0 \quad \text{unless } 2\gamma = 3\alpha$$

This requires α to be even. In this case,

$$\frac{g(p^{3\alpha}(n')^2,p^{3\alpha+1}d')}{\sqrt{\mathbb{N}p^{3\alpha+1}\mathbb{N}d'}} = \frac{\mathbb{N}p^{3\alpha/2}g((n')^2,pd')}{\sqrt{\mathbb{N}p\mathbb{N}d'}} = \mathbb{N}p^{3\alpha/2}G((n')^2,pd').$$

Lastly, if $p|d_2$, then we may write $d = p^{3\alpha+2}d'$ with $(d',p) = 1$ and $n = p^\gamma n'$ with $(n',p) = 1$. In this case,

$$G(n^2,d) = \frac{g(p^{2\gamma}(n')^2,p^{3\alpha+2}d')}{\sqrt{\mathbb{N}p^{3\alpha+2}\mathbb{N}d'}} = 0 \quad \text{unless } 2\gamma = 3\alpha+1$$

This requires α to be odd. In this case,

$$\begin{aligned}
\frac{g(p^{3\alpha+1}(n')^2,p^{3\alpha+2}d')}{\sqrt{\mathbb{N}p^{3\alpha+2}\mathbb{N}d'}} &= \frac{\mathbb{N}p^{(3\alpha+1)/2}g((n')^2,d')\overline{g((n')^2,p)}}{\sqrt{\mathbb{N}p\mathbb{N}d'}} \\
&= \mathbb{N}p^{(3\alpha+1)/2}G((n')^2,d')\overline{G((n')^2,p)}.
\end{aligned}$$

By removing all of the primes p from d such that $p \nmid d_0$, then primes from d which divide d_1 and lastly, p such that $p|d_2$, we have

$$\begin{aligned}
\sum_{n,d} \frac{G(n^2,d)\psi_1(n)\psi_2(d)}{\mathbb{N}n^{s_2}\mathbb{N}d^w} &= \sum_{\substack{d \\ d=d_1d_2^2d_3^3}} \frac{G(1,d_1)\overline{G(1,d_2)}L(s_2,\chi_{d_0}\psi_1)\psi_2(d)\bar{\psi}_1(d_2)}{(d_1d_2)^w} \\
&\prod_{\substack{p\nmid d_0 \\ p^\alpha || d_3 \\ \alpha \text{ even}}} \frac{\phi(p^{3\alpha})}{\mathbb{N}p^{3\alpha/2+3\alpha s_2/2+3\alpha w}} \cdot \prod_{\substack{p\nmid d_0 \\ p^\alpha || d_3 \\ \alpha \text{ odd}}} \Big(\bar{\chi}_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha/2-(3\alpha+1)s_2/2-3\alpha w} - \\
&\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha/2-1-(3\alpha-1)s_2/2-3\alpha w}\Big) \cdot \prod_{\substack{p|d_1 \\ p^\alpha || d_3 \\ \alpha \text{ even}}} \mathbb{N}p^{3\alpha/2-3\alpha s_2/2-3\alpha w} \\
&\prod_{\substack{p|d_2 \\ p^\alpha || d_3 \\ \alpha \text{ odd}}} \mathbb{N}p^{(3\alpha+1)/2-(3\alpha+1)s_2/2-(3\alpha+1)w} \qquad (3.16)
\end{aligned}$$

Comparing this with the assertion in the proposition, this completes the proof. □

3.4 Additional Limiting Methods: A Rankin-Selberg Argument

We have determined all of the information about correction coefficients from limits of variables in

$$Z_4(s_1, s_2, w) = Z_1(1 - s_1, s_2, w + s_1 - 1/2).$$

Of course, we could similarly have transformed $Z_1(s_1, s_2, w)$ using the L-series in the variable s_2. We have named the resulting object $Z_5(s_1, s_2, w) = Z_1(s_1, 1 - s_2, w + s_2 - 1/2)$. But the result is just the same as $Z_4(s_1, s_2, w)$ with the variables s_1 and s_2 interchanged. Since we already know that the correction coefficients must be symmetric in s_1 and s_2, this gives no new information about the coefficients.

On the other hand, we might begin with $Z_1(s_1, s_2, w)$ and transform both arguments of the L-series, s_1 and s_2, according to the natural functional equations. What happens when we take limits of variables in the resulting object $Z_6(s_1, s_2, w) = Z_1(1-s_1, 1-s_2, w+s_1+s_2-1)$ and compare with the Dirichlet series suggested by square-free heuristics? Do we obtain new information about the correction coefficients? As we will demonstrate, the answer is no.

Recall that in the opening section of this chapter, we found that considering only the restricted sum over square-free integers, we have

$$Z_1(1 - s_1, 1 - s_2, w + s_1 + s_2 - 1) = \sum_{d \text{ sq. free}} \frac{L(1 - s_1, \chi_d)L(1 - s_2, \chi_d)}{d^{w+s_1+s_2-1}} =$$

$$\sum_{d \text{ sq. free}} \frac{L(s_1, \bar{\chi}_d)L(s_2, \bar{\chi}_d)G(1, d)^2}{\mathbb{N}d^w} = \sum_{\substack{m,n,d \\ \text{sq. free}}} \frac{\overline{G_6(m^2n^2, d)}}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^w} = Z_6(s_1, s_2, w)$$

Then, at least for square-free integers, we have

$$\lim_{\Re(s_1)\to\infty} Z_6(s_1, s_2, w) \stackrel{\text{def}}{=} Z_6(\infty, s_2, w) = \sum_{\substack{m,n,d \\ \text{sq. free}}} \frac{\overline{G_6(n^2, d)}}{\mathbb{N}n^{s_2}\mathbb{N}d^w}$$

Taking this to be true for all integers, we can solve for correction coefficients using the equality

$$\lim_{\Re(s_1)\to\infty} Z_1(1 - s_1, 1 - s_2, w + s_1 + s_2 - 1) = \sum_{n,d} \frac{\overline{G_6(n^2, d)}}{\mathbb{N}n^{s_2}\mathbb{N}d^w}.$$

We could decompose the Dirichlet series on the right according to primes dividing d as we did for a similar comparison of

$$\lim_{\Re(s_1)\to\infty} Z_1(1 - s_1, s_2, w + s_1 - 1/2) = \sum_{n,d} \frac{G_3(n^2, d)}{\mathbb{N}n^{s_2}\mathbb{N}d^w}$$

and show by direct examination that the resulting correction coefficients from the two equalities are the same. However, we offer an alternate approach here using a Rankin-Selberg unfolding argument. The result is posed for Dirichlet series without additional twists by ψ_i in order to preserve the clarity of the method, but can easily be extended to series with such additional characters added.

Proposition 3.5. *The following Dirichlet series are equal:*

$$\sum_{\substack{m,d\in\mathcal{O}_K\\ m,d\equiv 1(3)}} \frac{\overline{G_6(m^2,d)}}{\mathbb{N}m^s\mathbb{N}d^w} = \sum_{\substack{m,d\in\mathcal{O}_K\\ m,d\equiv 1\ (3)}} \frac{G_3(m^2,d)}{\mathbb{N}m^s\mathbb{N}d^w}$$

Proof. We first recall several definitions. The metaplectic Eisenstein series on the n-fold cover of $GL(2)$ has Fourier series

$$E^{(n)}(z,s) = y^{2s} + \frac{\zeta(2ns-n)}{\zeta(2ns-n+1)}y^{2-2s} + y\sum_{m\neq 0} A_m(s)\mathbb{N}m^{s-1/2}K_{2s-1}(4\pi\,|m|\,y)\exp(mx)$$

where K is a Bessel function and

$$A_m(s) = \sum_d \frac{g_n(m,d)}{(\mathbb{N}d)^{2s}}.$$

More generally, one can define an Eisenstein series on a cover of an appropriately restricted congruence subgroup Γ of $SL(2,\mathcal{O}_K)$ where K is a number field containing n^{th} roots of unity. For example, our series above with cubic Gauss sum is built out of Fourier coefficients of Eisenstein series on $\Gamma(3)$, the principle congruence subgroup of level 3. It consists of 2×2 matrices in $SL(2,\mathcal{O}_K)$ congruent to the identity matrix mod 3. Then the Fourier coefficient is precisely a sum over integers $d\equiv 1\ (3)$
$(d,6)=1$. For a detailed account of these Eisenstein series, we refer the reader to Kubota's book [18]. We further define the higher degree theta functions as a residue, $\Theta^{(n)}(z) = \mathrm{Res}_{2s=1+\frac{1}{n}}E^{(n)}(z,s)$. So for $n=2$, the Fourier expansion has form

$$\Theta^{(2)}(z) = y^{1/2} + \sum_{m\neq 0}\frac{\tau_2(m)}{(\mathbb{N}m)^{1/2}}W(|m|\,y)\exp(mx)$$

where $W(y) = yK_{1/2}(4\pi y) = y\sqrt{\frac{1}{8y}}e^{-4\pi y}$ and

$$\tau_2(m) = \begin{cases}(\mathbb{N}m_2)^{1/2} & \text{if } m = m_2^2,\\ 0 & \text{otherwise.}\end{cases}$$

This is carefully written in [13]. Again, we should take the theta function to be on $\Gamma(3)$ in order to appropriately restrict the sum over m to integers congruent to 1 mod 3. From these definitions, we see that, in particular,

$$L(w,E^{(n)}(z,s)) = \sum_{\substack{m\in\mathcal{O}_K\\ m\equiv 1\ (3)\\ (m,6)=1}} \frac{A_m^{(n)}(s)\mathbb{N}m^{s-1/2}}{\mathbb{N}m^w} \quad\text{and}\quad L(w,\Theta^{(n)}(z)) = \sum_{\substack{m\in\mathcal{O}_K\\ m\equiv 1\ (3)\\ (m,6)=1}} \frac{\tau_n(m)}{\mathbb{N}m^w}$$

so that the L-series of the Rankin-Selberg convolution of a cubic Eisenstein series and a quadratic theta function has form

$$L(w,E^{(3)}(z,s)\times\Theta^{(2)}(z)) = \sum_{\substack{m\in\mathcal{O}_K\\ m\equiv 1\ (3)\\ (m,6)=1}} \frac{\tau_2(m)A_m^{(3)}(s)\mathbb{N}m^{s-1/2}}{\mathbb{N}m^w} =$$

$$= \sum_{\substack{m \in \mathcal{O}_K \\ m \equiv 1\ (3) \\ (m,6)=1}} \frac{A^{(3)}_{m^2}(s)}{\mathbb{N}m^{2w-2s+1/2}} = \sum_{\substack{m,d \in \mathcal{O}_K \\ m,d \equiv 1\ (3) \\ (d,6)=1}} \frac{G_3(m^2,d)}{\mathbb{N}d^{2s-1/2}\mathbb{N}m^{2w-2s+1/2}}$$

Rankin-Selberg convolutions for $GL(2)$ automorphic forms are discussed carefully in Section 1.6 of [2]. Notice that, up to a change in variables, we have realized one of the series in the statement of our proposition via convolution. Recall that we produce such an L-series by taking the Mellin transform

$$\int_0^\infty \int_0^1 \overline{\Theta^{(2)}(z)} E^{(3)}(z,s) y^{2w} d\mu$$

where $d\mu = \dfrac{dxdy}{y^3}$. Substituting the form of the Fourier transforms yields

$$\int_0^\infty \int_0^1 \sum_{m,m'} \frac{\tau_2(m) A^{(3)}_{m'}(s)(\mathbb{N}m')^{s-1/2}}{(\mathbb{N}m)^{1/2}(\mathbb{N}m')^{1/2}} \exp((m-m')x) y^{2w} W_\theta(|m|\, y) W_E(|m'|\, y) \frac{dxdy}{y^3}.$$

Integrating over x, we obtain

$$\int_0^\infty \sum_m \frac{\tau_2(m) A^{(3)}_m(s)(\mathbb{N}m)^{s-1/2}}{\sqrt{(\mathbb{N}m)}^2} y^{2w} W_\theta(|m|\, y) W_E(|m|\, y) \frac{dy}{y^3}.$$

Now letting $y = |m|^{-1} y'$, we have

$$\sum_m \frac{\tau_2(m) A^{(3)}_m(s)(\mathbb{N}m)^{s-1/2}}{\mathbb{N}m^w} \int_0^\infty y^{2w-2} W_\theta(y) W_E(y) \frac{dy}{y}.$$

We often refer to the sum and integral as the arithmetic and infinite pieces, respectively, where the arithmetic piece is used in the L-series of the convolution. This is the arithmetic representation of the convolution we used in the L-series above.

Using the usual Rankin-Selberg technique of unfolding and refolding the integral, we obtain an alternate representation of our original integral

$$\int_0^\infty \int_0^1 \overline{\Theta^{(2)}(z)} E^{(3)}(z,s) y^{2w} \frac{dxdy}{y^3} = \int_{\Gamma_\infty \backslash \mathcal{H}} \overline{\Theta^{(2)}(z)} E^{(3)}(z,s) y^{2w} \frac{dxdy}{y^3}$$

where Γ_∞ denotes the subgroup of Γ stabilizing the cusp at ∞ which consists of all integer translations and $\mathcal{H}$ is the upper-half plane. Now unfolding this integral, the above can be written as

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \iint_{\gamma(D)} \overline{\Theta^{(2)}(z)} E^{(3)}(z,s) y^{2w} \frac{dxdy}{y^3} = \iint_D \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \overline{\Theta^{(2)}(\gamma z)} E^{(3)}(\gamma z,s) \Im(\gamma(z))^{2w} \frac{dxdy}{y^3}$$

where D is a fundamental domain for $\Gamma \backslash \mathcal{H}$. However, $\Theta^{(2)}$ and $E^{(3)}$ are automorphic functions on the metaplectic group; that is, $\Theta^{(2)}(\gamma z) = \kappa_2(\gamma)\Theta^{(2)}(z)$ and $E^{(3)}(\gamma z, s) =$

$\overline{\kappa_3(\gamma)}E^{(3)}(z,s)$, where κ_n denotes the n^{th} order Kubota symbol. Performing a substitution, the above integral becomes

$$\iint_D \overline{\Theta^{(2)}(z)}E^{(3)}(z,s)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\overline{\kappa_6(\gamma)}\Im(\gamma(z))^{2w}d\mu = \iint_D \overline{\Theta^{(2)}(z)}E^{(3)}(z,s)\overline{E^{(6)}(z,w)}d\mu.$$

Now unfolding the Eisenstein series $E^{(3)}(z,s)$,

$$\begin{aligned}
\iint_D \overline{\Theta^{(2)}(z)}E^{(3)}(z,s)\overline{E^{(6)}(z,w)}d\mu &= \iint_D \overline{\Theta^{(2)}(z)E^{(6)}(z,w)}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\kappa_3(\gamma)\Im(\gamma(z))^{2s}d\mu \\
&= \iint_D \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\overline{\Theta^{(2)}(\gamma z)E^{(6)}(\gamma z,w)} \\
&\qquad \cdot \kappa_2(\gamma)\overline{\kappa_6(\gamma)}\kappa_3(\gamma)\Im(\gamma(z))^{2s}d\mu \\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\iint_{\gamma(D)}\overline{\Theta^{(2)}(z)E^{(6)}(z,w)}y^{2s}d\mu \\
&= \int_0^\infty\int_0^1 \overline{\Theta^{(2)}(z)E^{(6)}(z,w)}y^{2s}d\mu \\
&= \int_0^\infty\int_0^1 \sum_{m,m'}\frac{\tau_2(m)\bar{A}^{(6)}_{m'}(s)(\mathbb{N}m')^{w-1/2}}{(\mathbb{N}m)^{1/2}(\mathbb{N}m')^{1/2}} \\
&\qquad \cdot \exp((m-m')x)y^{2s}W_\theta(|m|\,y)W_{E6}(|m'|\,y)\frac{dxdy}{y^3} \\
&= \int_0^\infty \sum_m \frac{\tau_2(m)\bar{A}^{(6)}_m(s)(\mathbb{N}m)^{w-1/2}}{(\mathbb{N}m)^{1/2}(\mathbb{N}m)^{1/2}}y^{2s} \\
&\qquad \cdot W_\theta(|m|\,y)W_{E6}(|m|\,y)\frac{dy}{y^3}
\end{aligned}$$

As before, letting $y=|m|^{-1}y'$, we have

$$\sum_m \frac{\overline{G_6(m^2,d)}}{\mathbb{N}m^{2s-2w+1/2}\mathbb{N}d^{2w-1/2}}\int_0^\infty y^{2s-2}W_\theta(y)W_{E6}(y)\frac{dy}{y}$$

. Tracing back along this long string of equalities among integrals, we can equate the arithmetic parts of the convolution so that, for the respective L-series,

$$\sum_m \frac{\overline{G_6(m^2,d)}}{\mathbb{N}m^{2s-2w+1/2}\mathbb{N}d^{2w-1/2}} = \sum_m \frac{G_3(m^2,d)}{\mathbb{N}m^{2w-2s+1/2}\mathbb{N}d^{2s-1/2}}$$

Finally, letting $S=2s-2w+1/2$ and $W=2w-1/2$, then $W+S-1/2=2s-1/2$ and $1-S=2w-2s+1/2$. Substituting this change of variables into the above gives the result. □

3.5 Limits at Infinity and the Coefficients of $Q(w,m,n)$

We have exhausted all of the limiting methods associated to functional equations in s_1 and s_2. In this section, we will apply these techniques to the series Z_1 with the order of summation reversed. Recall that it takes form

$$Z_1(s_1, s_2, w) = \sum_{\substack{m,n \in \mathcal{O}_K \\ m,n \equiv 1\ (3) \\ (mn,6)=1}} \frac{L(w, \chi_{\underline{mn}_0} \psi_2) \psi_1(mn) Q(w, m, n)}{\mathbb{N}m^{s_1} \mathbb{N}n^{s_2}}$$

Remember that $\underline{mn}_0$ denotes the cube-free part of the product of integers mn. This object inherits a natural functional equation as $w \to 1 - w$ from the L-series with argument w in the numerator. Accordingly, the above should be equal to

$$\sum_{\substack{m,n \in \mathcal{O}_K \\ m,n \equiv 1\ (3)}} \frac{L(1-w, \overline{\chi}_{\underline{mn}_0} \bar{\psi}_2) G(1, \underline{mn}_1) \overline{G(1, \underline{mn}_2)} G(\psi_2) \bar{\psi}_2(\underline{mn}_2) Q(w, m, n)}{\mathbb{N}m^{s_1} \mathbb{N}n^{s_2} \mathbb{N}\underline{mn}_1^{1/2-w} \mathbb{N}\underline{mn}_2^{1/2-w}} \tag{3.17}$$

Just as we tried in the case of functional equations in the s_1 and s_2 variables, we want to attempt to relate this to some ideal Dirichlet series with additional functional equations. Such a perfect series would determine the coefficients of Q and thus complete $Z(s_1, s_2, w)$ so that exact transformation properties hold. To find this object, we again rely on the intuition provided by the square-free integers. If the above was summed over m, n which were square-free, then neglecting bad primes and congruence conditions, we would have:

$$\sum_{m,n} \frac{L(1-w, \overline{\chi}_{mn}) G(1, mn)}{\mathbb{N}m^{s_1+w-1/2} \mathbb{N}n^{s_2+w-1/2}} = \sum_{m,n,d} \frac{\overline{\chi}_{mn}(d) G(1, mn)}{\mathbb{N}m^{s_1+w-1/2} \mathbb{N}n^{s_2+w-1/2} \mathbb{N}d^{1-w}} \tag{3.18}$$

The ideal object associated to the above should similarly be described in terms of the variables $(s_1 + w - 1/2, s_2 + w - 1/2, 1 - w)$, but we can perform a change of variables on (3.18) with $s_i \mapsto s_i + w - 1/2$ and $w \mapsto 1 - w$ (since this transformation is an involution) so that the ideal object is expressed in terms of (s_1, s_2, w). Under the square-free heuristic, we now have $Z_1(s_1 + w - 1/2, s_2 + w - 1/2, 1 - w) =$

$$\sum_{m,n,d} \frac{\overline{\chi}_{mn}(d) G(1, mn)}{\mathbb{N}m^{s_1} \mathbb{N}n^{s_2} \mathbb{N}d^{w}} = \sum_{m,n,d} \frac{G(d, mn)}{\mathbb{N}m^{s_1} \mathbb{N}n^{s_2} \mathbb{N}d^{w}} \quad \text{if } (d, mn) = 1 \text{ for all } m, n, d$$

Recall that we defined the resulting Dirichlet series as $Z_3(s_1, s_2, w)$ in (3.3). Then $Z_3(s_1, s_2, w)$ contains a cubic Gauss sum in the numerator and its cube-free part is essentially the Mellin transform of a Rankin-Selberg convolution of an ordinary (non-metaplectic) Eisenstein series with a $GL(2)$ automorphic form. More explicitly, we have

$$\begin{aligned} Z_3(s_1, s_2, w) &= \sum_{d,m,n} \frac{G_3(d, mn)}{\mathbb{N}d^w \mathbb{N}m^{s_1} \mathbb{N}n^{s_2}} \\ (\text{writing } M = mn) &= \sum_{d,M} \frac{G_3(d, M) \sigma_{s_1 - s_2}(M)}{\mathbb{N}d^w \mathbb{N}m^{s_1}} \\ (\text{more symmetrically}) &= \sum_{d,M} \frac{\chi_d(M) G_3(1, M) \sigma_{s_1-s_2}(M) \mathbb{N}M^{(s_1-s_2)/2}}{\mathbb{N}M^{(s_1+s_2)/2}} \end{aligned}$$

From this final incarnation, we see that the numerator is realized as a Rankin-Selberg convolution of a twisted cubic theta function and a non-metaplectic Eisenstein series $E(z, (s_1 - s_2 + 1)/2)$. This object has a $GL(4)$ functional equation as $s_i \to 1 - s_i$ so that $w \to w + 2s_1 + 2s_2 - 2$

and poles at $2s_i - 1/2 = 1/2 + 1/3$ according to the usual Fourier analysis which transforms the argument of the Eisenstein series.

Simply taking this to be the definition of the $Z_1(s_1, s_2, w)$ under the transformation with $w \mapsto 1 - w$ turns out to be too simple to provide us with the rest of the necessary functional equations. However, in taking the limit as $\Re(w) \to \infty$, we reduce to a much simpler object. Here it is reasonable to assume that the Dirichlet series suggested by the square-free heuristic yields the correct definition. Let us assume this is the case.

Now taking the limit as $Re(w) \to \infty$ in the square-free case, all terms associated to d's with $\mathbb{N}d > 1$ vanish, and we are left with

$$\lim_{\Re(w)\to\infty} \sum_{m,n,d} \frac{\overline{\chi}_{mn}(d) G(1, mn)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^{w}} = \sum_{m,n} \frac{G(1, mn)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}}.$$

According to the above discussion, the appropriate generalization of this series to a sum over all integers is a Rankin-Selberg convolution of an *untwisted* cubic theta function and an Eisenstein series. We take this to be the ideal object and we require that it is equal to (3.17) under the involution $(s_1, s_2, w) \mapsto (s_1 + w - 1/2, s_2 + w - 1/2, 1 - w)$. That is, we must find coefficients of $Q(w, m, n)$ so that

$$\lim_{Re(w)\to\infty} \sum_{m,n} \frac{L(w, \overline{\chi}_{\underline{mn}_0}\bar{\psi}_2) G(1, \underline{mn}_1)\overline{G(1, \underline{mn}_2)}\bar{\psi}_2(\underline{mn}_2) Q(1 - w, m, n)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}\underline{mn}_2^{w-1/2}\mathbb{N}\underline{mn}_3^{3w-3/2}} = \sum_{m,n} \frac{\tau_3(mn)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} \tag{3.19}$$

where $\tau_3(m)$ denotes the m^{th} Fourier coefficient of the cubic theta function.

We want to take the limit of the left-hand side of (3.19) above. Recall that $Q(w, m, n)$ was expressed as

$$Q(w, m, n) = \prod_{p^\beta || M_3} \sum_{l} b_l(m, n, p^\beta)\mathbb{N}p^{-lw} = \sum_{f | M_3^\infty} b_f(m, n, M_3)\mathbb{N}f^{-w}$$

where $mn = M_0 M_3^3$ with M_0 square-free. The notation above also implicitly states that $\beta = ord_p(\underline{mn}_3) = ord_p(M_3)$. Further, define $\beta_2 = ord_p(\underline{mn}_2) \in \{0, 1\}$. We refrain from indexing the β by p to streamline the notation. We may now rewrite the left-hand side of (3.19) as:

$$\lim_{Re(w)\to\infty} \sum_{m,n} \frac{\overline{\chi}_{\underline{mn}_0}(d)) G(1, \underline{mn}_1)\overline{G(1, \underline{mn}_2)} G(\psi_2)\bar{\psi}_2(\underline{mn}_2) \prod_{p^\beta || M_3} \sum_l b_l(m, n, p^\beta)\mathbb{N}p^{-l+lw}}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}\underline{mn}_2^{w-1/2}\mathbb{N}\underline{mn}_3^{3w-3/2}\mathbb{N}d^{w}}$$

Hence, the terms that survive in the limit will be those which have $\mathbb{N}d = 1$ and are also associated to $b_l(m, n, p^\beta)$ with $l - 3\beta - \beta_2 = 0$. Indeed, if $l - 3\beta - \beta_2 < 0$, then these terms go to 0 in the limit. If $l - 3\beta - \beta_2 > 0$, then the object does not converge. This immediately implies that if our correction factor is to produce a coherent theory, we must have $b_l(m, n, p^\beta) = 0$ for all $l > 3\beta + \beta_2$. Now our (3.19) reads:

$$\sum_{m,n} \frac{G(1, \underline{mn}_1)\overline{G(1, \underline{mn}_2)} G(\psi_2)\bar{\psi}_2(\underline{mn}_2)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} \prod_{p^\beta || M_3} b_{3\beta}(m, n, p^\beta)\mathbb{N}p^{(-3\beta-\beta_2)/2} = \sum_{m,n} \frac{\tau_3(mn)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}}$$

To finish, we need an explicit determination of the right-hand side. To determine the coefficient $b_{3\beta}(m, n, p^\beta)$, we can fix a prime p with m and n chosen so that $ord_p(\underline{mn}_3) =$

$ord_p(M_3) = \beta$. But $\tau_3(mn)$, the fourier coefficient of the cubic theta function on the field $K = \mathbb{Q}(\sqrt{-3})$, has the following transformation properties:

$$\tau_3(mp^2) = 0, \tau_3(mp) = G(m,p)\tau_3(m), \tau_3(mp^3) = \mathbb{N}p^{1/2}\tau_3(m)$$

This implies that if $p|\underline{mn_2}$, then the right-hand side of our equation is 0, while $G(1, \underline{mn_2})$ is non-zero, so $b_{3\beta+1} = 0$. If $p \not| \underline{mn_2}$, then

$$\sum_{m,n} \frac{G(1,mn)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} = \sum_{m,n} \frac{G(1,\underline{mn_1})\mathbb{N}p^{\beta/2}}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}}.$$

Cancelling common factors on both sides leaves:

$$b_{3\beta}(m,n,p^\beta)\mathbb{N}p^{-3\beta/2} = \mathbb{N}p^{\beta/2}$$

And so,

$$b_{3\beta}(m,n,p^\beta) = \mathbb{N}p^{2\beta}$$

This completes the method of taking limits as we have exhausted all possible natural functional equations of the initial object $Z(s_1, s_2, w)$.

3.6 A Summary of Assumptions on the Correction Factor

We need to show that correction factors exist which satisfy all of the necessary properties to guarantee exact functional equations. To close the chapter, we now summarize the collection of assumptions we have placed on these Dirichlet polynomials $P(s_1, s_2, d)$ and $Q(w, m, n)$.

Assumption 2. *The Dirichlet polynomials $P(s_1, s_2, d; \psi_1, \psi_1)$ and $Q(w, m, n; \psi_2)$ should satisfy the following properties:*

- *They are finite, Eulerian Dirichlet polynomials depending only on the indicated quantities.*
- $P(s_1, s_2, d) = P(s_2, s_1, d)$ *and* $Q(w, m, n) = Q(w, n, m)$.
- $P(s_1, s_2, d) = 1$ *if d is square-free.*
- $Q(w, m, n) = 1$ *if mn is square-free.*
- *They can be chosen so that the interchange equality (3.1)=(3.2) is satisfied.*
- *They can be chosen so that Z_3, Z_4, Z_5 and Z_6, defined according to (3.3)-(3.6), satisfy additional functional equations into themselves.*
- *If we expand P and Q as Euler products of form*

  $$P(s_1, s_2, d) = \prod_{p^\alpha || d_3} \sum_{i,j} a_{i,j}(d_0, p^\alpha)\mathbb{N}p^{-is_1 - js_2}$$

  and

  $$Q(w, m, n) = \prod_{p^\beta || M_3} \sum_{l} b_l(m, n, p^\beta)\mathbb{N}p^{-lw}$$

then the coefficients $a_{i,j}$ and b_l should agree with the coefficients determined by the limiting methods. In total, these are

$$a^{(\alpha)}_{i,0}(d_0,p) = a^{(\alpha)}_{0,i}(d_0,p) = \begin{cases} 1 & \text{if } i=0, \alpha \geq 0 \\ -\chi_{d_0}(p)\psi_1(p) & \text{if } i=1, \alpha > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$a^{(\alpha)}_{3\alpha,i}(d_0,p) = \begin{cases} \mathbb{N}p^{3\alpha} - \mathbb{N}p^{3\alpha-1}, & \text{if } p \nmid d_0, i = 3\alpha/2 \\ 0 & \text{if } p \nmid d_0, i \neq 3\alpha/2 \\ \mathbb{N}p^{3\alpha} & \text{if } p | d_1, i = 3\alpha/2, \\ 0 & \text{if } p|d_1, i \neq 3\alpha/2. \end{cases}$$

If α even and $p|d_2$, $a^{(\alpha)}_{3\alpha+1,i}(d_0,p) = 0$ for all i. If α is odd, then

$$a^{(\alpha)}_{3\alpha,i}(d_0,p) = \begin{cases} \bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{3\alpha} & \text{if } p \nmid d_2, i = (3\alpha+1)/2 \\ -\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha-1} & \text{if } p \nmid d_2, i = (3\alpha-1)/2 \\ 0 & \text{if } p \nmid d_2, i \neq (3\alpha+1)/2, (3\alpha-1)/2 \end{cases}$$

If α is odd and $p|d_2$,

$$a^{(\alpha)}_{3\alpha+1,i}(d_0,p) = \begin{cases} \mathbb{N}p^{3\alpha+1} & \text{if } p|d_2, i = (3\alpha+1)/2 \\ 0 & \text{if } p|d_2, i \neq (3\alpha+1)/2. \end{cases}$$

and

$$b_{3\beta}(m,n,p^\beta) = \mathbb{N}p^{2\beta} \quad \text{if } p \nmid \underline{mn}_2, \text{ for any choice of } m,n \text{ with } ord_p(mn) = \beta.$$

$$b_{3\beta+1}(m,n,p^\beta) = 0 \quad \text{if } p | \underline{mn}_2, \text{ for any choice of } m,n \text{ with } ord_p(mn) = \beta.$$

Chapter 4

Preparing to Interchange the Order of Summation

4.1 Restrictions on Coefficients from Interchanging Summation

Now that we have determined information about the correction factors imposed by functional equations in the variables s_i and w, we want to examine the effects of the interchange inequality on the form of the correction coefficients for both $P(s_1, s_2; d)$ and $Q(w; m, n)$.

Recall that $Z_1(s_1, s_2, w)$ took the form

$$\sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\ (3) \\ (d,6)=1}} \frac{L_6(s_1, \chi_{d_0}\psi_1) L_6(s_2, \chi_{d_0}\psi_1)\psi_2(d) P(s_1, s_2, d)}{\mathbb{N}d^w}$$

But since the correction factor can be expanded as

$$P(s_1, s_2, d) = \prod_{\substack{p^\alpha || d_3 \\ p^{\alpha_2} || d_2}} \sum_{i,j} a_{i,j}(d_0, p^\alpha) \mathbb{N}p^{-is_1 - js_2} = \sum_{e_1 e_2 | (d_2 d_3)^\infty} a_{e_1,e_2}(d_0, d_3^3) \mathbb{N}e_1^{-s_1} \mathbb{N}e_2^{-s_2},$$

we may rewrite Z_1 as

$$\begin{aligned} Z_1(s_1, s_2, w) &= \sum_{\substack{m,n,d \in \mathcal{O}_K \\ m,n,d \equiv 1\ (3) \\ (mnd,6)=1}} \frac{\chi_{d_0}(mn)\psi_1(mn)\psi_2(d)}{\mathbb{N}m^{s_1} \mathbb{N}n^{s_2} \mathbb{N}d^w} \sum_{e_1 e_2 | (d_2 d_3)^\infty} a_{e_1,e_2}(d_0, d_3^3) \mathbb{N}e_1^{-s_1} \mathbb{N}e_2^{-s_2} \\ &= \sum_{\substack{M,N \in \mathcal{O}_K \\ M,N \equiv 1\ (3) \\ (MN,6)=1}} \sum_{\substack{d \equiv 1\ (3) \\ me_1 = M \\ ne_2 = N \\ e_i | (d_2 d_3)^\infty}} \frac{\chi_{d_0}(mn)\psi_1(mn)\psi_2(d) a_{e_1,e_2}(d)}{\mathbb{N}M^{s_1} \mathbb{N}N^{s_2} \mathbb{N}d^w} \end{aligned} \tag{4.1}$$

where, in the last step, we have combined all contributions associated to the variables s_1 and s_2. Note that if $(d, MN) = 1$, then $e_1 = e_2 = 1$ since we require $e_1 e_2 | (d_2 d_3)^\infty$, so in particular, $a_{e_1,e_2}(d) = a_{1,1}(d) = 1$ for all such d. Motivated by this reordering, in what follows we will renormalize the correction coefficients according to the following definition. Define $\tilde{a}_{e_1,e_2}(d)$ by

$$a_{e_1,e_2}(d) \stackrel{\text{def}}{=} \chi_{d_0}(e_1e_2)\psi_1(e_1e_2)\tilde{a}_{e_1,e_2}(d).$$

Note that for all of the coefficients we have determined (compare, for example, Assumption 2 which concludes Chapter 3), the renormalized coefficients $\tilde{a}_{e_1,e_2}(d)$ are real-valued. This further suggests that such a definition is natural. Incorporating this notation into (4.1), we have

$$\sum_{\substack{M,N\in\mathcal{O}_K \\ M,N\equiv 1\ (3) \\ (MN,6)=1}} \sum_{\substack{d\equiv 1\ (3) \\ me_1=M \\ ne_2=N \\ e_i|(d_2d_3)^\infty}} \frac{\chi_{d_0}(MN)\psi_1(MN)\psi_2(d)\tilde{a}_{e_1,e_2}(d)}{\mathbb{N}M^{s_1}\mathbb{N}N^{s_2}\mathbb{N}d^w} \tag{4.2}$$

Moreover, if $(d, MN) = 1$, then $\chi_{d_0}(MN) = \chi_{\underline{MN}_0}(d)$ by cubic reciprocity, where $\underline{MN}_0$ denotes the cube-free part of the product MN. Reinterpreting the sum over all integers d as a sum over integers d relatively prime to MN multiplied by all possible powers of divisors of MN, we can rewrite the above (4.2) as

$$\sum_{\substack{M,N\equiv 1\ (3) \\ (MN,6)=1}} \sum_{\substack{d\equiv 1\ (3) \\ (d,MN)=1}} \frac{\chi_{\underline{MN}_0}(d)\psi_2(d)}{\mathbb{N}M^{s_1}\mathbb{N}N^{s_2}\mathbb{N}d^w} \prod_{\substack{p \\ p^{k_1}||M \\ p^{k_2}||N}} \sum_{\substack{\alpha\geq 0 \\ \alpha=\alpha_1+2\alpha_2+3\alpha_3}} \frac{\chi_{p^{\alpha_1+2\alpha_2}}(MN)\psi_1(MN)\psi_2(p^\alpha)}{\mathbb{N}p^{-\alpha w}} \cdot \sum_{\substack{c_1,c_2 \\ c_1\leq k_1, c_2\leq k_2 \\ 0\leq c_1,c_2\leq 3\alpha_3+\alpha_2}} \tilde{a}^{(\alpha_3)}_{c_1,c_2}(p^\alpha)$$

Performing this sum over d such that $(d, MN) = 1$, we have $Z_1(s_1, s_2, w) =$

$$\sum_{\substack{M,N\in\mathcal{O}_K \\ M,N\equiv 1\ (3) \\ (MN,6)=1}} \frac{L_{3MN}(w, \chi_{\underline{MN}_0}\psi_2)}{\mathbb{N}M^{s_1}\mathbb{N}N^{s_2}} \prod_{\substack{p \\ p^{k_1}||M \\ p^{k_2}||N}} \sum_{\substack{\alpha\geq 0 \\ \alpha=\alpha_1+2\alpha_2+3\alpha_3}} \frac{\chi_{p^{\alpha_1+2\alpha_2}}(MN)\psi_1(MN)\psi_2(p^\alpha)}{\mathbb{N}p^{-\alpha w}} \cdot \sum_{\substack{c_1,c_2 \\ c_1\leq k_1, c_2\leq k_2 \\ 0\leq c_1,c_2\leq 3\alpha_3+\alpha_2}} \tilde{a}^{(\alpha_3)}_{c_1,c_2}(p^\alpha) \tag{4.3}$$

where $L_{3MN}(w, \chi_{\underline{MN}_0})$ denotes the L-series with Euler factors corresponding to $p|3MN$ removed. Compare this with the form of $Z_1(s_1, s_2, w)$ with the order of summation reversed. With the outer sum over m and n, we defined the Dirichlet series $Z_1(s_1, s_2, w) =$

$$\sum_{\substack{m,n\in\mathcal{O}_K \\ m,n\equiv 1\ (3) \\ (mn,6)=1}} \frac{L_6(w, \chi_{\underline{mn}_0}\psi_2)Q(w, m, n)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} = \sum_{m,n} \frac{L_6(w, \chi_{\underline{mn}_0}\psi_2)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} \prod_{p^\beta||M_3} \sum_{l\geq 0} b_l(m, n, p^\beta)\mathbb{N}p^{-lw}$$

If we abuse the previous notation slightly, and substitute $M = m$ and $N = n$ from the formulation of $Z_1(s_1, s_2, w)$ in (4.3), we obtain the similar looking

$$\sum_{\substack{m,n\in\mathcal{O}_K \\ m,n\equiv 1\ (3) \\ (mn,6)=1}} \frac{L_{3mn}(w, \chi_{\underline{mn}_0}\psi_2)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} \prod_{\substack{p \\ p^{k_1}||m \\ p^{k_2}||n}} \sum_{\substack{\alpha\geq 0 \\ \alpha=\alpha_1+2\alpha_2+3\alpha_3}} \frac{\chi_{p^{\alpha_1+2\alpha_2}}(mn)\psi_1(mn)\psi_2(p^\alpha)}{\mathbb{N}p^{-\alpha w}} \sum_{\substack{c_1,c_2 \\ c_1\leq k_1, c_2\leq k_2 \\ 0\leq c_1,c_2\leq 3\alpha_3+\alpha_2}} \tilde{a}^{(\alpha_3)}_{c_1,c_2}(p^\alpha)$$

$$= \sum_{m,n} \frac{L_6(w, \chi_{\underline{mn}_0}\psi_2)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}} \prod_{\substack{p \\ p^{k_1}||m \\ p^{k_2}||n}} \left[1 - \frac{\chi_{\underline{mn}_0}(p)\psi_2(p)}{\mathbb{N}p^w}\right] \cdot$$

$$\sum_{\substack{\alpha \geq 0 \\ \alpha=\alpha_1+2\alpha_2+3\alpha_3}} \frac{\chi_{p^{\alpha_1+2\alpha_2}}(mn)\psi_1(mn)\psi_2(p^\alpha)}{\mathbb{N}p^{-\alpha w}} \sum_{\substack{c_1,c_2 \\ c_1 \leq k_1, c_2 \leq k_2 \\ 0 \leq c_1, c_2 \leq 3\alpha_3+\alpha_2}} \tilde{a}^{(\alpha_3)}_{c_1,c_2}(p^\alpha)$$

by adding back in the missing Euler factors corresponding to primes $p|mn$. This implies that, in order for the interchange equality hold, for any choice of m, n with $p^{k_1}|m$ and $p^{k_2}|n$,

$$\sum_l b_l(m,n,p^\beta)\mathbb{N}p^{-lw} = \left[1 - \frac{\chi_{\underline{mn}_0}(p)\psi_2(p)}{\mathbb{N}p^w}\right] \sum_{\substack{\alpha \geq 0 \\ \alpha=\alpha_1+2\alpha_2+3\alpha_3}} \frac{\chi_{p^{\alpha_1+2\alpha_2}}(mn)\psi_1(mn)\psi_2(p^\alpha)}{\mathbb{N}p^{-\alpha w}} \sum_{\substack{c_1,c_2 \\ c_1 \leq k_1, c_2 \leq k_2 \\ 0 \leq c_1, c_2 \leq 3\alpha_3+\alpha_2}} \tilde{a}^{(\alpha_3)}_{c_1,c_2}(p^\alpha) \quad (4.4)$$

We now exploit our previous investigations of the coefficients $b_l(m,n;p^\beta)$. Recall that in the previous chapter, we determined that

$$b_{3\beta}(m,n;p^\beta) = \mathbb{N}p^{2\beta}, \quad b_l(m,n;p^\beta) = 0 \quad \text{if } l > 3\beta, \text{ for any choice of } m, n$$

According to the latter result, the contribution from the right-hand side of (4.4) at $\mathbb{N}p^{-lw}$ must be 0 for $l > 3\beta + \beta_2$.

If $k_1+k_2 \not\equiv 0$ (3), then $p|\underline{mn}_0$, so the inverse of the Euler factor above is trivial. Then, in this case, our condition on the vanishing of the b_l implies that for fixed $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3$ with $\alpha > 3\beta_3$,

$$\chi_{p^{\alpha_1+2\alpha_2}}(mn)\psi_1(mn)\psi_2(p^\alpha) \sum_{\substack{c_1,c_2 \\ c_1 \leq k_1, c_2 \leq k_2 \\ 0 \leq c_1, c_2 \leq 3\alpha_3+\alpha_2}} \tilde{a}^{(\alpha_3)}_{c_1,c_2}(p^\alpha) = 0 \quad \text{if } \alpha > 3\beta_3 \quad (4.5)$$

If instead $k_1 + k_2 \equiv 0$ (3), then this inverse of an Euler factor may appear non-trivially. Fix an $\alpha > 3\beta_3$ with $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3$ and $\alpha - 1 = \gamma_1 + 2\gamma_2 + 3\gamma_3$. (We have been forced to introduce notation for the decompositions of both α and $\alpha - 1$ since they may change depending on the residue class of r mod 3.) Then by determining the contribution to $\mathbb{N}p^{-\alpha w}$ on the right-hand side of (4.4), we have:

$$\chi_{p^{\alpha_1+\alpha_2}}(mn)\psi_1(mn)\psi_2(p^\alpha) \sum_{\substack{c_1,c_2 \\ c_1 \leq k_1, c_2 \leq k_2 \\ 0 \leq c_1, c_2 \leq 3\alpha_3+\alpha_2}} \tilde{a}^{(\alpha)}_{c_1,c_2}(p^\alpha) -$$

$$\chi_{\underline{mn}_0}(p)\psi_2(p)\chi_{p^{\gamma_1+2\gamma_2}}(mn)\psi_1(mn)\psi_2(p^{\alpha-1}) \cdot \sum_{\substack{c_1,c_2 \\ c_1 \leq k_1, c_2 \leq k_2 \\ 0 \leq c_1, c_2 \leq 3t_3+t_2}} \tilde{a}^{(\alpha-1)}_{c_1,c_2}(p^{\alpha-1}) = 0.$$

Note in particular that the cubic characters attached to each sum match identically. Factoring these from both terms, we obtain

$$\chi_{p^{\alpha_1+\alpha_2}}(mn)\psi_1(mn)\psi_2(p^\alpha)\left[\sum_{\substack{c_1,c_2\\ c_1\le k_1, c_2\le k_2\\ 0\le c_1,c_2\le 3\alpha_3+\alpha_2}} \tilde{a}^{(\alpha)}_{c_1,c_2}(p^\alpha) - \sum_{\substack{c_1,c_2\\ c_1\le k_1, c_2\le k_2\\ 0\le c_1,c_2\le 3t_3+t_2}} \tilde{a}^{(\alpha-1)}_{c_1,c_2}(p^{\alpha-1})\right] = 0. \tag{4.6}$$

The characters ψ_i appearing in (4.5) and (4.6) will always be non-zero as our primes are congruent to 1 mod 3. However, since $p|mn$, then if either α_1 or $\alpha_2 \neq 0$, these identities (4.5) and (4.6) are trivial. But if $\alpha \equiv 0$ (3), then our two identities reduce to the combined result, for fixed $\alpha = 3\alpha_3 > 3\beta_3$ and any fixed $k_1, k_2 > 0$,

$$\sum_{\substack{c_1,c_2\\ c_1\le k_1, c_2\le k_2\\ 0\le c_1,c_2\le 3\alpha_3+\alpha_2}} \tilde{a}^{(\alpha)}_{c_1,c_2}(p^\alpha) = \begin{cases} \displaystyle\sum_{\substack{c_1,c_2\\ c_1\le k_1, c_2\le k_2\\ 0\le c_1,c_2\le 3t_3+t_2}} \tilde{a}^{(\alpha-1)}_{c_1,c_2}(p^{\alpha-1}) & \text{if } k_1+k_2\equiv 0\ (3)\\ 0 & \text{if } k_1+k_2\not\equiv 0\ (3)\end{cases} \tag{4.7}$$

These sums, as written with their large list of restrictions, are rather unwieldy. With a little work, we can determine necessary conditions which are much simpler to comprehend. As usual, this will depend on the residue of k_1+k_2 mod 3 and we break our analysis into cases accordingly. Suppose first that $k_1+k_2 \equiv 0$ (3). Then using the above equality (4.7) when $\alpha = 3\alpha_3 > k_1+k_2 = 3\beta_3$, we have

$$\sum_{\substack{c_1,c_2\ge 0\\ c_1\le k_1, c_2\le k_2}} \tilde{a}^{(\alpha)}_{c_1,c_2}(p^\alpha) = \sum_{\substack{c_1,c_2\ge 0\\ c_1\le k_1, c_2\le k_2}} \tilde{a}^{(\alpha-1)}_{c_1,c_2}(p^{\alpha-1}) \tag{4.8}$$

where the last condition on the sums in (4.7) is redundant in this case, and hence omitted. But if $k_1+k_2 \equiv 0$ (3), then for $p^{k_1}||m$ and $p^{k_2-1}||n$, we are in the simpler case of (4.7) where

$$\sum_{\substack{c_1,c_2\ge 0\\ c_1\le k_1, c_2\le k_2-1}} \tilde{a}^{(\alpha)}_{c_1,c_2}(p^\alpha) = 0 \quad \text{since } 3\alpha_3 > 3\beta_3 \text{ and } k_1+k_2-1\equiv 2\ (3).$$

Using this relation on both sides of the above equality (4.8) to remove these terms, we are left with

$$\sum_{\substack{c_1\\ 0\le c_1\le k_1}} \tilde{a}^{(\alpha)}_{c_1,k_2}(p^\alpha) = \sum_{\substack{c_1\\ 0\le c_1\le k_1}} \tilde{a}^{(\alpha-1)}_{c_1,c_2}(p^{\alpha-1}) \tag{4.9}$$

for $3\alpha_3 > k_1+k_2$. Moreover, the cases $p^{k_1-1}||m$ and $p^{k_2}||n$ and $p^{k_1-1}||m$ and $p^{k_2-1}||n$ have similarly simple stability relations where sums of correction coefficients are 0. Comparing the relation from each case using (4.7) yields:

$$\sum_{c_1\le k_1-1} \tilde{a}^{(\alpha)}_{c_1,k_2}(p^\alpha) = 0 \quad \text{for } \alpha = 3\alpha_3 > 3\beta_3$$

Removing these terms from the previous equality (4.9) at last yields the desired result that, for a fixed choice of d with $p \not| d_0$, $a^{(\alpha)}_{k_1,k_2}(p^\alpha) = a^{(\alpha-1)}_{k_1,k_2}(p^{\alpha-1})$ for all $\alpha = 3\alpha_3 > 3\beta_3$ when $k_1+k_2 \equiv 0$ (3).

In fact, this information tells us something about all the correction coefficients regardless of their residue class mod 3. Proceeding inductively, consider the case where $k_1 + k_2 = 1$. The coefficients included in such a sum must be 0 according to the stability relation. But we know the coefficient of weight 0 (that is, coefficients corresponding to choices of k_1, k_2 such that $k_1 + k_2 = 0$) is stable, so similarly coefficients of weight 1 (i.e. $k_1 + k_2 = 1$) must be stable. Similarly for weight 2, the coefficients sum to 0 and the same principle holds. In fact, it is clear that it will hold for all weights 3k+1 and 3k+2 and since all the previous coefficients are inductively stable, then the fact that the sum of these coefficients is 0 proves the induction hypothesis needed to show stability for all coefficients.

We now repeat this process in the case where $3\alpha_3 = 3\beta_3 = k_1 + k_2$. In this case, we know from taking the limit at infinity of the w variable that $b_{3\beta}(m, n; p^\beta) = \mathbb{N}p^{2\beta}$. Since $k_1 + k_2 \equiv 0$ (3) in this case, then from interchanging the order of summation, we have the relation, for $k_1 + k_2 = 3\beta_3 = \alpha$ so that the analogue of (4.6) has trivial cubic characters,

$$\sum_{\substack{c_1, c_2 \geq 0 \\ c_1 \leq k_1, c_2 \leq k_2}} \tilde{a}^{(\beta_3)}_{c_1,c_2}(p^{3\beta_3}) - \sum_{\substack{c_1, c_2 \geq 0 \\ c_1 \leq k_1, c_2 \leq k_2}} \tilde{a}^{(\beta_3 - 1)}_{c_1,c_2}(p^{3\beta_3 - 1}) = \mathbb{N}p^{2\beta_3} \tag{4.10}$$

Just as before, the case where $p^{k_1 - 1} || m$ and $p^{k_2} || n$ has $k_1 + k_2 - 1 \equiv 2$ (3) so it satisfies the simpler relation that the sum of twisted correction coefficients is 0. Subtracting these terms from (4.10) by the identical method to the above gives a much simpler relation.

$$a^{(\beta_3)}_{k_1,k_2}(p^{3\beta_3}) - a^{(\beta_3 - 1)}_{k_1,k_2}(p^{\beta - 1}) = p^{2\beta_3}$$

We must ensure that these conditions are satisfied when we make our determination of the $a_{i,j}$ in the subsequent chapter. If they are satisfied, then we are guaranteed that the b_l vanish for large enough l which shows that our correction factor $Q(w; m, n)$ is finite. Lastly, we need to ensure that the size of the b_l are not so large as to affect the convergence of the entire object $Z_2(s_1, s_2, w)$ and the convergence of the new object given by an additional functional equation with $w \mapsto 1 - w$ performed on Z_2. This is the goal of the next section.

4.2 Growth Conditions on the Polynomial $Q(w; m, n)$

By determining bounds on the size of the correction coefficients with respect to the power of primes dividing m and n, we will achieve two important ends. First, we will be able to show that the convergence of the correction polynomial does not interfere with the determination of a region of absolute convergence for the entire object $Z_1(s_1, s_2, w)$. That is, the growth estimates on the L-series are the limiting factors which determine the region of absolute convergence. Second, we will obtain enough control over the object $Z_3(s_1, s_2, w) = Z_1(s_1 + w - 1/2, s_2 + w - 1/2, 1 - w)$ to obtain an additional functional equation $Z_3(s_1, s_2, w) \to Z_3(1 - s_1, 1 - s_2, w + 2s_1 + 2s_2 - 4)$ which extends the region of analytic continuation! This is detailed in Chapter 5.

Our intuition on the correct bound for the size of the coefficients again comes from taking the limit as $\Re(w) \to \infty$ and $\Re(s_i) \to \infty$. In taking the limit in the variable s_1, we found that the coefficient of largest size is the one associated to the highest non-vanishing power of p^{-w}, namely $b_{3\beta_3}(1, n; p^\beta) = p^{2\beta_3}$. Similarly, taking the limit in s_2, we determined the largest coefficient is $b_{3\beta}(m, 1; p^\beta) = p^{2\beta}$. Then taking the limit in w, we found that $b_{3\beta_3}(m, n; p^\beta) = p^{2\beta_3}$ independent of the choice of m, n such that $ord_p(mn) = 3\beta_3$. This leads us to conjecture that the largest coefficient for any choice of m and n might similarly be $b_{3\beta_3}(m,n;p^\beta)$. We take this as an assumption and determine the consequences for the correction coefficients of $P(s_1,s_2,d)$.

Assumption 3. *Given a fixed* $\beta = ord_p(mn)$*, we have for any such* m *and* n*,*

$$\left|b_l(m,n;p^\beta)\right| \ll p^{2l} \quad \textit{for all } l.$$

where the implied constant is independent of β *and* l*.*

Note that the assumption is already known for $l \geq 3\beta$ according to the limiting methods of the previous chapter. There we found $b_{3\beta}(m,n;p^\beta) = \mathbb{N}p^{2\beta_3}$ and $b_l(m,n;p^\beta) = 0$ for $l > 3\beta_3$ and any choice of m and n. Hence we only need to guarantee this assumption for $0 < l < 3\beta_3$. But using (4.4), and repeating our simple trick of subtracting and adding the contributions of similar terms, we find that for any choice of m,n with $p^{k_1}||m$ and $p^{k_2}||n$, $k_1,k_2 > 0$, and any $l > 0$,

$$b_l(m,n;p^\beta) - b_l(m,n/p;p^{\beta-1}) - b_l(m/p,n;p^{\beta-1}) + b_l(m/p,n/p;p^{\beta-2}) = \tilde{a}^{(l)}_{k_1,k_2}(p^l)$$

Hence, our assumption will follow from a simple inductive argument if we can show that for any value of k_1 and k_2,

$$\left|\tilde{a}^{(l)}_{k_1,k_2}(p^l)\right| \ll \mathbb{N}p^{2l} \quad \text{for all } l. \tag{4.11}$$

4.3 Summary of Necessary Conditions on Correction Coefficients

We now collect all of the assumptions we have made on the form of the correction factors P and Q and their coefficients. This is the final revision to such a collection and each will be demonstrated in the following chapter.

Assumption 4. *The Dirichlet polynomials* $P(s_1,s_2,d;\psi_1,\psi_1)$ *and* $Q(w,m,n;\psi_2)$ *should satisfy the following properties:*

- *They are finite, Eulerian Dirichlet polynomials depending only on the indicated quantities.*
- $P(s_1,s_2,d) = P(s_2,s_1,d)$ *and* $Q(w,m,n) = Q(w,n,m)$*.*
- $P(s_1,s_2,d) = 1$ *if* d *is square-free.*
- $Q(w,m,n) = 1$ *if* mn *is square-free.*
- *They can be chosen so that the interchange equality (3.1)=(3.2) is satisfied.*
- *They can be chosen so that* Z_3, Z_4, Z_5 *and* Z_6*, defined according to (3.3)-(3.6), satisfy additional functional equations into themselves.*
- *If we expand* P *and* Q *as Euler products of form*

$$P(s_1,s_2,d) = \prod_{p^\alpha||d_3} \sum_{i,j} a_{i,j}(d_0,p^\alpha)\mathbb{N}p^{-is_1-js_2}$$

and

$$Q(w,m,n) = \prod_{p^\beta || M_3} \sum_{l} b_l(m,n,p^\beta) \mathbb{N}p^{-lw}$$

then the coefficients $a_{i,j}$ and b_l should agree with the coefficients determined by the limiting methods.

- *For fixed $\alpha = 3\alpha_3 > 3\beta_3$ and any fixed $k_1, k_2 > 0$, the normalized correction coefficients $\tilde{a}^{(\alpha)}_{c_1,c_2}(p^\alpha)$ of $P(s_1,s_2,d)$ satisfy (4.7).*

- *The normalized correction coeffients satisfy a mild growth hypothesis. Precisely, for any value of $k_1, k_2 > 0$,*

$$\left|\tilde{a}^{(l)}_{k_1,k_2}(p^l)\right| \ll \mathbb{N}p^{2l} \quad \textit{for all } l.$$

where the implied constant is independent of l.

Chapter 5

The "Ideal Object" Approach

5.1 Polynomial Combinations of Double Dirichlet Series

5.1.1 An Improved Definition for $Z_4(s_1, s_2, w)$

As we saw in (3.4) in Chapter 3, the heuristic obtained from sums of square-free integers for the functional equation $(s_1, s_2, w) \mapsto (1 - s_1, s_2, w + s_1 - 1/2)$ suggests that

$$Z_1(1 - s_1, s_2, w + s_1 - 1/2) \stackrel{\text{def}}{=} Z_4(s_1, s_2, w) = \sum_{m,n,d} \frac{G(mn^2, d)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^{w}}.$$

But this failed because it does not lead to symmetric contributions to the correction factor $P(s_1, s_2, d)$. However, the suggested series does possess the desired functional equation $Z_4(s_1, s_2, w) \to Z_4(s_1 + w - 1/2, s_2 + 2w - 1, 1 - w)$. We would like to determine a function of three complex variables which retains this functional equation but also defines a correction polynomial which satisfies the required properties of the previous sections.

Because the correction polynomial $P(s_1, s_2, d)$ is Eulerian, we can restrict our attention to powers of $\mathbb{N}p^{-s_1}$ or $\mathbb{N}p^{-s_2}$ in $Z_4(s_1, s_2, w)$ and still obtain complete information about the $a_{i,j}^{(\alpha_3)}(d_0, p)$ which comprise the Dirichlet polynomial P. Even in this special case, the form suggested by our previous heuristic is not consistent with our required properties. That is,

$$Z_4(s_1, w; p^k) := [\ \mathbb{N}p^{-ks_2} \text{ coefficient of } Z_4(s_1, s_2, w)\ ] \neq \sum_{m,d} \frac{G(mp^{2k}, d)\bar{\psi}_1(mp^{2k})\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^{w}} \stackrel{\text{def}}{=} D_4(s_1, w; p^{2k})$$

since it again fails to give symmetric contributions for the $a_{i,j}$.

In order to satisfy the functional equation $(s_1, s_2, w) \mapsto (s_1 + w - 1/2, s_2 + 2w - 1, 1 - w)$, the terms in $Z_4(s_1, w; p^k)$ must satisfy

$$Z_4(s_1, w; p^k)\mathbb{N}p^{-ks_2} = Z_4(s_1 + w - 1/2, 1 - w; p^k)\mathbb{N}p^{-ks_2 - 2kw + k}.$$

Our basic assumption, which motivates the collection of functions we consider, is that there aren't very many objects which possess this functional equation. In fact, the only such functions with the correct transformation properties indicated above should take form:

$$\sum_{l=0}^{2k} R_l(s_1, w; 2k) D(s_1, w; p^{2k-l}) = \sum_{l=0}^{2k} R_l(s_1, w; 2k) \sum_{\substack{m,d \equiv 1\ (3) \\ (md,6)=1}} \frac{G(mp^{2k-l}, d)\bar{\psi}_1(mp^{2k-l})\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^{w}}$$

where the $R_l(s_1, w; 2k)$ are finite Dirichlet polynomials in $\mathbb{N}p^{-s_1}$ and $\mathbb{N}p^{-w}$ with the property

$$R_l(s_1, w; 2k) = \mathbb{N}p^{l/2-lw} R_l(s_1 + w - 1/2, 1 - w).$$

Note that the transformation condition on each of the $R_l(s_1, w)$ immediately implies the desired transformation property for $Z_4(s_1, w; k)$. A further hypothesis on the Dirichlet polynomials $R_l(s_1, s_2, w)$ guarantees that such combinations are especially nice: all monomials $\mathbb{N}p^{-is_1-jw}$ occurring in $R_l(s_1, w; 2k)$ must have $i \leq j$. We will discuss the ramifications of this assumption in later sections.

In short, we have just proposed that the correct definition for $Z_4(s_1, s_2, w)$ is only a slight generalization of the one suggested by the square-free heuristics and still possesses all the essential characteristics. With this in mind, we will spend the rest of the chapter trying to prove the following result:

Theorem 5.1 (Main Theorem, Version 1). *For every value of $k \geq 0$, there exists a finite collection of finite Dirichlet polynomials $R_l(s_1, w; 2k)$ for $l = 0, \ldots, 2k$, satisfying the properties:*

1) $R_l(s_1, w; 2k) = \mathbb{N}p^{l/2-lw} R_l(s_1 + w - 1/2, 1 - w)$, and

2) All monomials $\mathbb{N}p^{-is_1-jw}$ occurring in $R_l(s_1, w; 2k)$ must have $i \leq j$

so that the definition

$$Z_4(s_1, w; p^k) \stackrel{def}{=} \sum_{l=0}^{2k} R_l(s_1, w; 2k) D(s_1, w; p^{2k-l})$$

defines a consistent set of correction coefficients $a_{i,j}^{(\alpha_3)}(d_0, p)$ of $P(s_1, s_2, d)$ such that the following properties hold:

1) The correction factor is a finite, Eulerian Dirichlet polynomial

2) The correction factor is symmetric in the variables s_1 and s_2.

3) The correction factor is trivial if d is cube-free. More specifically, each Euler factor at the prime p depends on the divisibility of d by powers of p^3.

4) The values of the coefficients agree with those determined by the method of taking variables to infinity.

5) Large collections of the $a_{i,j}$ sum to 0 according to (4.7).

6) The coefficients $a_{i,j}$ satisfy mild growth conditions according to (4.11).

We will begin the proof of this theorem in the next section. While the current statement of the theorem strongly reflects the thinking process used to obtain it, it is quite difficult to conceptualize a strategy for its proof. To conclude this section, we will work to simplify this statement into an equivalent but more combinatorial claim and prove several supporting lemmas which describe basic constraints on the form of these linear combinations.

5.1.2 Deconstructing $D_4(s_1, w; p^l)$

We begin this process with a more careful description of the decomposition of a generic Gauss sum object $D_4(s_1, w; p^l)$. Again, we'll want to use the method of decomposing the series according to the primes dividing d. This follows identically to the previous decomposition of $D(s_1, w)$ used in the method of taking variables to infinity. The only difference is the analysis at the distinguished prime p, so we begin our decomposition of $D_4(s_1, w; p^l)$ with this.

Proposition 5.2. *The p-th Euler factor of the series $D_4(s_1, w; p^l)$ as a Dirichlet series in d (called the p-part of $D_4(s_1, w; p^l)$) is given by the following cases. First, if $p \nmid d_0$, then the p-part takes the form:*

$$\bar{\chi}_{d_0}(p^l)\bar{\psi}(p^l)L^{(p)}(s_1, \bar{\chi}_{d_0}\bar{\psi}_1) + \sum_{0<3\alpha_3\leq l} \bar{\chi}_{d_0}(p^l)\bar{\psi}_1(p^l)L^{(p)}(s_1, \bar{\chi}_{d_0}\bar{\psi}_1)(\mathbb{N}p^{3\alpha_3/2} - \mathbb{N}p^{3\alpha_3/2-1})\mathbb{N}p^{-3\alpha_3 w} +$$
$$\sum_{3\alpha_3>l} L^{(p)}(s_1, \bar{\chi}_{d_0}\bar{\psi}_1)\left(\mathbb{N}p^{3\alpha_3/2-(3\alpha_3-l)s_1-3\alpha_3 w} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha_3/2-1-(3\alpha_3-1-l)s_1-3\alpha_3 w}\right)$$

If $p|d_1$, then the p-part of the series is

$$\sum_{3\alpha_3\geq l} \mathbb{N}p^{3\alpha_3/2-(3\alpha_3-l)s_1}$$

If $p|d_2$, then the p-part of the series is

$$\bar{\psi}_1(p)\sum_{3\alpha_3+1\geq l} \mathbb{N}p^{3\alpha_3/2-(3\alpha_3+1-l)s_1}$$

Proof: Pick and fix a value of d_0. As a first case, suppose that $p \nmid d_0$. Then write $d = p^{3\alpha_3}d'$ with $(p, d') = 1$ and similarly, write $m = p^\gamma m'$ with $(p, m') = 1$. Then

$$\begin{aligned} D_4(s_1, w; p^l) &= \sum_{\substack{m,d\in\mathcal{O}_K \\ m,d\equiv 1\ (3) \\ (md,6)=1}} \frac{G(p^l m, d)\bar{\psi}_1(mp^l)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} \\ &= \sum_{\substack{m',d' \\ \alpha_3,\gamma\geq 0}} \frac{G(p^{l+\gamma}(m'), p^{3\alpha_3}d')\bar{\psi}_1(p^{\gamma+l}m')\psi_2(d)}{(\mathbb{N}m')^{s_1}(\mathbb{N}d')^w\mathbb{N}p^{\gamma s_1+3\alpha_3 w}} \end{aligned} \tag{5.1}$$

But

$$G(p^{l+\gamma}(m'), p^{3\alpha_3}d') = \frac{g(p^{l+\gamma}(m'), p^{3\alpha_3}d')}{\sqrt{\mathbb{N}p^{3\alpha_3}\mathbb{N}d'}} = \frac{g(p^{l+\gamma}(m'), d')}{\sqrt{\mathbb{N}d'}}\frac{g(p^{l+\gamma}(m'), p^{3\alpha_3})}{\sqrt{\mathbb{N}p^{3\alpha_3}}} =$$

$$= \bar{\chi}_{d_0}(p^{l+\gamma})\frac{g(m', d')}{\sqrt{\mathbb{N}d'}}\frac{g(p^{l+\gamma}, p^{3\alpha_3})}{\sqrt{\mathbb{N}p^{3\alpha_3}}} = \frac{\bar{\chi}_{d_0}(p^{l+\gamma})}{\sqrt{\mathbb{N}p^{3\alpha_3}}}G(m', d')\frac{g(p^{l+\gamma}, p^{3\alpha_3})}{\sqrt{\mathbb{N}p^{3\alpha_3}}}$$

Then we may rewrite the above (5.1) as

$$D_4(s_1, w; p^l) = \sum_{\substack{m',d'\equiv 1\ (3) \\ (m'd',6p)=1}} \frac{G(m', d')\bar{\psi}_1(m')\psi_2(d)}{(\mathbb{N}m')^{s_1}(\mathbb{N}d')^w}\sum_{\alpha_3\geq 0}\left[\sum_{\gamma\geq 0}\frac{\bar{\chi}_{d_0}(p^{l+\gamma})\bar{\psi}_1(p^{l+\gamma})}{\mathbb{N}p^{\gamma s_1+3\alpha_3 w}}\frac{g(p^{l+\gamma}, p^{3\alpha_3})}{\sqrt{\mathbb{N}p^{3\alpha_3}}}\right]$$

But we know that

$$g(p^{l+\gamma}, p^{3\alpha_3}) = \begin{cases} \phi(p^{3\alpha_3}), & \text{if } l+\gamma \geq 3\alpha_3 \\ -\mathbb{N}p^{3\alpha_3-1}, & \text{if } l+\gamma = 3\alpha_3 - 1, \alpha_3 > 0 \\ 0, & \text{otherwise} \end{cases}$$

so we can evaluate the bracketed sum above according to a similar case method. That is, summing the geometric sum, we obtain $\sum_{\gamma\geq 0}\frac{\bar{\chi}_{d_0}(p^{l+\gamma})\bar{\psi}_1(p^{l+\gamma})}{\mathbb{N}p^{\gamma s_1+3\alpha_3 w}}\frac{g(p^{l+\gamma},p^{3\alpha_3})}{\sqrt{\mathbb{N}p^{3\alpha_3}}} =$

$$= \begin{cases} \bar{\chi}_{d_0}(p^l)\bar{\psi}_1(p^l)L^{(p)}(s_1,\bar{\chi}_{d_0}\bar{\psi}_1)\frac{\phi(p^{3\alpha_3})}{\sqrt{\mathbb{N}p^{3\alpha_3}}}\mathbb{N}p^{-3\alpha_3 w} & \text{if } l\geq 3\alpha_3 \\ L^{(p)}(s_1,\bar{\chi}_{d_0}\bar{\psi}_1)\frac{\phi(p^{3\alpha_3})}{\sqrt{\mathbb{N}p^{3\alpha_3}}}\mathbb{N}p^{-(3\alpha_3-l)s_1-3\alpha_3 w} - & \\ \qquad \bar{\chi}_{d_0}(p^{3\alpha_3-1})\bar{\psi}_1(p^{3\alpha_3-1})\frac{\mathbb{N}p^{3\alpha_3-1}}{\sqrt{\mathbb{N}p^{3\alpha_3}}}\mathbb{N}p^{-(3\alpha_3-1-l)s_1-3\alpha_3 w} & \text{if } l<3\alpha_3 \end{cases}$$

where $L^{(p)}(s_1,\bar{\chi}_{d_0}\bar{\psi}_1)$ denotes the pth Euler factor of the indicated L-series. By removing this factor from both terms in the latter case $l<3\alpha_3$, we have some cancellation which leads to the somewhat simpler looking cases for our sum:

$$\sum_{\gamma\geq 0}\frac{\bar{\chi}_{d_0}(p^{l+\gamma})\bar{\psi}_1(p^{l+\gamma})}{\mathbb{N}p^{\gamma s_1+3\alpha_3 w}}\frac{g(p^{l+\gamma},p^{3\alpha_3})}{\sqrt{\mathbb{N}p^{3\alpha_3}}} = L^{(p)}(s_1,\bar{\chi}_{d_0}\bar{\psi}_1)\,\cdot$$

$$\cdot\begin{cases} \bar{\chi}_{d_0}(p^l)\bar{\psi}_1(p^l) & \text{if } \alpha_3=0 \\ \bar{\chi}_{d_0}(p^l)\bar{\psi}_1(p^l)(\mathbb{N}p^{3\alpha_3/2}-\mathbb{N}p^{3\alpha_3/2-1})\mathbb{N}p^{-3\alpha_3 w} & \text{if } l\geq 3\alpha_3 \\ (\mathbb{N}p^{3\alpha_3/2-(3\alpha_3-l)s_1}-\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha_3/2-1-(3\alpha_3-1-l)s_1)})\mathbb{N}p^{-3\alpha_3 w} & \text{if } l<3\alpha_3 \end{cases}$$

Substituting this into the above equation for $D(s_2,w;p^l)$ gives the result in the case $p\nmid d_0$.

If, instead, $p|d_1$, then we may write $d=p^{3\alpha_3+1}d'$ and $m=p^\gamma m'$ as before. Then

$$G(p^{l+\gamma}m',p^{3\alpha_3+1}d')=\frac{g(p^{l+\gamma}m',p^{3\alpha_3+1}d')}{\sqrt{\mathbb{N}p^{3\alpha_3+1}\mathbb{N}d'}}=0 \text{ unless } l+\gamma=3\alpha_3.$$

In this case,

$$G(p^{l+\gamma}m',p^{3\alpha_3+1}d')=\frac{g(p^{3\alpha_3}m',p^{3\alpha_3+1}d')}{\sqrt{\mathbb{N}p^{3\alpha_3+1}\mathbb{N}d'}}=\mathbb{N}p^{3\alpha_3/2}\frac{g(m',pd')}{\sqrt{\mathbb{N}p\mathbb{N}d'}}=\mathbb{N}p^{3\alpha_3/2}G(m',pd')$$

Hence, substituting this into the original series, we have

$$\sum_{\substack{m,d\equiv 1\ (3)\\(md,6)=1}}\frac{G(p^lm,d)\bar{\psi}_1(p^lm)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w}=\sum_{\substack{m',d'\\(m'd',6p)=1}}\frac{G(m',pd')\bar{\psi}_1(m')\psi_2(pd')}{(\mathbb{N}m')^{s_1}(\mathbb{N}p\mathbb{N}d')^w}\sum_{3\alpha_3\geq l}\mathbb{N}p^{3\alpha_3/2-(3\alpha_3-l)s_1}$$

Lastly, if $p|d_2$, then write $d=p^{3\alpha_3+2}d'$ and $m=p^\gamma m'$. Then $G(p^{l+\gamma}m',p^{3\alpha_3+2}d')=0$ unless $l+\gamma=3\alpha_3+1$. In this case,

$$\begin{aligned} G(p^{l+\gamma}m',p^{3\alpha_3+2}d') &= \frac{g(p^{3\alpha_3+1}m',p^{3\alpha_3+2}d')}{\sqrt{\mathbb{N}p^{3\alpha_3+2}\mathbb{N}d'}} \\ &= \frac{g(p^{3\alpha_3+1}m',p^{3\alpha_3+2})}{\sqrt{\mathbb{N}p^{3\alpha_3+2}}}\frac{g(p^{3\alpha_3+1}m',d')}{\sqrt{\mathbb{N}d'}}\chi_{d'(p^2)}\bar{\chi}_p(d') \\ &= \mathbb{N}p^{3\alpha_3/2}\frac{\overline{g(m',p)}}{\sqrt{\mathbb{N}p}}\bar{\chi}_{d'}(p)\frac{g(m',d')}{\sqrt{\mathbb{N}d'}}\chi_{d'(p^2)}\bar{\chi}_p(d') \\ &= \mathbb{N}p^{3\alpha_3/2}\overline{G(m',p)}G(m',d') \quad \text{(using cubic reciprocity)} \end{aligned}$$

Now substituting this result into the original series, we get

$$\sum_{\substack{m,d\equiv 1\ (3)\\(md,6)=1}} \frac{G(p^l m,d)\bar{\psi}_1(p^l m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} =$$

$$\sum_{\substack{m',d'\\(m'd',6p)=1}} \frac{G(m',d')\overline{G(m',p)}\bar{\psi}_1(m')\bar{\psi}_1(p)\psi_2(d'p^2)}{(\mathbb{N}m')^{s_1}(\mathbb{N}p^2\mathbb{N}d')^w} \sum_{3\alpha_3+1\geq l} \mathbb{N}p^{3\alpha_3/2-(3\alpha_3+1-l)s_1}.$$

Note that the p-part of this series is precisely the one given in the statement of the proposition, and the result follows. □

5.1.3 The Coefficient of $\mathbb{N}p^{-ks_2}$ in $Z_4(s_1, s_2, w)$

Recall that our goal in the previously stated theorem was to find an appropriately chosen linear combination of Dirichlet polynomials $R_l(s_1, w; 2k)$ and series containing Gauss sums $D_4(s_1, w; p^{2k-l})$ so that the definition

$$Z_4(s_1, w; p^k) \stackrel{\text{def}}{=} \sum_{l=0}^{2k} R_l(s_1, w; 2k) D(s_1, w; p^{2k-l})$$

leads to well-behaved correction coefficients of $P(s_1, s_2, d)$. Now that we know the general form of terms arising from $D_4(s_1, w; p^l)$ for any choice of l, we turn to a careful investigation of the other side of this definition: the coefficient of $\mathbb{N}p^{-ks_2}$ in $Z_4(s_1, s_2, w)$. As we discovered earlier in (3.11), the expanded form of $Z_4(s_1, s_2, w)$ is

$$\sum_{\substack{d,m,n\in\mathcal{O}_K\\d,m,n\equiv 1\ (3)\\(dmn,6)=1}} \frac{\bar{\chi}_{d_0}(mn^2)\bar{\psi}_1(m)\psi_1(n)\psi_2(d)G(\chi_{d_1})G(\bar{\chi}_{d_2})\bar{\psi}_1(d_2) \prod_{\substack{p^\alpha || d_3\\ p\nmid d_2}} \left[a^{(\alpha_3)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2} + \cdots\right]}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^w}.$$

$$\cdot \prod_{\substack{p^\alpha || d_3\\ p | d_2}} \left[a^{(\alpha_3)}_{3\alpha+1,0}\mathbb{N}p^{-3\alpha/2-1/2} + a^{(\alpha_3)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2+1/2-s_1} + \cdots\right] \quad (5.2)$$

Using this form, we'd like to determine the terms which contribute to the coefficient of $\mathbb{N}p^{-ks_2}$ for a distinguished prime p. Note that this immediately implies that n is a power of p.

Grouping all such terms we have, for $p \nmid d_2$, $Z_4(s_1, s_2, w) =$

$$\sum_{\substack{d,m\equiv 1\ (3)\\ (dm,6)=1}} \frac{\bar{\chi}_{d_0}(m)\bar{\psi}_1(m)\psi_2(d)\bar{\psi}_1(d_2)G(1,d_1)\overline{G(1,d_2)}}{\mathbb{N}m^{s_1}\mathbb{N}d^w} \prod_{\substack{q^{\alpha_3}||d_3\\ q\nmid d_2\\ q\neq p}} \Big[a^{(\alpha_3)}_{3\alpha_3,0}\mathbb{N}q^{-3\alpha_3/2} + \cdots$$

$$\cdots + a^{(\alpha_3)}_{0,0}\mathbb{N}q^{3\alpha_3/2}\Big] \prod_{\substack{q^{\alpha_3}||d_3\\ q|d_2\\ q\neq p}} \Big[a^{(\alpha_3)}_{3\alpha_3+1,0}\mathbb{N}q^{-3\alpha_3/2-1/2} + \cdots + a^{(\alpha_3)}_{3\alpha_3,0}\mathbb{N}q^{3\alpha_3/2+1/2-(3\alpha_3+1)s_1} + \cdots\Big]$$

$$\sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j}) \sum_i a^{(\alpha_3)}_{i,j}\mathbb{N}p^{-3\alpha_3/2+(3\alpha_3-i)-(3\alpha_3-i)s_1} \tag{5.3}$$

If $p|d_2$, we have the very similar looking expression for $Z_4(s_1, s_2, w)$:

$$\sum_{\substack{d,m\equiv 1\ (3)\\ (dm,6)=1}} \frac{\bar{\chi}_{d_0}(m)\bar{\psi}_1(m)\psi_2(d)\bar{\psi}_1(d_2)G(1,d_1)\overline{G(1,d_2)}}{\mathbb{N}m^{s_1}\mathbb{N}d^w} \prod_{\substack{q^{\alpha_3}||d_3\\ q\nmid d_2\\ q\neq p}} \Big[a^{(\alpha_3)}_{3\alpha_3,0}\mathbb{N}q^{-3\alpha_3/2} + \cdots$$

$$\cdots + a^{(\alpha_3)}_{0,0}\mathbb{N}q^{3\alpha_3/2}\Big] \prod_{\substack{q^{\alpha_3}||d_3\\ q|d_2\\ q\neq p}} \Big[a^{(\alpha_3)}_{3\alpha_3+1,0}\mathbb{N}q^{-3\alpha_3/2-1/2} + \cdots + a^{(\alpha_3)}_{3\alpha_3,0}\mathbb{N}q^{3\alpha_3/2+1/2-(3\alpha_3+1)s_1}\Big]$$

$$\sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j}) \sum_i a^{(\alpha_3)}_{i,j}\mathbb{N}p^{-3\alpha_3/2-1/2+(3\alpha_3+1-i)-(3\alpha_3+1-i)s_1}.$$

Notice that the sum over d and m is precisely the series we investigated in taking the limit as $\Re(s_2) \to \infty$ and all of the products over primes $q \neq p$ similarly involve the identical correction coefficients and associated Dirichlet monomials. But as we argued in Chapter 3, we know by comparison with the two-variable situation that this should just be $D(s_1, w; 1)$ where the sum is taken over integers relatively prime to p. Hence, in the case where $p \nmid d_0$, we may rewrite the above (5.3) as

$$\sum_{\substack{m,d\equiv 1\ (3)\\ (md,6p)=1}} \frac{G(m,d)\bar{\psi}_1(m)\psi_2(d)}{m^{s_1}d^w} L^{(p)}(s_1, \bar{\chi}_{d_0}\bar{\psi}_1)$$

$$\sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j}) \sum_i a^{(\alpha_3)}_{i,j}\mathbb{N}p^{-3\alpha_3/2+(3\alpha_3-i)-(3\alpha_3-i)s_1-3\alpha_3 w}$$

A similar rewriting can be done for the cases $p|d_1$ and $p|d_2$. Now recall, for example, that in the case where $p \nmid d_0$, the series $D_4(s_1, w; p^l)$ could be rewritten as

$$\sum_{\substack{m,d\equiv 1\ (3)\\ (md,6p)=1}} \frac{G(m,d)\bar{\psi}_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^{w}} \Big[\bar{\chi}_{d_0}(p^l)\bar{\psi}_1(p^l)L^{(p)}(s_1, \bar{\chi}_{d_0}\bar{\psi}_1)+$$

$$\sum_{0<3\alpha_3\leq l} \bar{\chi}_{d_0}(p^l)\bar{\psi}_1(p^l)L^{(p)}(s_1, \bar{\chi}_{d_0}\bar{\psi}_1)(\mathbb{N}p^{3\alpha_3/2} - \mathbb{N}p^{3\alpha_3/2-1})\mathbb{N}p^{-3\alpha_3 w}+$$

$$\sum_{3\alpha_3>l} L^{(p)}(s_1, \bar{\chi}_{d_0}\bar{\psi}_1)\Big(\mathbb{N}p^{3\alpha_3/2-(3\alpha_3-l)s_1-3\alpha_3 w}-$$

$$\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha_3/2-1-(3\alpha_3-1-l)s_1-3\alpha_3 w}\Big)\Big]$$

Then, in the case where $p \nmid d_0$, the desired equality

$$Z_4(s_1, w; p^k) = \sum_{l=0}^{2k} R_l(s_1, w; 2k)D(s_1, w; p^{2k-l})$$

is equivalent to the following relation, after cancelling common terms on both sides.

$$\sum_{\alpha_3\geq 0} \mathbb{N}p^{-3\alpha_3 w} \sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j}) \sum_{i} a_{i,j}^{(\alpha_3)} \mathbb{N}p^{-3\alpha_3/2+(3\alpha_3-i)-(3\alpha_3-i)s_1}$$

$$= \sum_{l=0}^{2k} R_l(s_1, w; 2k) \Big[\bar{\chi}_{d_0}(p^{2k-l})\bar{\psi}_1(p^{2k-l}) +$$

$$\sum_{0<3\alpha_3\leq 2k-l} \bar{\chi}_{d_0}(p^{2k-l})\bar{\psi}_1(p^{2k-l})(\mathbb{N}p^{3\alpha_3/2} - \mathbb{N}p^{3\alpha_3/2-1})\mathbb{N}p^{-3\alpha_3 w} +$$

$$\sum_{3\alpha_3>2k-l} \Big(\mathbb{N}p^{3\alpha_3/2-(3\alpha_3-l)s_1-3\alpha_3 w} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha_3/2-1-(3\alpha_3-1-l)s_1-3\alpha_3 w}\Big)\Big]$$

Following the identical procedure and making the same corresponding cancellations in the cases $p|d_1$ and $p|d_2$, we arrive at a completely combinatorial equivalent version of Theorem 2.

Theorem 5.3 (Main Theorem, Version 2). *Let p be a fixed prime. Then for every $k \geq 0$, there exists a choice of finite Dirichlet polynomials $R_l(s_1, w; 2k)$ for $0 \leq l \leq 2k$ in the variables $\mathbb{N}p^{-s_1}$ and $\mathbb{N}p^{-w}$, with $R_l(s_1, w; 2k) = \mathbb{N}p^{l/2-lw}R_l(s_1 + w - 1/2, 1 - w)$, so that, if $p \nmid d_0$,*

$$\sum_{\alpha_3\geq 0} \mathbb{N}p^{-3\alpha_3 w} \sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j}) \sum_{i} a_{i,j}^{(\alpha_3)} \mathbb{N}p^{-3\alpha_3/2+(3\alpha_3-i)-(3\alpha_3-i)s_1}$$

$$= \sum_{l=0}^{2k} R_l(s_1, w; 2k) \Big[\bar{\chi}_{d_0}(p^{2k-l})\bar{\psi}_1(p^{2k-l}) +$$

$$\sum_{0<3\alpha_3\leq 2k-l} \bar{\chi}_{d_0}(p^{2k-l})\bar{\psi}_1(p^{2k-l})(\mathbb{N}p^{3\alpha_3/2} - \mathbb{N}p^{3\alpha_3/2-1})\mathbb{N}p^{-3\alpha_3 w} +$$

$$\sum_{3\alpha_3>2k-l} \Big(\mathbb{N}p^{3\alpha_3/2-(3\alpha_3-l)s_1-3\alpha_3 w} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha_3/2-1-(3\alpha_3-1-l)s_1-3\alpha_3 w}\Big)\Big]$$

if $p|d_1$, then

$$\sum_{\alpha_3\geq 0}\mathbb{N}p^{-(3\alpha_3+1)w}\sum_{j=0}^{k}\bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j})\sum_{i}a_{i,j}^{(\alpha_3)}\mathbb{N}p^{-3\alpha_3/2+(3\alpha_3-i)-(3\alpha_3-i)s_1}=$$
$$=\sum_{l=0}^{2k}R_l(s_1,w;2k)\left[\sum_{3\alpha_3\geq 2k-l}\mathbb{N}p^{3\alpha_3/2-(3\alpha_3-(2k-l))s_1-(3\alpha_3+1)w}\right]$$

and if $p|d_2$, then

$$\sum_{\alpha_3\geq 0}\mathbb{N}p^{-3\alpha_3 w}\sum_{j=0}^{k}\bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j})\sum_{i}a_{i,j}^{(\alpha_3)}\mathbb{N}p^{-3\alpha_3/2-1/2+(3\alpha_3+1-i)-(3\alpha_3+1-i)s_1}=$$
$$=\sum_{l=0}^{2k}R_l(s_1,w;2k)\left[\sum_{3\alpha_3+1\geq l}\mathbb{N}p^{3\alpha_3/2-(3\alpha_3+1-(2k-l))s_1-(3\alpha_3+2)w}\right]$$

define a consistent set of correction coefficients $a_{i,j}^{(\alpha_3)}(d_0,p)$ of $P(s_1,s_2,d)$ such that the following properties hold:

1) The correction factor is a finite, Eulerian Dirichlet polynomial

2) The correction factor is symmetric in the variables s_1 and s_2.

3) The correction factor is trivial if d is cube-free. More specifically, each Euler factor at the prime p depends on the divisibility of d by powers of p^3.

4) The values of the coefficients agree with those determined by the method of taking variables to infinity.

5) Large collections of the $a_{i,j}$ sum to 0 according to (4.7).

6) The coefficients $a_{i,j}$ satisfy mild growth conditions according to (4.11).

We have now reduced the main theorem of the chapter to a purely combinatorial question about a finite induction of certain finite Dirichlet polynomials $R_l(s_1,w;2k)$.

5.1.4 Lemmas on the Form of the Dirichlet Polynomials

These lemmas will further restrict the form of the Dirichlet Polynomials $R_l(s_1,w;2k)$ used in proving the Main Theorem according to the properties we wish them to satisfy.

Lemma 5.4 (Congruence Property). *The Dirichlet multipliers $R_l(s_1,w;2k)$ used to define the correction coefficients of $P(s_1,s_2,d)$ must only contain powers of $\mathbb{N}p^{-w}$ which are congruent to 0 mod 3.*

Proof: The Dirichlet polynomials must be chosen to solve for the correction coefficients in the pth Euler factor coming from $Z_4(s_1,w;p^k)$. These contributions are listed in the statement of the Main Theorem above. The only terms involving $\mathbb{N}p^{-w}$ in $Z_4(s_1,w;p^k)$ are of form $\mathbb{N}p^{-(3\alpha_3+\epsilon)w}$ where $\epsilon=\mathrm{ord}_p(d_0)$. Hence, the Dirichlet polynomials $R_l(s_1,w;2k)$ must be chosen so that the only those powers of $\mathbb{N}p^{-w}$ appear on the right-hand side. Since each Dirichlet polynomial R_l is associated to a distinct set of monomials in $\mathbb{N}p^{-s_1}$ and $\mathbb{N}p^{-w}$, this immediately implies that there can be no nontrivial power of $\mathbb{N}p^{-w}$ occurring in R_l which is not congruent to ϵ mod 3, else it would appear on the left-hand side. $\square$

Lemma 5.5 (Transformation Property). *Suppose that we have a Dirichlet polynomial of form*

$$R_l(s_1, w; 2k) = c_m \mathbb{N}p^{-js_2-3mw} + \cdots + c_M \mathbb{N}p^{-js_2-3Mw}$$

for some fixed $j > 0$*. Then* $R_l(s_1, w; 2k) = \mathbb{N}p^{(1/2-w)[j-3(M+m)]} R_l(s_1 + w - 1/2, 1 - w; 2k)$.

This will follow from an easy formula describing how each of the multipliers transform under $(s_1, w) \mapsto (s_1 + w - 1/2, 1 - w)$. Note that since powers of $\mathbb{N}p^{-js_1}$ remain fixed under this transformation, we must only check that for each $R_l(s_1, w)$, terms of a fixed $\mathbb{N}p^{-js_2}$ transform into each other up to a power of $\mathbb{N}p^{1/2-w}$. Hence, in conjunction with the previous result, this lemma applies in full generality to the Dirichlet polynomials $R_l(s_1, w; 2k)$.

Proof: Suppose, for a given fixed $j > 0$, that $\mathbb{N}p^{-3mw}$ is the smallest occurring power of $\mathbb{N}p^{-w}$ in $R_l(s_1, w; 2k)$ and $\mathbb{N}p^{-3Mw}$ is the largest such power. Then, under the transformation $(s_1, w) \mapsto (s_1 + w - 1/2, 1 - w)$, the terms

$$c_m \mathbb{N}p^{-js_1-3mw} + \cdots + c_M \mathbb{N}p^{-js_1-3Mw} \longrightarrow$$
$$c_m \mathbb{N}p^{(j/2-3m)-js_1-(j-3m)w} + \cdots + c_M \mathbb{N}p^{(j/2-3M)-js_1-(j-3M)w}$$

In particular, if this Dirichlet polynomial is to transform into itself up to a power of $\mathbb{N}p^{1/2-w}$, then the largest negative power of $\mathbb{N}p^{-w}$ on the right-hand side must transform into the largest negative power on the left-hand side. Hence, the term $c_m \mathbb{N}p^{(j/2-3m)-js_1-(j-3m)w}$ must be taken to $c_M \mathbb{N}p^{-js_1-3Mw}$ by a power of $\mathbb{N}p^{1/2-w}$. Equating these, we find that this factor must be $\mathbb{N}p^{(1/2-w)[j-3(M+m)]}$. □

As an immediate corollary, any single term $c_l \mathbb{N}p^{-js_2-lw}$ with $j > 0$ and $c_l \neq 0$ appearing in the Dirichlet polynomial implies that the entire polynomial transforms by at least $\mathbb{N}p^{(j-(l+3))/2-(j-(l+3))w}$. Note that the transformation property also provides a relation between c_m and c_M, though we will not be concerned with that here.

The previous lemma gave an indication of how the transformation property for the Dirichlet polynomial restricts the possible choices for $R_l(s_1, w; 2k)$, since each such R_l must transform by $\mathbb{N}p^{l(1/2-w)}$. In the above notation, this means that m, M, and j must satisfy $3M + 3m - j = l$. The following lemma shows how the other required property of $R_l(s_1, w; 2k)$, namely the condition that each monomial $\mathbb{N}p^{-is_1-jw}$ has $i \leq j$, places restrictions on the form of the coefficients of the correction factor $P(s_1, s_2, d)$.

Lemma 5.6 (Stability over α_3). *Suppose that the correction coefficients of* $P(s_1, s_2, d)$ *are defined by a choice of Dirichlet polynomials* $R_l(s_1, w; 2k)$ *according to the Main Theorem. Given an integer* $k > 0$*, then for any* $i, r > 0$*, we have*

$$a_{i,k}^{(\alpha_3)}(d_0, p) = a_{i,k}^{(\alpha_3+r)}(d_0, p) \ \text{if } 3\alpha_3 > 2k$$

Proof: Recall that, according to the Main Theorem, we have for $p \nmid d_0$,

$$\sum_{\alpha_3 \geq 0} \mathbb{N}p^{-3\alpha_3 w} \sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j}) \sum_i a_{i,j}^{(\alpha_3)} \mathbb{N}p^{-3\alpha_3/2+(3\alpha_3-i)-(3\alpha_3-i)s_1}$$
$$= \sum_{l=0}^{2k} R_l(s_1, w; 2k) \Bigg[\bar{\chi}_{d_0}(p^l) + \sum_{0<3\alpha_3 \leq 2k-l} \bar{\chi}_{d_0}(p^l)(\mathbb{N}p^{3\alpha_3/2} - \mathbb{N}p^{3\alpha_3/2-1})\mathbb{N}p^{-3\alpha_3 w} +$$
$$\sum_{3\alpha_3 > 2k-l} \left(\mathbb{N}p^{3\alpha_3/2-(3\alpha_3-l)s_1-3\alpha_3 w} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha_3/2-1-(3\alpha_3-1-l)s_1-3\alpha_3 w} \right) \Bigg] \quad (5.4)$$

Similar but far simpler equalities hold for $p|d_1$ and $p|d_2$. The lemma will follow from showing that the value of

$$\sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j}) \sum_i a_{i,j}^{(\alpha_3)} = \sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j}) \sum_i a_{i,j}^{(\alpha_3+r)}$$

for any $r > 0$ whenever $3\alpha_3 > 2k$. We do this by studying the right-hand side of the above equality. Pick any value β with $3\beta > 2k$. Then contributions on the right-hand side to $\mathbb{N}p^{-3\beta w}$ come from products of monomials $\mathbb{N}p^{-is_1-3\lambda w}$ in $R_l(s_1, w; 2k)$ with bracketed terms involving $\mathbb{N}p^{-3\gamma w}$ such that $\lambda+\gamma = \beta$. We want to show that $3\gamma > 2k-l$; this will guarantee that the contribution from the bracketed terms in (5.4) comes from the final sum. But the appearance of the monomial $\mathbb{N}p^{-is_1-3\lambda w}$ implies that R_l transforms by at least $\mathbb{N}p^{-3\lambda w}$. Indeed, since $i < 3k$ for all appearing monomials $\mathbb{N}p^{-is_1-3kw}$, then writing down the list of terms in R_l including $\mathbb{N}p^{-is_1}$ we have

$$c(i, m, l)\mathbb{N}p^{-is_1-3mw} + \cdots + c(i, \lambda, l)\mathbb{N}p^{-is_1-3\lambda w} + \cdots + c(i, M, l)\mathbb{N}p^{-is_1-3Mw}$$

with $m \leq \lambda \leq M$, and the $c(i, r, l)$ constants depending on the indicated quantities. According to the transformation property, this transforms by $\mathbb{N}p^{-(3M+3m-i)w} \geq \mathbb{N}p^{-3\lambda w}$ since $i \leq m$. So R_l transforms by at least $\mathbb{N}p^{-3\lambda w}$. But then $3\lambda \leq l$ since R_l transforms by $\mathbb{N}p^{-lw}$ by definition. Putting this all together, we have $3\gamma = 3\beta - 3\lambda \geq 3\beta - l > 2k - l$. In short, all contributions of bracketed terms to $\mathbb{N}p^{-3\beta w}$ come from the sum over $3\gamma > 2k - l$. Writing down the total contribution to $\mathbb{N}p^{-3\beta w}$ from the right-hand side, we have

$$\sum_{\gamma+\lambda=\beta} \underbrace{\sum_i c(i, \lambda, l)\mathbb{N}p^{-is_1-3\lambda w}}_{\text{coming from } R_l(s_1,w;2k)}$$
$$\underbrace{\left[\mathbb{N}p^{3\gamma/2-(3\gamma-(2k-l))s_1-3\gamma w} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\gamma/2-1-(3\gamma-1-(2k-l))s_1-3\gamma w}\right]}_{\text{coming from the sum over } 3\gamma > 2k-l \text{ from bracketed terms}}$$

where $c(i, \lambda, l)$ is the constant from R_l associated to the indicated monomial. For any fixed power of $\mathbb{N}p^{-s_1}$, we have a finite number of terms contributing to, say, $\mathbb{N}p^{-(3\beta-i)s_1-3\beta w}$ for any i. This set of terms, according to the equality in the Main Theorem, should correspond to terms on the left-hand side of form:

$$\sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j}) a_{i,j}^{(\beta)} \mathbb{N}p^{-3\beta/2+(3\beta-i)-(3\beta-i)s_1-3\beta w}$$

To finish the argument in this case, note that the contribution to $\mathbb{N}p^{-3(\beta+1)w}$ on the right-hand side comes from products of terms including the very same $\sum_i c(i,\lambda,l)\mathbb{N}p^{-is_1-3\lambda w}$ from R_l but now paired with $\mathbb{N}p^{-3(\gamma+1)w}$ from the bracketed sums. (The Dirichlet polynomials can't contain a power of $\mathbb{N}p^{-3(\beta+1)w}$ since this would transform by at least $\mathbb{N}p^{-3(\beta+1)w}$ and we chose $\beta > 2k$.) Hence, all terms contributing to $\mathbb{N}p^{-3(\beta+1)w}$ from R_l are in one-to-one correspondence with terms of R_l contributing to $\mathbb{N}p^{-3\beta w}$. However, to add to a total of $\mathbb{N}p^{-3(\beta+1)w}$, the contribution from R_l must now be paired with bracketed terms containing $\mathbb{N}p^{-3(\gamma+1)w}$. So the total contribution to $\mathbb{N}p^{-3(\beta+1)w}$ on the right-hand side is now just expressed as

$$\sum_{\gamma+\lambda=\beta} \underbrace{\sum_i c(i,\lambda,l)\mathbb{N}p^{-is_1-3\lambda w}}_{\text{coming from } R_l(s_1,w;2k)} \cdot$$

$$\underbrace{\left[\mathbb{N}p^{(3\gamma+3)/2-(3\gamma+3-(2k-l))s_1-3(\gamma+1)w} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{(3\gamma+3)/2-1-(3\gamma+3-1-(2k-l))s_1-3(\gamma+1)w}\right]}_{\text{coming from the sum over } 3(\gamma+1)>2k-l}$$

Now it is clear that, comparing contributions at β and $\beta+1$, terms in this finite sum with associated monomial $\mathbb{N}p^{-(3\beta+3-i)s_1-3(\beta+1)w}$ are the same as those associated to $\mathbb{N}p^{-(3\beta-i)s_1-3\beta w}$, but with an additional factor of $\mathbb{N}p^{3/2}$. In the case of $\beta+1$, the equality in the Main Theorem gives terms on the left-hand side of form

$$\sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j})a_{i,j}^{(\beta+1)}\mathbb{N}p^{-3(\beta+1)/2+(3\beta+3-i)-(3\beta+3-i)s_1-3(\beta+1)w}$$

Comparing the terms on the left-hand side at β, we see that the identical coefficients appear with an associated monomial that also differs by a factor of $\mathbb{N}p^{3/2}$. Hence the contribution to

$$\sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j})\psi_1(p^{k-j})a_{i,j}^{(\beta+1)}$$

is stable from β to $\beta+1$. Since the same argument would show a stable contribution if k were replaced by $k-1$, then the $a_{i,k}$ coefficient must also be stable in this case. This completes the case where $p|d_0$.

Rather than repeat this argument for the cases $p|d_1$ and $p|d_2$, we merely note that the same procedure can be identically carried out provided that we can guarantee that all contributions from the right-hand side's bracketed terms come from the sums with $3\alpha_3 \geq 2k-l$ and $3\alpha_3+1 \geq 2k-l$, respectively. But all the other coefficients are 0 in these cases, so this condition is automatically satisfied and the claims follow by identical argument to the above. □

5.2 Proving the Existence of a Consistent Set of Correction Coefficients

5.2.1 Outline of the Method

The object of this chapter is to determine a natural definition for $Z_4(s_1,w;p^k)$, the coefficient of $\mathbb{N}p^{-ks_2}$ in $Z_4(s_1,s_2,w)$. As previously discussed, we want this series to satisfy the transformation property $Z_4(s_1, w; p^k) = Z_4(s_1 + w - 1/2, 1 - w; p^k)p^{k-2kw}$, since this will lead to a simple proof that $Z_4(s_1, s_2, w) = Z_4(s_1 + w - 1/2, s_2 + 2w - 1, 1 - w)$. Our investigations in the earlier sections led us to conjecture that taking the definition

$$Z_4(s_1, w; p^k) = \sum_{l=0}^{2k} R_l(s_1, w; 2k) D(s_1, w; p^{2k-l}) = \sum_{l=0}^{2k} R_l(s_1, w; 2k) \sum_{m,d} \frac{G(mp^{2k-l}, d)}{\mathbb{N}m^{s_1} \mathbb{N}d^w}$$

for appropriately chosen Dirichlet polynomials $R_l(s_1, w; 2k)$ would produce such a series with all the necessary properties. After analyzing both sides of this definition, we reduced this conjecture to the existence of a finite set of Dirichlet polynomials $R_l(s_1, w; 2k)$ so that a given set of equations can be solved in a consistent way. We stated this as the Main Theorem; its proof is the sole goal of this section. We restate the theorem here as a reminder.

Theorem 5.7 (Main Theorem). *Let p be a fixed prime. Then for every $k \geq 0$, there exists a choice of finite Dirichlet polynomials $R_l(s_1, w; 2k)$ for $0 \leq l \leq 2k$ in the variables $\mathbb{N}p^{-s_1}$ and $\mathbb{N}p^{-w}$, with $R_l(s_1, w; 2k) = \mathbb{N}p^{l/2-lw} R_l(s_1 + w - 1/2, 1 - w)$, so that the definition*

$$Z_4(s_1, w; p^k) = \sum_{l=0}^{2k} R_l(s_1, w; 2k) D(s_1, w; p^{2k-l}) = \sum_{l=0}^{2k} R_l(s_1, w; 2k) \sum_{m,d} \frac{G(mp^{2k-l}, d)}{\mathbb{N}m^{s_1} \mathbb{N}d^w},$$

or equivalently, the definition according to the following case method: if $p|d_0$,

$$\sum_{\alpha_3 \geq 0} \mathbb{N}p^{-3\alpha_3 w} \sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j}) \psi_1(p^{k-j}) \sum_i a_{i,j}^{(\alpha_3)} \mathbb{N}p^{-3\alpha_3/2 + (3\alpha_3 - i) - (3\alpha_3 - i)s_1}$$
$$= \sum_{l=0}^{2k} R_l(s_1, w; 2k) \Big[\bar{\chi}_{d_0}(p^{2k-l}) \bar{\psi}_1(p^{2k-l}) +$$
$$\sum_{0 < 3\alpha_3 \leq 2k-l} \bar{\chi}_{d_0}(p^{2k-l}) \bar{\psi}_1(p^{2k-l}) (\mathbb{N}p^{3\alpha_3/2} - \mathbb{N}p^{3\alpha_3/2-1}) \mathbb{N}p^{-3\alpha_3 w} +$$
$$\sum_{3\alpha_3 > 2k-l} \left(\mathbb{N}p^{3\alpha_3/2 - (3\alpha_3 - l)s_1 - 3\alpha_3 w} - \chi_{d_0}(p) \psi_1(p) \mathbb{N}p^{3\alpha_3/2 - 1 - (3\alpha_3 - 1 - l)s_1 - 3\alpha_3 w} \right) \Big]$$

if $p|d_1$, then

$$\sum_{\alpha_3 \geq 0} \mathbb{N}p^{-(3\alpha_3+1)w} \sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j}) \psi_1(p^{k-j}) \sum_i a_{i,j}^{(\alpha_3)} \mathbb{N}p^{-3\alpha_3/2 + (3\alpha_3 - i) - (3\alpha_3 - i)s_1} =$$
$$= \sum_{l=0}^{2k} R_l(s_1, w; 2k) \left[\sum_{3\alpha_3 \geq 2k-l} \mathbb{N}p^{3\alpha_3/2 - (3\alpha_3 - (2k-l))s_1 - (3\alpha_3+1)w} \right]$$

and if $p|d_2$, then

$$\sum_{\alpha_3 \geq 0} \mathbb{N}p^{-3\alpha_3 w} \sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j}) \psi_1(p^{k-j}) \sum_i a_{i,j}^{(\alpha_3)} \mathbb{N}p^{-3\alpha_3/2 - 1/2 + (3\alpha_3 + 1 - i) - (3\alpha_3 + 1 - i)s_1} =$$
$$= \sum_{l=0}^{2k} R_l(s_1, w; 2k) \left[\sum_{3\alpha_3 + 1 \geq l} \mathbb{N}p^{3\alpha_3/2 - (3\alpha_3 + 1 - (2k-l))s_1 - (3\alpha_3+2)w} \right]$$

defines a consistent set of correction coefficients $a_{i,j}^{(\alpha_3)}(d_0, p)$ *of* $P(s_1, s_2, d)$ *such that the following properties hold:*

1) The correction factor is a finite, Eulerian Dirichlet polynomial

2) The correction factor is symmetric in the variables s_1 *and* s_2*.*

3) The correction factor is trivial if d *is cube-free. More specifically, each Euler factor at the prime* p *depends on the divisibility of* d *by powers of* p^3*.*

4) The values of the coefficients agree with those determined by the method of taking variables to infinity.

5) Large collections of the $a_{i,j}$ *sum to 0 according to (4.7).*

6) The coefficients $a_{i,j}$ *satisfy mild growth conditions according to (4.11).*

Both formulations of the theorem will be useful to us: we want to remember that the structure of the right-hand side is largely dictated by the presence of the series containing Gauss sums, but also that this ultimately reduces to a finite set of terms corresponding to any choice of d and corresponding α_3. Before settling on this definition of $Z_4(s_1, w; p^k)$, we attempted to show that the series

$$\sum_{m,d} \frac{G(mp^{2k}, d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} = Z_4(s_1, w; p^k)$$

as was suggested by the square-free heuristic. In our new notation, this would mean that $R_0(s_1, w; 2k) = 1$ for all k and $R_l(s_1, w; 2k) = 0$ for all $l \neq 0$. As we remarked earlier, there were immediate problems with this. However, the assumption that $R_0(s_1, w; 2k) = 1$ seems reasonable and we make this more precise in the following discussion.

Assumption 5. *Keeping all of the previous notation as before, so that* $R_l(s_1, w; 2k)$ *denotes a finite Dirichlet polynomial with specified transformation property, which serves to define the coefficients of the correction factor* $P(s_1, s_2, d)$*. Then* $R_0(s_1, w; 2k) = 1$*.*

Justification: We know from the transformation property, together with the assumption that all monomials take form $\mathbb{N}p^{-is_1-jw}$ with $i \leq j$, that the value of $R_0(s_1, w; 2k)$ must be a constant. However, we don't know that it is non-zero. If we take $R_0(s_1, w; 2k) = 1$, what does this imply about the coefficients $a_{i,j}^{(\alpha_3)}(d_0, p)$? Choose α_3 to be the largest integer such that $2k \geq 3\alpha_3$. Then referring again to the terms involved in the equality of the Main Theorem, for $p \nmid d_0$ we have a right-hand side contribution from the $l = 0$ term of $\bar{\chi}_{d_0}(p^{2k})\bar{\psi}_1(p^{2k})(\mathbb{N}p^{3\alpha_3/2} - \mathbb{N}p^{3\alpha_3/2-1})\mathbb{N}p^{-3\alpha_3 w}$. Comparing this with the left-hand side, this implies that

$$\sum_{j=0}^{k} \bar{\chi}_{d_0}(p^{2k-2j})\bar{\psi}_1(p^{2k-2j})a_{3\alpha_3,j}^{(\alpha_3)}\mathbb{N}p^{-3\alpha_3/2} = \bar{\chi}_{d_0}(p^{2k})\bar{\psi}_1(p^{2k})(\mathbb{N}p^{3\alpha_3/2} - \mathbb{N}p^{3\alpha_3/2-1}).$$

Recall that by taking the $\Re(s_1) \to \infty$, we found that

$$a_{3\alpha_3,j}^{(\alpha_3)}(d_0, p) = \begin{cases} \mathbb{N}p^{3\alpha_3} - \mathbb{N}p^{3\alpha_3-1} & \text{if } \alpha_3 \text{ is even, } j = 3\alpha_3/2. \\ \bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{3\alpha_3} & \text{if } \alpha_3 \text{ is odd, } j = (3\alpha_3+1)/2. \\ -\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha_3-1} & \text{if } \alpha_3 \text{ is odd, } j = (3\alpha_3-1)/2. \\ 0 & \text{otherwise.} \end{cases}$$

Substituting these values into the left-hand side reveals a perfect match of character values and powers of primes for both even and odd α_3. One can show that no other single choice of $R_l(s_1, w; 2k)$ and attached Dirichlet series will provide the same data, but it is difficult to rule out the possibility of a complex combination of them. Ultimately, this is of no concern to us. The important point is that, because all of the right-hand side contributions from choosing $R_0(s_1, w; 2k) = 1$ are known to occur on the left-hand side, then setting $R_0(s_1, w; 2k) = 1$ is a viable choice which will reduce our work when we try to find a consistent determination of the $a_{i,j}$ according to the choice of these Dirichlet polynomials.

5.2.2 Proof of the Main Theorem

We are now ready to show that appropriate choices for the rest of the Dirichlet polynomials $R_l(s_1, w; 2k)$ for $l > 0$ exist.

Proof of the Main Theorem: First note that since the contributions on the right-hand side of our definition are all determined locally for each prime p and will contain a finite list of terms for each α_3 provided that $R_l(s_1, w; 2k)$ is finite for each l. Hence, the correction factor determined by this process with automatically be finite and Eulerian, satisfying the first property.

The collection of $a_{i,j}^{(\alpha_3)}(d_0, p)$ determined by the choice of $R_l(s_1, w; 2k)$ will be consistent provided that we can prove the following inductive step. We must show that the coefficients determined by the $\mathbb{N}p^{-ks_2}$ coefficient (i.e. the choices of $R_l(s_1, w; 2k)$ for $l = 0, \ldots, 2k$) agree with the coefficients determined by the $\mathbb{N}p^{-(k+1)s_2}$ coefficient (i.e. the choices of $R_l(s_1, w; 2k+2)$ for $l = 0, \ldots, 2k+2$.)

Suppose that we have a consistent set of coefficients determined by the choices of $R_l(s_1, w; 2t)$ for $t \leq k$. According to the definition in the Main Theorem, these $R_l(s_1, w; 2t)$ will determine the coefficients $a_{i,j}^{(\alpha_3)}(d_0, p)$ for $j \leq k$ and any $i, \alpha_3 > 0$. Further, if we require that the $a_{i,j}^{(\alpha_3)}(d_0, p)$ are symmetric in i and j, then we have in fact determined all $a_{i,j}^{(\alpha_3)}(d_0, p)$ with either i or $j \leq k$. Now let's look more closely at the implied conditions on $R_l(s_1, w; 2k+2)$.

According to the equality in the main theorem, the sum total of terms of the form $\mathbb{N}p^{-(3\alpha_3 - i)s_1 - 3\alpha_3 w}$ on the right-hand side (in the coefficient of $\mathbb{N}p^{-(k+1)s_2)}$) should correspond to the set of terms

$$\sum_{j=0}^{k+1} \bar{\chi}_{d_0}(p^{2k-2j}) \bar{\psi}_1(p^{2k-2j}) a_{i,j}^{(\alpha_3)}(d_0, p) \mathbb{N}p^{-3\alpha_3/2 + (3\alpha_3 - i)}$$

on the left-hand side. If $i > k$, then this list of terms includes the previously undetermined coefficient $a_{i,k+1}^{(\alpha_3)}(d_0, p)$ (note this is the only term in the list which is previously undetermined since we require symmetry in the indexes i and j), so the right-hand side need not conform to any prescribed value in these cases. However, for $i \leq k$, this is not the case. For all such terms $\mathbb{N}p^{-(3\alpha_3 - i)s_1 - 3\alpha_3 w}$, the left-hand side has been completely inductively determined so we must ensure that terms on the right-hand side appropriately agree with these values.

At first glance, this appears quite difficult since we must have agreement over all values of α_3. However, according to the lemma on stability in α_3, the $\mathbb{N}p^{-(k+1)s_2}$ coefficient produces coefficients which are stable with respect to α_3 whenever $3\alpha_3 > 2k+2$. Moreover, the sums of correction factor coefficients we are trying to agree with, $a_{i,j}^{(\alpha_3)}(d_0, p)$ with i or $j \leq k$, are stable for $3\alpha_3 > 2k$. Hence, if we agree with contributions from the left-hand side for these terms up to the smallest α_3 such that $3\alpha_3 > 2k+2$, then we are done, since the right-hand side will be forced by its structure to produce the correct stable values of $a_{i,j}^{(\alpha_3)}(d_0,p)$ for all α_3 ever after.

In short, our conclusion is that we have a consistent determination of Dirichlet polynomials $R_l(s_1, w; 2k+2)$ which produces a correction factor symmetric in the variables s_1 and s_2 if we can correct for the previously determined terms associated to $\mathbb{N}p^{-(3\alpha_3-i)s_1-3\alpha_3 w}$ for $3\alpha_3 \leq 2k+2$ and $0 < i \leq k$. Our plan is to do this by induction on i. That is, we correct for all of the terms containing $\mathbb{N}p^{-(3\alpha_3-i)s_1}$ without worrying about the effect on terms containing $\mathbb{N}p^{-(3\alpha_3-(i+1))s_1}$.

We do the case $i=1$ to begin. Then we need to match all of the potential left-hand side contributions to $\mathbb{N}p^{-(3\alpha_3-1)s_1-3\alpha_3 w}$ for all $\alpha_3 > 0$. These terms occur in the form:

$$\sum_{\alpha_3>0}\sum_{j=0}^{k}\bar{\chi}_{d_0}(p^{2k-2j})\bar{\psi}_1(p^{2k-2j})a_{1,j}^{(\alpha_3)}(d_0,p)\mathbb{N}p^{-3\alpha_3/2+(3\alpha_3-1)-(3\alpha_3-1)s_1-3\alpha_3 w}$$

But we know that, in general, the coefficients of $a_{i,j}^{(\alpha_3)}(d_0,p)$ are stable in α_3 for $3\alpha_3 > 2\min\{i,j\}$. Moreover, $a_{1,j}^{(\alpha_3)}(d_0,p) = 0$ if $j > 3$ and

$$\sum_{j=0}^{k}\bar{\chi}_{d_0}(p^{2k-2j})\bar{\psi}_1(p^{2k-2j})a_{1,j}^{(\alpha_3)}(d_0,p) = 0$$

if $k \geq 3$ and $\alpha_3 \geq 1$, by our previous assumptions. Hence, we only need to correct for this term if $\alpha_3 = 1$ and $k = 1, 2$. If $k = 2$, then we take the contribution from the left-hand side, call it $c(i,\alpha_3,2k)\mathbb{N}p^{-(3\alpha_3-i)s_1-3\alpha_3 w} = c(1,1,4)\mathbb{N}p^{-2s_1-3w}$ and match it on the right-hand side using $c(1,1,4)\mathbb{N}p^{-2s_1-3w}D(s_1,w;1)$ which possesses the correct transformation property at $k=2$ and gives the correct stable contribution for every α_3 ever after. The case $k=1$ is more of a problem. No terms have a transformation small enough to be added to a Dirichlet polynomial. Instead, we will show in the examples of the subsequent section that we do not need to correct for these terms.

We pick and fix a value i and find potential contributions from the left-hand side to $\mathbb{N}p^{-(3\alpha_3-i)s_1}$ whenever $\alpha_3 \leq 2k+5$. In general, we know that these terms occur in the form:

$$\sum_{\alpha_3>0}\sum_{j=0}^{k+1}\bar{\chi}_{d_0}(p^{2(k+1)-2j})\bar{\psi}_1(p^{2(k+1)-2j})a_{i,j}^{(\alpha_3)}(d_0,p)\mathbb{N}p^{-3\alpha_3/2+(3\alpha_3-i)-(3\alpha_3-i)s_1-3\alpha_3 w}$$

But we know that, in general, the coefficients of $a_{i,j}^{(\alpha_3)}(d_0,p)$ are stable in α_3 for $3\alpha_3 > 2\min\{i,j\}$. So, in particular, these coefficients in the above sum are stable in α_3 for $3\alpha_3 > 2i$. If α_3 is such that $3\alpha_3 \leq 2i$, then we look to add a term to the right-hand side of form $c(i,\alpha_3,2k+2)\mathbb{N}p^{-(3\alpha_3-i)s_1-3\alpha_3 w}$ for some constant c. If such a term occurred alone in one of the Dirichlet polynomials, it would transform by $6\alpha_3 - (3\alpha_3 - i) = 3\alpha_3 + i$. If $3\alpha_3 + i \leq 2k+2$ then this is a legitimate transformation size to occur in a Dirichlet polynomial and we may add the term to the appropriate series containing a Gauss sum as $c(i,\alpha_3,2k+2)\mathbb{N}p^{-(3\alpha_3-i)s_1-3\alpha_3 w}D_4(s_1,w;p^{2k+2-(3\alpha_3+i)})$. The terms generated by this addition on the right-hand side obviously include the desired term. All subsequent terms produced by this addition have higher values of α_3 and correspond to terms $\mathbb{N}p^{-(3\alpha_3-r)s_1}$ with $r > i$.

However, it may be the case that even though $3\alpha_3 \leq 2i$ and $i \leq k$, we still have $3\alpha_3 + i > 2k + 2$, in which case we need to work a little harder to correct the missing term. In general, the class of terms we may add to a Dirichlet polynomial containing $\mathbb{N}p^{-(3\alpha_3-i)s_1-3\alpha_3 w}$ take the form

$$\mathbb{N}p^{-(3\alpha_3-i)s_1}(c_m\mathbb{N}p^{-3mw} + \cdots + c_M\mathbb{N}p^{-3Mw})$$

for some appropriately chosen coefficients $c_m, \ldots, c_M$, with $m \leq \alpha_3 \leq M$, and $3\alpha_3 - i \leq 3m$. Such a collection of terms transforms by $3M + 3m - (3\alpha_3 - i)$. This is minimized when $M = \alpha_3$ and $3m$ chosen as close to $3\alpha_3 - i$ as possible, so that $3m - (3\alpha_3 - i) \leq 2$. In this case, the set of terms transforms by $3\alpha_3 + 2 \leq 2i + 2 \leq 2k + 2$. So there is always some such collection of terms which can be attached to a series containing Gauss sums. Precisely, for any such collection of terms which transforms by less than $2k + 3$, it should look like

$$\mathbb{N}p^{-(3\alpha_3-i)s_1}\Big[c(i,\alpha_3,2k+2)\mathbb{N}p^{-3/2(\alpha_3-m)-3mw} + \cdots + c(i,\alpha_3,2k+2)\mathbb{N}p^{-3\alpha_3 w} + \cdots \\ \cdots + c(i,\alpha_3,2k+2)\mathbb{N}p^{3/2(M-\alpha_3)-3Mw}\Big] D(s_1, w; p^{2k+2-(3M+3m-(3\alpha_3-i))})$$

where $c(i, \alpha_3, 2k + 2)$ is the coefficient of the left-hand side term $\mathbb{N}p^{-(3\alpha_3-i)s_1-3\alpha_3 w}$ for which we are trying to correct and the other monomials have corresponding coefficients to guarantee that the terms transform into itself. Unfortunately, in adding this term, we have introduced some lower order terms which are jeopardizing the validity of our induction. To fix this problem, if $m < \alpha_3$, then also add the following term to the right-hand side:

$$-\mathbb{N}p^{-(3\alpha_3-i)s_1}\Big[c(i,\alpha_3,2k+2)\mathbb{N}p^{-3/2(\alpha_3-m)-3mw} + \cdots \\ \cdots + c(i,\alpha_3,2k+2)\mathbb{N}p^{-3/2-3(\alpha_3-1)w}\Big] D(s_1, w; p^{2k+2-(3\alpha_3-3+3m-(3\alpha_3-i))})$$

Notice we have simply removed the top terms from previous terms in the Dirichlet polynomial. This has the effect of reducing the transformation property by $3(M - \alpha_3 + 1)$, so it is associated to a Gauss sum series with prime power correspondingly shifted by $3(M - \alpha_3 + 1)$.

Lastly, we must check that when α_3 is chosen to be the smallest integer such that $3\alpha_3 > 2i$ (that is, the smallest value of α_3 for which we have stability ever after), we can similarly correct for terms appearing on the left-hand side. In this case, $2i < 3\alpha_3 \leq 2i + 3$. Again, we can perform the same type of addition of terms as in the previous case, but only if $3\alpha_3 + 3m - (3\alpha_3 - i) \leq 2k + 2$. Note that we can find the minimum transformation of such terms according to the following case method:

$$\text{minimum size of transformation} = 3\alpha_3 + 3m - (3\alpha_3 - i) = \begin{cases} 2i+2 & \text{if } i \equiv 1(3) \\ 2i+3 & \text{if } i \equiv 0(3) \\ 2i+4 & \text{if } i \equiv 2(3) \end{cases}$$

Since we know that $i \leq k$, this gives us a means for correcting left-hand side contributions for all i excepting $i = k$ in the cases where $i \equiv 0, 2(3)$.

However, we do want to be careful in selecting these additional monomials so that the corrected value gives a stable collection of correction coefficients at every α_3 ever after. To ensure this, consider the collection of terms added to one of the $R_l(s_1, w; 2k + 2)$ which corrects for $a_{i,j}$ at a fixed i. This set of terms should contain $c(3\alpha_3)\mathbb{N}p^{-(3\alpha_3-i)s_1-3\alpha_3 w}$ for some appropriately chosen constant $c(3\alpha_3)$. If this collection of terms transforms by less than $2k$ then it will be associated to a Gauss sum series $D_4(s_1, w; p^k)$ with $k \geq 3$ and hence, will not produce the stable set of terms we want. Then add a collection of terms containing $c(3\alpha_3)\mathbb{N}p^{3/2-(3\alpha_3+3-i)s_1-(3\alpha_3+3)w}$, that is, $\mathbb{N}p^{3/2-3s_1-3w} \cdot \{$ previously used term $\}$. This will give the appropriate stable contribution for $a_{i,j}$ at $\alpha_3 + 1$ and the collection of terms transforms by at least 3 more than the previous collection. Repeat this process until α_3 is so large that this transforms by at least $2k$. Then the following final additions, covering all possible cases, give the desired stability in α_3 for the $a_{i,j}$. Suppose that the collection of terms added to an R_l contains the desired term $\mathbb{N}p^{-(3\alpha_3-i)s_1-3\alpha_3 w}$ and transforms by at least $2k$. Then to guarantee stability in the $a_{i,j}$ for all successive α_3, add the following terms:

$$\left[c(m)\mathbb{N}p^{-(3\alpha_3-i)s_1-3mw} + \cdots + c(3\alpha_3)\mathbb{N}p^{-(3\alpha_3-i)s_1-3\alpha_3 w}\right] G(m, d)$$

if transforming by $2k + 2$,

$$\left[c(m)\mathbb{N}p^{-(3\alpha_3-i-2)s_1-3mw} + \cdots + c(3\alpha_3)\mathbb{N}p^{-(3\alpha_3-i-2)s_1-(3\alpha_3-3)w}\right] G(mp^3, d) - \left[c(m)\mathbb{N}p^{-(3\alpha_3-i-5)s_1-3mw} + \cdots + c(3\alpha_3)\mathbb{N}p^{-(3\alpha_3-i-5)s_1-(3\alpha_3-6)w}\right] G(mp^6, d)$$

if transforming by $2k + 1$, and

$$\left[c(m)\mathbb{N}p^{-(3\alpha_3-i)s_1-3mw} + \cdots + c(3\alpha_3)\mathbb{N}p^{3/2-(3\alpha_3-i)s_1-3\alpha_3 w}\right] G(p^2m, d)$$

if transforming by $2k$.

In the last remaining case, where $i = k$ and $k \equiv 0, 2(3)$, these terms can be corrected for by a similar addition and subtraction of similar terms as above. Matching the transformations appropriately, then we have the result. This completes the induction.

Notice that we have satisfied all the desired properties, since the decision by the previous lemma to fix $R_0(s_1, w; 2k) = 1$ for all k, together with the base case of the induction, guarantee that our inductive process will always agree with the coefficients determined by methods of taking variables to infinity. They also give the appropriate stability conditions by construction. In particular, if $\alpha_3 = 0$, this forces all coefficients to vanish except for $a_{0,0} = 1$ in this case. Hence, the correction factors are trivial if d is cube-free. Moreover, by appropriately choosing the first undetermined coefficient in each step of the induction, we can guarantee that the coefficients sum to zero in precisely the appropriate range according to the stability of the Gauss sum for $3\alpha_3 > 2k$. Such a choice is further consistent with the mild growth hypothesis by simple induction. The fact that the Dirichlet polynomial combinations lead to additional functional equations for Z_4, Z_5, and Z_6 will be the topic of the next chapter. The astute reader will now notice that all of the conditions of Assumption 4 have been verified. □

5.3 Examples: Computing the Polynomials $R_l(s_1, w; 2k)$ for Small k

The previous section gave a proof that there exist combinations of Dirichlet polynomials $R_l(s_1, w; p^{2k})$ and Dirichlet series $D_4(s_1, w; p^{2k-l})$ for $l = 0, \ldots, 2k$ which define each prime power coefficient $\mathbb{N}p^{-ks_2}$ of $Z_4(s_1, w; p^k)$. The combinatorics involved are difficult to grasp in such generality and it is rather spectacular to see the fully realized Dirichlet polynomials actually occurring in practice. For these reasons, we will compute several examples for small k, remarking at both the coefficients they determine and the relative uniqueness of the process as we compute each case.

Case 0: k=0. For this case, we want to find the coefficient of 1^{-s_2} in $Z_4(s_1, s_2, w)$. According to our set-up, this should be

$$R_0(s_1, w; p^{2k}) D_4(s_1, w; p^{2k}) = R_0(s_1, w; 1) D_4(s_1, w; 1)$$

for some appropriately chosen Dirichlet polynomial R_0. Recall that for each such Dirichlet polynomial, we required $R_l(s_1, w; p^{2k}) = R_l(s_1 + w - 1/2, 1 - w; p^{2k}) \mathbb{N}p^{l/2 - lw}$. So in our case, we must have

$$R_0(s_1, w; 1) = R_0(s_1 + w - 1/2, 1 - w; 1)$$

which immediately implies that R_0 must be constant. This constant must be 1 in order to satisfy the desired properties of the correction factor, including triviality for cube-free integers among others.

Of course, we already knew this must be the case since the coefficient of 1^{-s_2} was already determined by letting $\Re(s_2) \to \infty$ in Section 3.2.2. There we determined that the ideal object should be $\displaystyle\sum_{\substack{m,d \equiv 1 \ (3) \\ (md,6)=1}} \frac{G(m,d)\bar{\psi}_1(m)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^{w}}$. But this is just $D_4(s_1, w; 1)$ in our current notation.

Case 1: k=1. Here we seek the coefficient of $\mathbb{N}p^{-s_2}$ in $Z_4(s_1, s_2, w)$. Translating into a statement about Dirichlet combinations, we want to find polynomials R_0, R_1 and R_2 so that

$$\sum_{l=0}^{2} R_l(s_1, w; p^2) D_4(s_1, w; p^{2-l})$$

agrees with the correction coefficients determined in the case $k = 0$.

Again, R_0 must be a constant. Moreover, according to the transformation requirement,

$$R_1(s_1, w; p^2) = R_1(s_1 + w - 1/2, 1 - w; p^2) \mathbb{N}p^{1/2 - w}$$

and

$$R_2(s_1, w; p^2) = R_2(s_1 + w - 1/2, 1 - w; p^2) \mathbb{N}p^{1 - 2w}.$$

But by Lemma 9, since terms of $R_l(s_1, w; p^{2k})$ are known to contain only those powers of $\mathbb{N}p^{-w}$ congruent to 0 mod 3, then there are no Dirichlet polynomials R_1 and R_2 which satisfy the transformations above. All such Dirichlet polynomials transform by a larger power of $\mathbb{N}p^{-w}$. R_0 must then be 1 to satisfy the required properties. So, in summary, $R_0(s_1, w; p^2) = 1$ and $R_1(s_1, w; p^2) = R_2(s_1, w; p^2) = 0$.

Does this provide a definition of correction coefficients which agree with those determined by the case $k = 0$?

To answer this question, we need to refer to earlier propositions in the chapter citing the form of $D_4(s_1, w; p^2)$, particularly with respect to the Euler factor in the correction term corresponding to the distinguished prime p. We also need to recall the combinatorial dictionary between such factors and the corresponding correction coefficients.

According to Proposition 6.5, for example in the case of $p \nmid d_0$, the p^{th} Euler factor of the correction polynomial coming from $D_4(s_1, w; p^2)$ is

$$\begin{array}{ll} \bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2) & \text{if } \alpha_3 = 0. \\ \mathbb{N}p^{3\alpha_3/2-(3\alpha_3-2)s_1-3\alpha_3 w} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha_3/2-1-(3\alpha_3-3)s_1-3\alpha_3 w} & \text{if } \alpha_3 \geq 1. \end{array}$$

Following Theorem 8, this should be equal to terms coming from the coefficient of $\mathbb{N}p^{-s_2}$ coming from $Z_4(s_1, s_2, w)$. For each $\alpha_3 \geq 0$, we have terms associated to $\mathbb{N}p^{-3\alpha_3 w}$ of form

$$\sum_{j=0}^{1} \bar{\chi}_{d_0}(p^{2-2j})\bar{\psi}_1(p^{2-2j}) \sum_i a_{i,j}^{(\alpha_3)}(d_0, p)\mathbb{N}p^{-3\alpha_3/2+(3\alpha_3-i)-(3\alpha_3-i)s_1} =$$

$$= \sum_i \left[\bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2)a_{i,0}^{(\alpha_3)}(d_0, p) + a_{i,1}^{(\alpha_3)}(d_0, p)\right] \mathbb{N}p^{-3\alpha_3/2+(3\alpha_3-i)-(3\alpha_3-i)s_1}$$

Comparing the two sides of this equality, we have by matching coefficients of $\mathbb{N}p^{-s_1}$,

$$\begin{array}{lll} \left[\bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2)a_{0,0}^{(\alpha_3)}(d_0, p) + a_{0,1}^{(\alpha_3)}(d_0, p)\right] & = \bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2) & \text{if } \alpha_3 = 0, \\ \left[\bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2)a_{2,0}^{(\alpha_3)}(d_0, p) + a_{2,1}^{(\alpha_3)}(d_0, p)\right] & = \mathbb{N}p^2 & \text{if } \alpha_3 \geq 1, \\ \left[\bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2)a_{3,0}^{(\alpha_3)}(d_0, p) + a_{3,1}^{(\alpha_3)}(d_0, p)\right] & = -\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^2 & \text{if } \alpha_3 \geq 1, \\ \left[\bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2)a_{i,0}^{(\alpha_3)}(d_0, p) + a_{i,1}^{(\alpha_3)}(d_0, p)\right] & = 0 & \text{in all other cases of } i, \alpha_3. \end{array}$$

The only pair of coefficients determined by the previous case ($k = 0$), is

$$\left[\bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2)a_{0,0}^{(\alpha_3)}(d_0, p) + a_{0,1}^{(\alpha_3)}(d_0, p)\right].$$

But $a_{0,0}^{(\alpha_3)}(d_0, p) = 1$ for all α_3. Hence for $\alpha_3 = 0$, this implies $a_{0,1}^{(0)}(d_0, p) = 0$ and for $\alpha_3 \geq 1$, $a_{0,1}^{(\alpha_3)}(d_0, p) = -\chi_{d_0}(p)\psi_1(p)$. The important point is that this agrees with the determination of $a_{1,0}^{(\alpha_3)}(d_0, p) = -\chi_{d_0}(p)\psi_1(p)$ from the case $k = 0$. Since all other occurring pairs of coefficients contain at least one undetermined coefficient, we need not worry about consistency in these cases.

Thus far, we have seen that the Dirichlet polynomials agree with our original proposed definition of

$$Z_4(s_1, s_2, w) = \sum_{m,n,d} \frac{G(mn^2, d)\bar{\psi}_1(mn^2)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^w}$$

inherited from the square-free heuristic. The next case will show this is not the correct object and that non-trivial Dirichlet polynomials can occur.

Case 2: k=2. We want to determine the coefficient of $\mathbb{N}p^{-2s_2}$ in $Z_4(s_1, s_2, w)$. Again, according to our earlier results, this means we must find Dirichlet polynomials $R_0, \ldots, R_4$, so that

$$\sum_{l=0}^{4} R_l(s_1, w; p^4)D_4(s_1, w; p^{4-l})$$

agrees with the correction coefficients determined in the cases $k = 0$ and $k = 1$. In our analysis of the $k = 1$ case, we determined that R_0 is always constant, and R_1 and R_2 must always be 0. Remember that, according to the required transformation property,

$$R_3(s_1, w; p^4) = R_3(s_1 + w - 1/2, 1 - w; p^4)\mathbb{N}p^{3/2-3w}$$

and

$$R_4(s_1, w; p^4) = R_4(s_1 + w - 1/2, 1 - w; p^4)\mathbb{N}p^{2-4w}.$$

Then no term involving $\mathbb{N}p^{-6w}$ can occur in either polynomial as it immediately implies the transformed series differs by at least $\mathbb{N}p^{3-6w}$. One can check that this restricts R_3 and R_4 to the following.

$$\begin{array}{rcll} R_3(s_1, w; p^4) & = & c_1(1 + \mathbb{N}p^{3/2-3w}) + c_2\mathbb{N}p^{-3s_1-3w} & \text{for any constants } c_1 \text{ and } c_2, \\ R_4(s_1, w; p^4) & = & c_3\mathbb{N}p^{-2s_1-3w} & \text{for any choice of constant } c_3. \end{array}$$

Writing $R_0 = c_0$ for any constant c_0, our task is now to solve for the undetermined constants $c_0, \ldots, c_3$. To find these values, we need to again refer back to the relevant propositions earlier in the chapter, remembering that R_i is associated to the Dirichlet series $D_4(s_1, w; p^{4-i})$.

By Proposition 6.5, in the case $p \nmid d_0$,

$$D_4(s_1, w; p^4) = \\ = \begin{cases} \bar{\chi}_{d_0}(p^4)\bar{\psi}_1(p^4) & \text{if } \alpha_3 = 0, \\ \bar{\chi}_{d_0}(p^4)\bar{\psi}_1(p^4)(\mathbb{N}p^{3/2} - \mathbb{N}p^{3/2-1}) & \text{if } \alpha_3 = 1, \\ \mathbb{N}p^{3\alpha_3/2-(3\alpha_3-4)s_1-3\alpha_3 w} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha_3/2-1-(3\alpha_3-5)s_1-3\alpha_3 w} & \text{if } \alpha_3 \geq 2. \end{cases}$$

$$D_4(s_1, w; p) = \\ = \begin{cases} \bar{\chi}_{d_0}(p)\bar{\psi}_1(p) & \text{if } \alpha_3 = 0, \\ \mathbb{N}p^{3\alpha_3/2-(3\alpha_3-1)s_1-3\alpha_3 w} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha_3/2-1-(3\alpha_3-2)s_1-3\alpha_3 w} & \text{if } \alpha_3 \geq 1. \end{cases}$$

$$D_4(s_1, w; 1) = \\ = \begin{cases} 1 & \text{if } \alpha_3 = 0, \\ \mathbb{N}p^{3\alpha_3/2-(3\alpha_3)s_1-3\alpha_3 w} - \chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{3\alpha_3/2-1-(3\alpha_3-1)s_1-3\alpha_3 w} & \text{if } \alpha_3 \geq 1. \end{cases}$$

Following Theorem 5.3, this should be equal to terms coming from the coefficient of $\mathbb{N}p^{-2s_2}$ coming from $Z_4(s_1, s_2, w)$. For each $\alpha_3 \geq 0$, we have terms associated to $\mathbb{N}p^{-3\alpha_3 w}$ of form

$$\sum_{j=0}^{2} \bar{\chi}_{d_0}(p^{4-2j})\bar{\psi}_1(p^{4-2j}) \sum_i a_{i,j}^{(\alpha_3)}(d_0, p)\mathbb{N}p^{-3\alpha_3/2+(3\alpha_3-i)-(3\alpha_3-i)s_1} = \\ \sum_i \left[\bar{\chi}_{d_0}(p^4)\bar{\psi}_1(p^4)a_{i,0}^{(\alpha_3)}(d_0, p) + \bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2)a_{i,1}^{(\alpha_3)}(d_0, p) + a_{i,2}^{(\alpha_3)}(d_0, p)\right] \\ \mathbb{N}p^{-3\alpha_3/2+(3\alpha_3-i)-(3\alpha_3-i)s_1}$$

Comparing the above terms with the proposed combination of Dirichlet polynomials and Dirichlet series, we have by matching coefficients of 1^{-s_1} at $\alpha_3 = 0$,

$$\bar{\chi}_{d_0}(p^4)\bar{\psi}_1(p^4)a_{0,0}^{(0)}(d_0,p) + \bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2)a_{0,1}^{(0)}(d_0,p) + a_{0,2}^{(0)}(d_0,p) = c_0\bar{\chi}_{d_0}(p^4)\bar{\psi}_1(p^4) + c_1\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)$$

But we have already determined the left-hand side coefficients by the case $k = 0$. From this, we know $a_{0,0}^{(0)} = 1$ and $a_{1,0}^{(0)} = a_{2,0}^{(0)} = 0$ in this case. This reduces the above equality to simply:

$$\bar{\chi}_{d_0}(p^4)\bar{\psi}_1(p^4) = c_0\bar{\chi}_{d_0}(p^4)\bar{\psi}_1(p^4) + c_1\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)$$

This implies $c_0 = 1$ and $c_1 = 0$, since the c_i are constants independent of the choice of d_0. To solve for c_2 (where we now know $R_3(s_1, w; p^4) = c_2\mathbb{N}p^{-3s_1-3w}$) and c_3 (where $R_4(s_1, w; p^4) = c_3\mathbb{N}p^{-2s_1-3w}$), we should examine the terms on both sides which are coefficients of $\mathbb{N}p^{-3s_1-3w}$ and $\mathbb{N}p^{-2s_1-3w}$, respectively.

For the coefficient of $\mathbb{N}p^{-3s_1-3w}$, we have

$$\left[\bar{\chi}_{d_0}(p^4)\bar{\psi}_1(p^4)a_{0,0}^{(1)}(d_0,p) + \bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2)a_{0,1}^{(1)}(d_0,p) + a_{0,2}^{(1)}(d_0,p)\right]\mathbb{N}p^{3/2} = \bar{\chi}_{d_0}(p)\bar{\psi}_1(p)(c_2)$$

as no other combination of Dirichlet series and Dirichlet polynomials produces a term containing $\mathbb{N}p^{-3s_1-3w}$. Again, we have determined all the correction coefficients on the left-hand side from the case $k = 0$. We know $a_{0,0}^{(1)} = 1$, $a_{0,1}^{(1)} = -\chi_{d_0}(p)\psi_1(p) = -\bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2)$, and $a_{0,2}^{(1)} = 0$. Substituting these values, we see that the left-hand side is 0, so $c_2 = 0$ as well.

The last coefficient to check is $\mathbb{N}p^{-2s_1-3w}$, which will determine c_3 and complete the case $k = 2$. Comparing contributions, we have

$$\left[\bar{\chi}_{d_0}(p^4)\bar{\psi}_1(p^4)a_{1,0}^{(1)}(d_0,p) + \bar{\chi}_{d_0}(p^2)\bar{\psi}_1(p^2)a_{1,1}^{(1)}(d_0,p) + a_{1,2}^{(1)}(d_0,p)\right]\mathbb{N}p^{1/2} = c_3$$

as no other combination of Dirichlet series and Dirichlet polynomials produces a term containing $\mathbb{N}p^{-2s_1-3w}$. However, we've determined all of the correction coefficients on the left-hand side by the case $k = 1$. There, we found that $a_{1,0}^{(1)}(d_0,p) = -\chi_{d_0}(p)\psi_1(p)$, $a_{1,1}^{(1)}(d_0,p) = \bar{\chi}_{d_0}(p)\bar{\psi}_1(p)$, and $a_{1,2}^{(1)}(d_0,p) = \mathbb{N}p^2$ (remembering that $a_{i,j} = a_{j,i}$). Substituting into the left-hand side, we are left with $\mathbb{N}p^{5/2} = c_3$. In summary, we have just determined that the coefficient of $\mathbb{N}p^{-2s_2}$ in $Z_4(s_1, s_2, w)$,

$$Z_4(s_1, w; p^2) = D_4(s_1, w; p^4) + \mathbb{N}p^{5/2-2s_1-3w}D_4(s_1, w; p).$$

We leave it to the reader to check that this does indeed agree with all previous determinations of the correction factor, noting that we have already checked many of these conditions.

Chapter 6

Functional Equations and Analytic Continuation

6.1 Overview

We have worked hard to guarantee the existence of Dirichlet polynomials which lead to a definition of the correction polynomial $P(s_1, s_2, d)$ possessing many desirable properties. Now we are ready to reap the rewards. We first show that the form of $Z_4(s_1, s_2, w)$ as a collection of series containing Gauss sums leads to a functional equation into itself. Exploiting the symmetry in the variables s_1 and s_2, we can show a very similar result using the transformation in the s_2 variable. Most importantly, we will also show that the the definition of $Z_4(s_1, s_2, w)$ leads to a definition of $Z_6(s_1, s_2, w)$ which has a functional equation into itself. It is this last functional equation which will provide a continuation of $Z_1(s_1, s_2, w)$ to a half-space. Finally, we show that our original assumptions forced by the interchange equality give an additional functional equation. This provides a continuation to a slightly larger region, and this will be sufficient for our moment information in the final chapter.

6.2 Functional Equations Related to $s_1 \mapsto 1 - s_1$

Recall that $Z_1(s_1, s_2, w)$ contains L-series with arguments s_1 and s_2 in the numerator, and so it should inherit a natural functional equation as each of these variables $s_i \mapsto 1 - s_i$. To further elucidate this property, we defined $Z_1(s_1, s_2, w)$ according to the equality $Z_1(s_1, s_2, w) = Z_4(1 - s_1, s_2, w + s_1 - 1/2)$, where $Z_4(s_1, s_2, w)$ is defined implicitly by its value at each of the prime powered coefficients $\mathbb{N}p^{-ks_2}$ for any prime p and any $k \geq 0$. We use the notation $Z_4(s_1, w; p^k)$ to denote this coefficient. In the previous chapter, we showed that for any fixed prime p and non-negative integer k, there exists a set of Dirichlet polynomials $R_l(s_1, w; 2k)$ such that $R_l(s_1, w; 2k) = \mathbb{N}p^{l/2 - lw} R_l(1 - s_1, w + s_1 - 1/2; 2k)$ and

$$Z_4(s_1, w; p^k) = D_4(s_1, w; p^{2k}) + \sum_{l=1}^{2k} R_l(s_1, w; 2k) D_4(s_1, w; p^{2k-l})$$

so that the resulting $Z_1(s_1, s_2, w)$ has correction coefficients with all the required properties. Here $D_4(s_1, w; p^r)$ denotes the Dirichlet series

$$D_4(s_1, w; p^r) = \sum_{m,d} \frac{G(mp^r, d)\bar{\psi}_1(mp^r)\psi_2(d)}{\mathbb{N}d^w \mathbb{N}m^{s_1}}$$

with $G(m,d)$, the usual normalized cubic Gauss sum of modulus d.

6.2.1 Reformulating Results of S. J. Patterson

According to Patterson (cf. [P]), the inner sum taken over integers d has a functional equation as $w \to 1-w$. We now formulate this precisely, translating from his results to our notation. Let

$$D(w,\mu m) = \zeta_K(3w-1/2) \sum_{d\equiv 1\ (3)} \frac{G(\mu m, d)}{\mathbb{N}d^w} = \zeta_K(3w-1/2) \sum_{d\equiv 1\ (3)} \frac{G(m,d)\psi(d)}{\mathbb{N}d^w}$$

where ψ corresponds to $\mu = \mu(\psi)$ as in our Definition 5 of Chapter 2. Further define

$$D^*(w,\mu m) = (2\pi)^{-2w}\Gamma(w+1/6)\Gamma(w-1/6)D(w,\mu m).$$

Then Patterson shows that

$$\begin{aligned} D^*(w,\mu m) = {} & (1-3^{3/2-3w})(1-3^{3w-5/2})^{-1}3^{7/2-9w}(\mathbb{N}m)^{1/2-w} \\ & \Big[D^*_\infty(1-w,\mu m) + D^*(1-w,\mu m)(e(\mu m)+e(-\mu m)+2\cdot 3^{3/2-3w})\Big] \end{aligned} \tag{6.1}$$

where

$$D^*_\infty(w,\mu m) = \sum_{\substack{2\le b\le 5+ord_{1-\omega}(\mu)\\ a=0,1,2}} 3^{b(1/2-w)}\overline{\left(\frac{\omega^a(1-\omega)^b}{m}\right)}\gamma(a,b,\mu)D^*(w,\omega^a(1-\omega)^b\mu m)$$

and the $\gamma(a,b,\mu)$ are explicit constants given in [23]. Translating back to our set-up, we may write

$$D^*_\infty(w,\mu m) = \sum_{\psi\in\Psi} \phi_1(w,\psi,\mu)\psi(m)D^*(w,\mu(\psi)\mu m)$$

where the $\phi_1(w,\psi,\mu)$ are related to the $\gamma(a,b,\mu)$ via (6.1) in the natural way where the constants $\phi_1(w,\psi,\mu) \ll 1$ for w with $\Re(w)$ bounded. Substituting this into (6.1), we obtain the stream-lined functional equation

$$D^*(w,\mu m) = (1-3^{3/2-3w})(1-3^{3w-5/2})^{-1}(\mathbb{N}m)^{1/2-w} \sum_{\psi\in\Psi} \phi_2(w,\psi,\mu)\psi(m)D^*(1-w,\mu(\psi)\mu m) \tag{6.2}$$

where the constants $\phi_2(w,\psi,\mu)$ are slight variants of the $\phi_1(w,\psi,\mu)$ according to (6.1) and similarly satisfy $\phi_2(w,\psi,\mu) \ll 1$ for w with $\Re(w)$ bounded.

We note that a similar result can be obtained for the Dirichlet series summed over d relatively prime to 6. However this adds additional Euler factors corresponding to the primary prime (-2) which further complicates the notation. Since we will not be concerned with the precise formulation of this functional equation (but rather the one for Z_6 into itself), we will carry out the functional equation for the series summed over all $d \equiv 1\ (3)$ and refer the reader to [11] for a precise formulation of the Euler factors corresponding to additional bad primes.

6.2.2 A Functional Equation for the Prime Power Coefficients

Now we must show that (6.2) implies that each of the series $D_4(s_1, w; p^r)$ has functional equation roughly of the form

$$D_4(s_1, w; p^r) \to D_4(s_1 + w - 1/2, 1 - w; p^r)\mathbb{N}p^{r/2 - rw}.$$

Because these functional equations permute the additional twists ψ_i, we will now introduce them into the notation for D_4 to emphasize the dependence on the ψ_i's. Write

$$D_4(s_1, w; p^r, \psi_1, \psi_2) \stackrel{\text{def}}{=} D_4(s_1, w; p^r) = \sum_{m,d} \frac{G(mp^r, d)\bar{\psi}_1(mp^r)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w}$$

We will further define

$$\begin{aligned} D_4^*(s_1, w; p^r, \psi_1, \psi_2) &\stackrel{\text{def}}{=} (2\pi)^{-2s_1 - w + 1/2}\Gamma(s_1)\Gamma(s_1 + w - 1/2) \\ &\quad \cdot \sum_{m \equiv 1\ (3)} \frac{D^*(w, \mu(\psi_2)m)\bar{\psi}_1(mp^r)}{\mathbb{N}m^{s_1}} \\ &= \Gamma_{D_4}(s_1, w)\zeta_K^*(3w - 1/2) \sum_{m,d} \frac{G(mp^r, d)\bar{\psi}_1(mp^r)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} \\ &= \Gamma_{D_4}(s_1, w)\zeta_K^*(3w - 1/2) D_4(s_1, w; p^r, \psi_1, \psi_2) \qquad (6.3) \end{aligned}$$

with

$$\Gamma_{D_4}(s_1, w) \stackrel{\text{def}}{=} (2\pi)^{-2s_1 - w + 1/2}\Gamma(s_1)\Gamma(s_1 + w - 1/2)(2\pi)^{-2w}\Gamma(w + 1/6)\Gamma(w - 1/6).$$

Moreover, define

$$\begin{aligned} Z_4^*(s_1, w; p^k, \psi_1, \psi_2) &\stackrel{\text{def}}{=} \Gamma_{D_4}(s_1, w)\zeta_K^*(3w - 1/2) Z_4(s_1, w; p^k, \psi_1, \psi_2) \qquad (6.4) \\ &= \sum_{l=0}^{2k} R_l(s_1, w; p^{2k}) D_4^*(s_1, w; p^{2k - l}, \psi_1, \psi_2) \end{aligned}$$

Using this newly defined notation, we want to show the following result.

Proposition 6.1. *Keeping all of the notation as above,*

$$Z_4^*(s_1, w; p^k, \psi_1, \psi_2) = \sum_{\psi \in \Psi} Z_4(s_1 + w - 1/2, 1 - w; p^k, \psi_1\bar{\psi}, \psi_2\psi)\mathbb{N}p^{k - 2kw} \qquad (6.5)$$

Proof: Applying the functional equation (6.2), we have the above (6.3) equal to

$$(2\pi)^{-2s_1 - w + 1/2}\Gamma(s_1)\Gamma(s_1 + w - 1/2)(1 - 3^{3/2 - 3w})(1 - 3^{3w - 5/2})^{-1}$$
$$\sum_{\psi \in \Psi} \phi_2(w, \psi, \mu(\psi_2)) \sum_{m \equiv 1\ (3)} \frac{D^*(1 - w, \mu(\psi_2)\mu(\psi)m)\psi(mp^r)\bar{\psi}_1(mp^r)\mathbb{N}p^{r/2 - rw}\mathbb{N}m^{1/2 - w}}{\mathbb{N}m^{s_1}}$$

So we have at last shown that

$$D_4^*(s_1, w; p^r, \psi_1, \psi_2) = \Gamma_{D_4}(s_1, w)\zeta_K^*(3w - 1/2) D_4(s_1, w; p^r, \psi_1, \psi_2)$$

$$= \sum_{\psi \in \Psi} \phi_2(w, \psi, \mu(\psi_2)) D_4^*(s_1 + w - 1/2, 1 - w; p^r, \psi_1 \bar{\psi}, \psi_2 \psi) \mathbb{N} p^{r/2 - rw}$$

Moreover, we have engineered the $R_l(s_1, w; p^{2k})$ so that

$$R_l(s_1, w; p^{2k}) = R_l(s_1 + w - 1/2, 1 - w; p^{2k}) \mathbb{N} p^{l/2 - lw}$$

and each $R_l(s_1, w; p^{2k})$ is paired with $D_4(s_1, w; p^{2k-l}, \psi_1, \psi_2)$. In total, we have

$$R_l(s_1, w; p^{2k}) D_4^*(s_1, w; p^{2k-l}, \psi_1, \psi_2) = R_l(s_1 + w - 1/2, 1 - w; p^{2k}) \mathbb{N} p^{l/2 - lw}$$
$$\sum_{\psi \in \Psi} \phi_2(w, \psi, \mu(\psi_2)) D_4^*(s_1 + w - 1/2, 1 - w; p^{2k-l}, \psi_1 \bar{\psi}, \psi_2 \psi) \mathbb{N} p^{(2k-l)/2 - (2k-l)w}$$

$$= R_l(s_1 + w - 1/2, 1 - w; p^{2k})$$
$$\sum_{\psi \in \Psi} \phi_2(w, \psi, \mu(\psi_2)) D_4^*(s_1 + w - 1/2, 1 - w; p^{2k-l}, \psi_1 \bar{\psi}, \psi_2 \psi) \mathbb{N} p^{k - 2kw}$$

Since $Z_4(s_1, w; p^k)$ is built from a finite sum of these combinations of Dirichlet polynomials and Dirichlet series, we must have

$$Z_4(s_1, w; p^k, \psi_1, \psi_2) \to \sum_{\psi \in \Psi} \phi_2(w, \psi, \mu(\psi_2)) Z_4(s_1 + w - 1/2, 1 - w; p^k, \psi_1 \bar{\psi}, \psi_2 \psi) \mathbb{N} p^{k - 2kw} \tag{6.6}$$

where the above is an equality upon adding the Gamma factors $\Gamma_{D_4}(s_1, w) \zeta_K^*(3w - 1/2)$ previously defined. □

6.2.3 A Global Functional Equation for $Z_4(s_1, s_2, w)$

To determine a functional equation for the entire series $Z_4(s_1, s_2, w)$, it suffices to show the analogue of (6.5) for any arbitrary integer N,

$$Z_4(s_1, w; N, \psi_1, \psi_2) \to \sum_{\psi \in \Psi} Z_4(s_1 + w - 1/2, 1 - w; N, \psi_1 \bar{\psi}, \psi_2 \psi) \mathbb{N} N^{1 - 2w}.$$

Indeed, if this is true, then regarding $Z_4(s_1, s_2, w)$ as a sum over N of form

$$Z_4(s_1, s_2, w) = \sum_{N \equiv 1 \ (3)} \frac{Z_4(s_1, w; N)}{\mathbb{N} N^{s_2}},$$

the global functional equation

$$Z_4(s_1, s_2, w; \psi_1, \psi_2) \to \sum_{\psi \in \Psi} Z_4(s_1 + w - 1/2, s_2 + 2w - 1, 1 - w; \psi_1 \bar{\psi}, \psi_2 \psi)$$

follows. We record this formally in the following result.

Proposition 6.2. *Define the completed Dirichlet series associated to $Z_4(s_1, s_2, w)$ by*

$$Z_4^*(s_1, s_2, w; \psi_1, \psi_2) \stackrel{def}{=} (2\pi)^{-2s_2 - 2w + 1} \Gamma(s_2) \Gamma(s_2 + 2w - 1) \sum_{N \equiv 1 \ (3)} \frac{Z_4^*(s_1, w; N)}{\mathbb{N} N^{s_2}}.$$

Then there exists a functional equation

$$Z_4^*(s_1, s_2, w; \psi_1, \psi_2) = \sum_{\psi \in \Psi} Z_4^*(s_1 + w - 1/2, s_2 + 2w - 1, 1 - w; \psi_1 \bar{\psi}, \psi_2 \psi). \tag{6.7}$$

Proof: Recall that by transforming $Z_1(s_1, s_2, w)$ under $(s_1, s_2, w) \to (1-s_1, s_2, w+s_1-1/2)$ we obtain $Z_4(s_1, s_2, w) =$

$$\sum_{\substack{d,m,n\in\mathcal{O}_K \\ d,m,n\equiv 1\ (3)}} \frac{\bar{\chi}_{d_0}(mn^2)\bar{\psi}_1(m)\psi_1(n)\psi_2(d)G(\chi_{d_1})G(\bar{\chi}_{d_2})\bar{\psi}_1(d_2)\prod_{\substack{p^\alpha || d_3 \\ p\nmid d_2}}\left[a^{(\alpha)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2}+\cdots\right]}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}\mathbb{N}d^w}\cdot$$

$$\cdot\prod_{\substack{p^\alpha || d_3 \\ p | d_2}}\left[a^{(\alpha)}_{3\alpha+1,0}\mathbb{N}p^{-3\alpha/2-1/2}+a^{(\alpha)}_{3\alpha,0}\mathbb{N}p^{-3\alpha/2+1/2-s_1}+\cdots\right] \quad (6.8)$$

We want to find the coefficient of $\mathbb{N}N^{-s_2}$ (noting that additional terms with exponent s_2 occur in the correction factor), for an arbitrary integer N. Suppose $N = p_1^{k_1}\cdots p_r^{k_r}$. Then the coefficient of $\mathbb{N}N^{-s_2}$ in $Z_4(s_1, s_2, w)$ is

$$\sum_d \frac{\bar{\chi}_{d_0}(m)\bar{\psi}_1(m)\bar{\psi}_1(d_2)G(1,d_1)\overline{G(1,d_2)}}{m^{s_1}d^w}\left[\sum_{l=0}^{k_1}\bar{\chi}_{d_0}(p_1^{2(k_1-l)})\bar{\psi}_1(p^{2(k_1-l)})\right.$$

$$\left.\sum_i a^{(\alpha)}_{i,l}(d_0,p_1)\mathbb{N}p_1^{-3\alpha/2+(3\alpha-i)-(3\alpha-i)s_1}\right]\cdots\left[\sum_{l=0}^{k_r}\bar{\chi}_{d_0}(p_r^{2(k_r-l)})\bar{\psi}_1(p^{2(k_r-l)})\cdot\right.$$

$$\left.\sum_i a^{(\alpha)}_{i,l}(d_0,p_r)\mathbb{N}p_r^{-3\alpha/2+(3\alpha-i)-(3\alpha-i)s_1}\right]\prod_{\substack{q^\alpha || d_3 \\ q\nmid d_2 N}}\left[a^{(\alpha)}_{3\alpha,0}\mathbb{N}q^{-3\alpha/2}+\cdots+a^{(\alpha)}_{0,0}\mathbb{N}q^{3\alpha/2-3\alpha s_1}\right]\cdot$$

$$\prod_{\substack{q^\alpha || d_3 \\ q | d_2 \\ q\nmid N}}\left[a^{(\alpha)}_{3\alpha+1,0}\mathbb{N}q^{-3\alpha/2-1/2}+\cdots+a^{(\alpha)}_{0,0}\mathbb{N}q^{3\alpha/2+1/2-(3\alpha-1)s_1}\right]$$

But we know that each of the bracketed terms corresponding to primes p dividing N satisfy the equality written in the second version of the main theorem of Section 5.1.3. In abbreviated form, it reads

$$\sum_{l=0}^{k}\bar{\chi}_{d_0}(p^{2(k-l)})\bar{\psi}_1(p^{2(k-l)})\sum_i a^{(\alpha)}_{i,l}(d_0,p)\mathbb{N}p^{-3\alpha/2+(3\alpha-i)-(3\alpha-i)s_1} =$$

$$\sum_{l=0}^{2k}R_l(s_1,w;2k)\left[\ p\text{-part of } D_4(s_1,w;p^{2k-l})\right]$$

From this identity, it is clear that the $\mathbb{N}N^{-s_2}$ coefficient of $Z_4(s_1, s_2, w)$ is

$$Z_4(s_1,w;N) = \sum_{l_1=0}^{k_1}\cdots\sum_{l_r=0}^{k_r}R_{l_1}(s_1,w;p_1^{k_1})\cdots R_{l_r}(s_1,w;p_r^{k_r})D_4(s_1,w;p_1^{2k_1-l_1}\cdots p_r^{2k_r-l_r})$$

where $N = p_1^{k_1}\cdots p_r^{k_r}$. We simply take all possible products of Dirichlet polynomials $R_l(s_1, w; p^{2k})$ occurring in the coefficients of the primes p dividing N. Then decomposing the Gauss sum series $D_4(s_1, w; p_1^{2k_1-l_1}\cdots p_r^{2k_r-l_r})$, we have exactly the desired contribution for each prime divisor p^{2k-l}. For those primes q not dividing N, $D_4(s_1, w; p_1^{2k_1-l_1} \cdots p_r^{2k_r-l_r})$ behaves just like $D_4(s_1, w; 1)$ and we see that our product terms associated to these primes q in the above expression are identical to those from the method of taking variables to infinity, which (as shown in Section 3.3.2) come from precisely $D_4(s_1, w; 1)$. Now applying the functional equations of the previous section to Dirichlet series with numerator $D_4(s_1, w; p_1^{2k_1-l_1} \cdots p_r^{2k_r-l_r}, \psi_1, \psi_2)$, it is further clear that, given the transformation property of each of the Dirichlet polynomials R_l,

$$Z_4(s_1, w; N, \psi_1, \psi_2) \to \sum_{\psi \in \Psi} Z_4(s_1 + w - 1/2, 1 - w; N, \psi_1 \bar{\psi}, \psi_2 \psi) \mathbb{N}N^{1-2w}.$$

Moreover, according to our definition in the statement of the proposition,

$$Z_4^*(s_1, s_2, w; \psi_1, \psi_2) \stackrel{\text{def}}{=} (2\pi)^{-2s_2-2w+1} \Gamma(s_2) \Gamma(s_2 + 2w - 1) \sum_{N \equiv 1\ (3)} \frac{Z_4^*(s_1, w; N)}{\mathbb{N}N^{s_2}},$$

the global functional equation

$$Z_4^*(s_1, s_2, w; \psi_1, \psi_2) = \sum_{\psi \in \Psi} Z_4^*(s_1 + w - 1/2, s_2 + 2w - 1, 1 - w; \psi_1 \bar{\psi}, \psi_2 \psi) \tag{6.9}$$

follows. □

6.3 Functional Equations Related to $s_2 \mapsto 1 - s_2$

In the previous section, we showed that using the natural functional equation $s_1 \to 1 - s_1$ for the L-series $L(s_1, \chi_{d_0})$ in $Z_1(s_1, s_2, w)$ leads to a definition of $Z_1(s_1, s_2, w) = Z_4(1 - s_1, s_2, w + s_1 - 1/2)$ where $Z_4(s_1, s_2, w)$, in turn, possesses a functional equation into itself. Now that we have fixed a definition of $Z_1(s_1, s_2, w)$ by choosing Dirichlet polynomials which determine the form of $Z_4(s_1, w; p^k)$, we now explore the consequences for the other natural functional equation $s_2 \to 1 - s_2$ associated to the other L-series occurring in the numerator of $Z_1(s_1, s_2, w)$. Recall that we defined $Z_5(s_1, s_2, w)$ according to

$$Z_5(s_1, s_2, w) \stackrel{\text{def}}{=} Z_1(s_1, 1 - s_2, w + s_2 - 1/2)$$

Because the $a_{i,j}^{(\alpha)}(d_0, p)$ which comprise the correction factor $P(s_1, s_2, d)$ are symmetric in the indices i and j, the original object

$$Z_1(s_1, s_2, w) = \sum_{d \equiv 1\ (3)} \frac{L(s_1, \chi_{d_0} \psi_1) L(s_2, \chi_{d_0} \psi_1) \psi_2(d) P(s_1, s_2, d)}{\mathbb{N}d^w}$$

retains its symmetry in s_1 and s_2. Then we may similarly define the $\mathbb{N}p^{-ks_1}$ coefficient of Z_5 according to

$$Z_5(s_2, w; p^k) = D(s_2, w; p^{2k}) + \sum_{l=1}^{2k} R_l(s_2, w; 2k) D(s_2, w; p^{2k-l})$$

where all of the earlier notation has been preserved in the above equation and the Dirichlet polynomials $R_l(s_2, w; 2k)$ are exactly those polynomials chosen to satisfy

$$Z_4(s_1, w; p^k) = D(s_1, w; p^{2k}) + \sum_{l=1}^{2k} R_l(s_1, w; 2k) D(s_1, w; p^{2k-l}).$$

Then repeating the arguments in the previous sections' Proposition 6.1 and Proposition 6.2 with the roles of s_1 and s_2 interchanged, we may obtain the analogue of (6.7) for Z_5. Again, introducing ψ_i's into the notation to emphasize the dependence of the functional equation on the choice of the ψ_i, we have that

$$Z_5(s_1, s_2, w; \psi_1, \psi_2) \to \sum_{\psi \in \Psi} \phi_2(w, \psi, \mu) Z_5(s_1 + 2w - 1, s_2 + w - 1/2, 1 - w; \psi_1 \bar{\psi}, \psi_2 \psi),$$

where the constants $\phi_2(w, \psi, \mu)$ are slight variants of the $\phi_1(w, \psi, \mu)$ according to (6.1) and similarly satisfy $\phi_2(w, \psi, \mu) \ll 1$ for w with $\Re(w)$ bounded. Again, we can make the above an equality by adding the analogous Gamma factors so that

$$Z_5^*(s_1, s_2, w) \stackrel{\text{def}}{=} \Gamma_{D_4}(s_2, w) \zeta_K^*(3w - 1/2) Z_5(s_1, s_2, w).$$

6.4 Functional Equations Related to $(s_1, s_2) \mapsto (1 - s_1, 1 - s_2)$

Just as in the previous sections, we recall the definition a new object according to the transformation of L-series in the numerator of $Z_1(s_1, s_2, w)$. In Section 3.1 we defined $Z_6(s_1, s_2, w)$ according to

$$Z_1(1 - s_1, 1 - s_2, w + s_1 + s_2 - 1) \stackrel{\text{def}}{=} Z_6(s_1, s_2, w)$$

Recall that from the square-free heuristics of Chapter 1, we find

$$Z_1(s_1, s_2, w) \approx \sum_d \frac{L(s_1, \chi_d) L(s_2, \chi_d)}{\mathbb{N}d^w} = \sum_d \frac{L(1 - s_1, \bar{\chi}_d) L(1 - s_2, \bar{\chi}_d) \mathbb{N}d^{1 - s_1 - s_2} G_3^2(1, d)}{\mathbb{N}d^w}$$

$$\text{(by Hasse-Davenport relation)} = \sum_{m,n,d} \frac{\overline{G_6(m^2 n^2, d)}}{\mathbb{N}m^{1-s_1} \mathbb{N}n^{1-s_2} \mathbb{N}d^{w + s_1 + s_2 - 1}}$$

Then we have

$$Z_6(s_1, s_2, w) \stackrel{\text{def}}{=} Z_1(1 - s_1, 1 - s_2, w + s_1 + s_2 - 1) \approx \sum_{m,n,d} \frac{\overline{G_6(m^2 n^2, d)}}{\mathbb{N}m^{s_1} \mathbb{N}n^{s_2} \mathbb{N}d^w}$$

For fixed m and n, the inner sum of the above Dirichlet series taken over integers d is the Fourier coefficient of a metaplectic Eisenstein series on the six-fold cover of $GL(2)$. Accordingly, it should have a functional equation roughly of the form

$$Z_6(s_1, s_2, w) \to Z_6(s_1 + 2w - 1, s_2 + 2w - 1, 1 - w)$$

Unlike the previous section, in which $Z_5(s_1, s_2, w)$ inherited the anticipated functional equation into itself almost immediately from the definition of $Z_4(s_1, s_2, w)$, we will have to do significantly more work to realize the functional equation of Z_6 into itself. In Section 3.1, we mentioned that defining

$$Z_6(s_1, s_2, w) = \sum_d \frac{\overline{G_6(m^2 n^2, d)}}{\mathbb{N}m^{1-s_1} \mathbb{N}n^{1-s_2} \mathbb{N}d^{w + s_1 + s_2 - 1}} \quad \text{where } d \text{ ranges over all integers}$$

leads to a determination of the correction factors which does not provide an analytic continuation to the whole plane, and worse, is inconsistent with the definition of $Z_4(s_1, s_2, w)$

we have offered above. Based on the definition of $Z_4(s_1, s_2, w)$, we might expect that particular combinations of Dirichlet polynomials and Dirichlet series similar to those used to define Z_4 would provide a definition of $Z_6(s_1, s_2, w)$ which possesses the appropriate additional functional equation. For any such potential combination, we must check that it is consistent with our previous definition of $Z_4(s_1, s_2, w)$. Then we are guaranteed that the correction coefficients are consistently defined and will thus retain the desired list of properties. As before, we reduce this to a question of combinatorics. To formulate these questions, we will need a careful analysis of our original series under the transformation $(s_1, s_2, w) \mapsto (1 - s_1, 1 - s_2, w + s_1 + s_2 - 1)$. First, we make the above assertions more precise.

6.4.1 Formulating the Consistency Claim

We want to show the following claim.

Theorem 6.3. *There exist Dirichlet polynomials $T_l(s_2, w; p^{2k})$ satisfying the transformation property $T_l(s_2, w; p^{2k}) = T_l(s_2+2w-1, 1-w; p^{2k})\mathbb{N}p^{l-2lw}$ such that the coefficient of $\mathbb{N}p^{-ks_1}$ in Z_6, $Z_6(s_2, w; p^k)$, takes form*

$$Z_6(s_2, w; p^k) = \sum_{l=0}^{k} T_l(s_2, w; p^{2k}) D_6(s_2, w; p^{2(k-l)}),$$

where $D_6(s_2, w; p^{2(k-l)})$ is defined by

$$D_6(s_2, w; p^{2k}) = \sum_{n,d \equiv 1\ (3)} \frac{\overline{G_6(p^{2k}n^2, d)}\bar{\psi}_1(p^{2k}n^2)\psi_2(d)}{\mathbb{N}n^{s_2}\mathbb{N}d^w}.$$

It requires some explanation to determine exactly what needs to be shown to prove this theorem. We defined $Z_1(s_1, s_2, w)$ according to the relation

$$Z_1(1 - s_1, s_2, w + s_1 - 1/2) = Z_4(s_1, s_2, w)$$

and then subsequently defined $Z_6(s_1, s_2, w)$ by the relation

$$Z_6(s_1, s_2, w) = Z_1(1 - s_1, 1 - s_2, w + s_1 + s_2 - 1).$$

Then by definition, we have the relation

$$Z_6(s_1, s_2, w) = Z_4(s_1, 1 - s_2, w + s_2 - 1/2).$$

There is ambiguity in the definition of Z_4 and Z_6 according to transformations of Z_1 because the correction factors have not been determined. However, once we assert a form for Z_4, we need to guarantee that Z_6 has a definition which gives the same correction coefficients and still possesses an additional functional equation into itself.

In a previous section of this chapter, we showed that

$$Z_4(s_1, s_2, w) = \sum_{N \equiv 1\ (3)} \left[\sum_{\substack{m,d,c,l \\ c\,l = N^2}} \frac{R_l(s_1, w; N) G(cm, d)\bar{\psi}_1(cm)\psi_2(d)}{\mathbb{N}m^{s_1}\mathbb{N}d^w} \right] \mathbb{N}N^{-s_2}$$

This was equivalent to showing that the coefficient of $\mathbb{N}p^{-k_2 s_2}$ in $Z_4(s_1, s_2, w)$, for any prime p and any positive integer k_2, satisfies

$$Z_4(s_1, w; p^{k_2}) = \sum_{l_2=0}^{2k_2} R_{l_2}(s_1, w; p^{2k_2}) D_4(s_1, w; p^{2k_2 - l_2})$$

Expanding these Dirichlet polynomials into a more explicit form, we may rewrite our series $Z_4(s_1, w; p^{k_2})$ as

$$\sum_{l_2=0}^{2k_2} \sum_{\substack{l_1 \\ V_{l_1}+v_{l_1}-l_1=l_2}} c_{l_1,l_2} \left[\mathbb{N}p^{3v_{l_1}/2-3v_{l_1}w} + \cdots + \mathbb{N}p^{3V_{l_1}/2-3V_{l_1}w}\right] \mathbb{N}p^{-l_1 s_1} D_4(s_1, w; p^{2k_2-l_2})$$

Then, in particular, the coefficient of $\mathbb{N}p^{-k_1 s_1 - k_2 s_2}$ must have p-part equal to the p-part of

$$Z_4(w; p^{k_1}, p^{k_2}) = \sum_{l_2=0}^{2k_2} \sum_{\substack{l_1=0 \\ V_{l_1}+v_{l_1}-l_1=l_2}}^{k_1} c_{l_1,l_2} \left[\mathbb{N}p^{3v_{l_1}/2-3v_{l_1}w} + \cdots + \mathbb{N}p^{3V_{l_1}/2-3V_{l_1}w}\right] \cdot D_4(w; p^{2k_2-l_2+k_1-l_1})$$

These are finite sums so we may reorder the terms according to fixed powers of $\mathbb{N}p^{-k_1 s_1}$ rather than $\mathbb{N}p^{-k_2 s_2}$. Hence, the p-part of the above must be equal to the p-part of $Z_4(s_2, w; p^{k_1}) =$

$$= \sum_{l_1=0}^{k_1} \sum_{\substack{l_2 \\ V_{l_1}+v_{l_1}-l_2=l_1}} c_{l_1,l_2} \left[\mathbb{N}p^{3v_{l_1}/2-3v_{l_1}w} + \cdots + \mathbb{N}p^{3V_{l_1}/2-3V_{l_1}w}\right] \mathbb{N}p^{-l_2 s_2} D_4(s_2, w; p^{k_1-l_1})$$

Note these pieces of the Dirichlet polynomials in s_2 satisfy the correct transformation property to that each of the terms corresponding to $D_4(s_2, w; p^{k_1-l_1})$ transform by $\mathbb{N}p^{-l_1 w}$. This shows that there is a collection of Dirichlet polynomials $S_l(s_2, w; M)$ so that

$$Z_4(s_1, s_2, w) = \sum_{M \equiv 1\ (3)} \left[\sum_{\substack{n,d,c,l \\ c\,l=M}} \frac{S_l(s_2, w; M) G(cn^2, d) \bar{\psi}_1(cn^2) \psi_2(d)}{\mathbb{N}n^{s_2} \mathbb{N}d^{w}} \right] \mathbb{N}M^{-s_1}$$

since each of the prime pieces which determine the correction coefficients match the prime power contributions of $Z_4(s_1, w; p^k)$. This new formulation of $Z_4(s_1, s_2, w)$ behaves well under the transformation $(s_1, s_2, w) \mapsto (s_1, 1 - s_2, w + s_2 - 1/2)$. Now it is clear that we need only show that there is a choice of Dirichlet polynomials $T_l(s_2, w; p^{2k})$ as in the theorem so that

$$\begin{aligned} Z_6(s_2, w; p^k) &\overset{\text{def}}{=} \sum_{l=0}^{k} T_l(s_2, w; p^{2k}) D_6(s_2, w; p^{2(k-l)}) \qquad (6.10) \\ &= \sum_{\substack{l=0 \\ n,d}}^{k} \frac{S_l(1 - s_2, w + s_2 - 1/2; p^k) G(p^{k-l}n^2, d) \bar{\psi}_1(p^{k-l}n^2) \psi_2(d)}{\mathbb{N}n^{1-s_2} \mathbb{N}d^{w+s_2-1/2}} \end{aligned}$$

We can then stitch these sums together, just as we did for $Z_4(s_1, s_2, w)$ in the previous section, to obtain a complete object $Z_6(s_1, s_2, w)$ which possesses the correct functional equation into itself.

6.4.2 Analysis of $D_6(s_2, w; p^{2k})$

In order to prove our claim, we need to analyze both components of the above equation (6.10). We first analyze the left-hand side by interchanging the order of summation in

$$D_6(s_2, w; p^{2k}) = \sum_{n,d \equiv 1\ (3)} \frac{\overline{G_6(p^{2k}n^2, d)}\bar{\psi}_1(p^k n)\psi_2(d)}{\mathbb{N}n^{s_2}\mathbb{N}d^w}.$$

In its current form, we can see the functional equation as $w \mapsto 1-w$ by thinking of this series as an outer sum over n so the inner sum over d looks like the $(p^{2k}n^2)^{th}$ Fourier coefficient of a metaplectic Eisenstein series. But we want to compare this series to Z_4 via Z_1. Interchanging the order of summation so that the outer sum is over d, we will see L-series coming from $D_6(s_2, w; p^{2k})$ and this will provide the link. We have already analyzed the series $D_6(s_2, w; 1)$ in Chapter 3 when taking limits of variables at infinity. As we have seen before, the analysis of $D_6(s_2, w; p^{2k})$ will only differ at the prime p.

Proposition 6.4. *Fix a positive integer k. Then the series $D_6(s_2, w; p^{2k})$ takes on the following forms according to the residue class of $ord_p(d)$ mod 6.*

$$D_6(s_2, w; p^{2k}) = \sum_{\substack{d \\ d=d_0d_3^3}} \frac{\overline{G_6(1,d_1)}G_6(1,d_2)L(s_2, \bar{\chi}_{d_0}\bar{\psi}_1)\bar{\psi}_1(d_2)\psi_2(d_1d_2^2)}{(\mathbb{N}d_1d_2)^w}$$

$$\cdot \prod_{\substack{q \text{ prime} \\ q^{\alpha_q}||d_3 \\ q\neq p}} \left[\begin{matrix}\textit{same product terms as} \\ \textit{those coming from } D_6(s_2,w;1)\end{matrix}\right] \cdot$$

$$\begin{cases} \displaystyle\sum_{\substack{\lambda \\ 2k<6\lambda}} \phi(p^{3\lambda})\mathbb{N}p^{-(6\lambda-2k)s_2/2-6\lambda w)} + \sum_{\substack{\lambda \\ 2k\geq 6\lambda}} \bar{\chi}_{d_0}(p^k)\bar{\psi}_1(p^k)\phi(p^{3\lambda})\mathbb{N}p^{-6\lambda w} & \text{if } ord_p(d) \equiv 0\ (6) \\ \displaystyle\sum_{\substack{\lambda \\ 3\lambda\geq k}} \mathbb{N}p^{3\lambda-(3\lambda-k)s_2-6\lambda w} & \text{if } ord_p(d) \equiv 1\ (6) \\ \displaystyle\sum_{\substack{\lambda \\ 3\lambda+1\geq k}} \bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{3\lambda+1-(3\lambda+1-k)s_2-(6\lambda+3)w}\left[1-\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{-s_2}\right] & \text{if } ord_p(d) \equiv 3\ (6) \\ \displaystyle\sum_{\substack{\lambda \\ 3\lambda+2\geq k}} \mathbb{N}p^{3\lambda+2-(3\lambda+2-k)s_2-(6\lambda+4)w} & \text{if } ord_p(d) \equiv 5\ (6) \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Pick and fix a value of d_0. Suppose $p \nmid d_0$ with $ord_p(d)$ even. Then write $d = p^{6\lambda}d'$ with $(p, d') = 1$ and $n = p^\gamma n'$ with $(p, n') = 1$. So

$$D_6(s_2, w; p^{2k}) = \sum_{n,d\equiv 1\ (3)} \frac{\overline{G_6(p^{2k+2\gamma}(n')^2, p^{6\lambda}d')}\bar{\psi}_1(p^{k+\gamma}n')\psi_2(d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w\mathbb{N}p^{\gamma s_2+6\lambda w}}$$

But

$$\begin{aligned} G_6(p^{2k+2\gamma}(n')^2, p^{6\lambda}d') &= \frac{g_6(p^{2k+2\gamma}(n')^2, p^{6\lambda}d')}{\sqrt{\mathbb{N}p^{6\lambda}\mathbb{N}d'}} = \frac{g_6(p^{2k+2\gamma}(n')^2, d')}{\sqrt{\mathbb{N}d'}}\frac{g_6(p^{2k+2\gamma}(n')^2, p^{6\lambda})}{\sqrt{\mathbb{N}p^{6\lambda}}} \\ &= \bar{\chi}_{d_0}(p^{2k+2\gamma})G_6((n')^2, d')\frac{g_6(p^{2k+2\gamma}, p^{6\lambda})}{\sqrt{\mathbb{N}p^{6\lambda}}} \end{aligned}$$

where χ_{d_0} still denotes the familiar cubic character. Then we may write

$$D_6(s_2,w;p^{2k}) = \sum_{n,d} \frac{\overline{G_6((n')^2,d')}\bar{\psi}_1(n')\psi_2(d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w} \cdot \sum_{\lambda\geq 0}\sum_{\gamma\geq 0} \frac{\bar{\chi}_{d_0}(p^{k+\gamma})\bar{\psi}_1(p^{k+\gamma})\overline{g_6(p^{2k+2\gamma},p^{6\lambda})}}{\sqrt{\mathbb{N}p^{6\lambda}}}\mathbb{N}p^{-\gamma s_2-6\lambda w}$$

We know, moreover, that

$$g_6(p^{2k+2\gamma},p^{6\lambda}) = \begin{cases} \phi(p^{6\lambda}), & \text{if } 2k+2\gamma \geq 6\lambda \\ -\mathbb{N}p^{6\lambda-1}, & \text{if } 2k+2\gamma = 6\lambda-1 \\ 0, & \text{otherwise} \end{cases}$$

where the middle case is evidently empty according to parity. Then we may evaluate the innermost sum according to the following case method:

$$\sum_{\gamma} \frac{\bar{\chi}_{d_0}(p^{k+\gamma})\bar{\psi}_1(p^{k+\gamma})\overline{g_6(p^{2k+2\gamma},p^{6\lambda})}}{\sqrt{\mathbb{N}p^{6\lambda}}}\mathbb{N}p^{-\gamma s_2-6\lambda w} =$$

$$= \begin{cases} \phi(p^{3\lambda})\bar{\chi}_{d_0}(p^k)\bar{\psi}_1(p^k)L^{(p)}(s_2,\bar{\chi}_{d_0}\bar{\psi}_1)\mathbb{N}p^{-6\lambda w}, & \text{if } 2k\geq 6\lambda \\ \phi(p^{3\lambda})L^{(p)}(s_2,\bar{\chi}_{d_0}\bar{\psi}_1)\mathbb{N}p^{-(6\lambda-2k)s_2/2-6\lambda w)}, & \text{if } 2k<6\lambda \end{cases}$$

Bringing all of this analysis together, we have that for $p\nmid d_0$ with $ord_p(d)$ even,

$$D_6(s_2,w;p^{2k}) = \sum_{n,d} \frac{\overline{G_6((n')^2,d')}\bar{\psi}_1(n')\psi_2(d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w} L^{(p)}(s_2,\bar{\chi}_{d_0}\bar{\psi}_1)$$
$$\sum_{\lambda\geq 0}\left[\sum_{\substack{k\\2k<6\lambda}} \phi(p^{3\lambda})\mathbb{N}p^{-(6\lambda-2k)s_2/2-6\lambda w)} + \sum_{\substack{k\\2k\geq 6\lambda}} \bar{\chi}_{d_0}(p^k)\bar{\psi}_1(p^k)\phi(p^{3\lambda})\mathbb{N}p^{-6\lambda w}\right]$$

This completes the first case depending on the divisibility of d_0. Suppose now that $p|d_1$. Write $d=p^{6\lambda+1}d'$ with $(p,d')=1$ and $n=p^\gamma n'$ with $(p,n')=1$. Then

$$D_6(s_2,w;p^{2k}) = \sum_{\substack{n',d'\\(n'd',p)=1\\\lambda,\gamma\geq 0}} \frac{\overline{G_6(p^{2k+2\gamma}(n')^2,p^{6\lambda+1}d')}\bar{\psi}_1(p^{k+\gamma}n')\psi_2(pd')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w\mathbb{N}p^{\gamma s_2+(6\lambda+1)w}}$$

But

$$G_6(p^{2k+2\gamma}(n')^2,p^{6\lambda+1}d') = 0 \quad \text{unless } 2k+2\gamma = 6\lambda$$

If this is the case, then we have

$$\begin{aligned} G_6(p^{2k+2\gamma}(n')^2,p^{6\lambda+1}d') &= \frac{g_6(p^{6\lambda}(n')^2,p^{6\lambda+1}d')}{\sqrt{\mathbb{N}p^{6\lambda+1}\mathbb{N}d'}} \\ &= \frac{\mathbb{N}p^{6\lambda}}{\sqrt{\mathbb{N}p^{6\lambda}}}\frac{g_6((n')^2,pd')}{\sqrt{\mathbb{N}p\mathbb{N}d'}} = \mathbb{N}p^{3\lambda}G_6((n')^2,pd') \end{aligned}$$

Then

$$D_6(s_2,w;p^{2k}) = \sum_{\substack{n',d' \\ (n'd',p)=1}} \frac{\overline{G_6((n')^2,pd')}\bar{\psi}_1(n')\psi_2(pd')}{(\mathbb{N}n')^{s_2}(\mathbb{N}p\mathbb{N}d')^w} \sum_{\substack{\lambda \\ 3\lambda\geq k}} \mathbb{N}p^{3\lambda-(3\lambda-k)s_2-6\lambda w}$$

Suppose instead that $p\nmid d_0$ but that we may write $d=p^{6\lambda+3}d'$ with $(p,d')=1$ and $n=p^\gamma n'$ with $(p,n')=1$. Then

$$D_6(s_2,w;p^{2k}) = \sum_{\substack{n',d' \\ (n'd',p)=1 \\ \lambda,\gamma\geq 0}} \frac{\overline{G_6(p^{2k+2\gamma}(n')^2,p^{6\lambda+3}d')}\bar{\psi}_1(p^{k+\gamma}n')\psi_2(d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w\mathbb{N}p^{\gamma s_2+(6\lambda+3)w}}$$

But

$$G_6(p^{2k+2\gamma}(n')^2,p^{6\lambda+3}d')=0 \quad \text{unless } 2k+2\gamma=6\lambda+2 \text{ (i.e. } \gamma=3\lambda+1-k)$$

If this is the case, then we have

$$\begin{aligned}
G_6(p^{2k+2\gamma}(n')^2,p^{6\lambda+3}d') &= \frac{g_6(p^{6\lambda+2}(n')^2,p^{6\lambda+3}d')}{\sqrt{\mathbb{N}p^{6\lambda+3}\mathbb{N}d'}} = \\
&= \bar{\chi}_{d_0}(p^2)\frac{g_6(p^{6\lambda+2}(n')^2,p^{6\lambda+3})}{\sqrt{\mathbb{N}p^{6\lambda+3}}}\frac{g_6(p^{6\lambda+2}(n')^2,d')}{\sqrt{\mathbb{N}d'}} \\
&= \chi_{d_0}(p)\frac{\mathbb{N}p^{6\lambda+2}}{\sqrt{\mathbb{N}p^{6\lambda+2}}}\frac{g_6((n')^2,d')}{\sqrt{\mathbb{N}d'}}\frac{g_2((n')^2,p)}{\sqrt{\mathbb{N}p}} \\
&= \chi_{d_0}(p)\mathbb{N}p^{3\lambda+1}G_6((n')^2,d')
\end{aligned}$$

Then

$$D_6(s_2,w;p^{2k}) = \sum_{\substack{n',d' \\ (n'd',p)=1}} \frac{\overline{G_6((n')^2,d')}\bar{\psi}_1(pn')\psi_2(d)}{(\mathbb{N}n')^{s_2}(\mathbb{N}p^3\mathbb{N}d')^w} \sum_{\substack{\lambda \\ 3\lambda+1\geq k}} \mathbb{N}p^{3\lambda+1-(3\lambda+1-k)s_2-6\lambda w}$$

Anticipating the removal of an Euler factor of a cubic L-series from the above sum, we may rewrite this as

$$\begin{aligned}
D_6(s_2,w;p^{2k}) = &\sum_{\substack{n',d' \\ (n'd',p)=1}} \frac{\overline{G_6((n')^2,d')}\bar{\psi}_1(n')\psi_2(d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w} L^{(p)}(s_2,\bar{\chi}_{d_0}\bar{\psi}_1) \\
&\sum_{\substack{\lambda \\ 3\lambda+1\geq k}} \bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{3\lambda+1-(3\lambda+1-k)s_2-(6\lambda+3)w}\left[1-\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{-s_2}\right]
\end{aligned}$$

Suppose lastly that $p|d_2$ and that we may write $d=p^{6\lambda+5}d'$ with $(p,d')=1$ and $n=p^\gamma n'$ with $(p,n')=1$. Then

$$D_6(s_2,w;p^{2k}) = \sum_{\substack{n',d' \\ (n'd',p)=1 \\ \lambda,\gamma\geq 0}} \frac{\overline{G_6(p^{2k+2\gamma}(n')^2,p^{6\lambda+5}d')}\bar{\psi}_1(p^{k+\gamma}n')\psi_2(p^2d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w\mathbb{N}p^{\gamma s_2+(6\lambda+5)w}}$$

But

$$G_6(p^{2k+2\gamma}(n')^2, p^{6\lambda+5}d') = 0 \quad \text{unless } 2k+2\gamma = 6\lambda+4 \text{ (i.e. } \gamma = 3\lambda+2-k)$$

If this is the case, then we have

$$\begin{aligned} G_6(p^{2k+2\gamma}(n')^2, p^{6\lambda+5}d') &= \frac{g_6(p^{6\lambda+4}(n')^2, p^{6\lambda+5}d')}{\sqrt{\mathbb{N}p^{6\lambda+5}\mathbb{N}d'}} \\ &= \chi_{d'}^{(6)}(p^5)\bar{\chi}_p^{(6)}(d')\frac{g_6(p^{6\lambda+4}(n')^2, p^{6\lambda+5})}{\sqrt{\mathbb{N}p^{6\lambda+5}}}\frac{g_6(p^{6\lambda+4}(n')^2, d')}{\sqrt{\mathbb{N}d'}} \\ &= \bar{\chi}_{d_0}^{(6)}(p^4)\chi_{d'}^{(6)}(p^5)\bar{\chi}_p^{(6)}(d')\frac{p^{6\lambda+4}}{\sqrt{\mathbb{N}p^{6\lambda+4}}}\frac{g_6((n')^2, d')}{\sqrt{\mathbb{N}d'}}\frac{g_2((n')^2, p)}{\sqrt{\mathbb{N}p}} \\ &= \mathbb{N}p^{3\lambda+2}\,\overline{G_6((n')^2, p)}G_6((n')^2, d') \end{aligned}$$

where the superscripts (6) denote a sixth order character. Then

$$D_6(s_2, w; p^{2k}) = \sum_{\substack{n', d' \\ (n'd', p)=1}} \frac{G_6((n')^2, p)\overline{G_6((n')^2, d')}\bar{\psi}_1(p^2n')\psi_2(p^2d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}p^5\mathbb{N}d')^w} \cdot \sum_{\substack{\lambda \\ 3\lambda+2\geq k}} \mathbb{N}p^{3\lambda+2-(3\lambda+2-k)s_2-6\lambda w}$$

Note that this completes our case analysis, since all other cases involve an integer d with $ord_p(d)$ even and $ord_p(d) \not\equiv 0$ (6), so there is no choice of n so that the Gauss sum $G_6(p^{2k}n^2, d)$ is non-trivial. It is clear that, according to the rules for decomposing Gauss sums, the analysis at all other primes $q \neq p$ yields the same finite collection of terms as $D_6(s_2, w; 1)$. □

6.4.3 Analysis of $D_4(s_2, w; p^{2k})$

Now we want to do the identical procedure on the Gauss sum series coming from the $\mathbb{N}p^{-ks_1}$ coefficient of $Z_4(s_1, s_2, w)$. Rather than state the results of this process at the outset, we launch headlong into the argument and summarize our findings at the end of the section.

Pick and fix a value of d_0. As a first case, suppose that $p \nmid d_0$. Then write $d = p^{3\alpha}d'$ with $(p, d') = 1$ and similarly, write $n = p^\gamma n'$ with $(p, n') = 1$. Then

$$D_4(s_2, w; p^k) = \sum_{n,d} \frac{G(p^kn^2, d)\bar{\psi}_1(p^kn^2)\psi_2(d)}{\mathbb{N}n^{s_2}\mathbb{N}d^w} = \sum_{\substack{n', d' \\ \alpha,\gamma\geq 0}} \frac{G(p^{k+2\gamma}(n')^2, p^{3\alpha}d')\bar{\psi}_1(p^{k+2\gamma}(n')^2)}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w\mathbb{N}p^{\gamma s_2+3\alpha w}} \tag{6.11}$$

But

$$G(p^{k+2\gamma}(n')^2, p^{3\alpha}d') = \frac{g(p^{k+2\gamma}(n')^2, p^{3\alpha}d')}{\sqrt{\mathbb{N}p^{3\alpha}\mathbb{N}d'}} = \frac{g(p^{k+2\gamma}(n')^2, d')}{\sqrt{d'}}\frac{g(p^{k+2\gamma}(n')^2, p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}} =$$

$$= \bar{\chi}_{d_0}(p^{k+2\gamma})\frac{g((n')^2, d')}{\sqrt{\mathbb{N}d'}}\frac{g(p^{k+2\gamma}, p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}} = \frac{\bar{\chi}_{d_0}(p^{k+2\gamma})}{\sqrt{\mathbb{N}p^{3\alpha}}}G((n')^2, d')\frac{g(p^{k+2\gamma}, p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}$$

Then we may rewrite the above (6.11) as

$$D(s_2,w;p^k) = \sum_{\substack{n',d'\\(n'd',p)=1}} \frac{G((n')^2,d')\bar{\psi}_1((n')^2)\psi_2(d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w} \sum_{\alpha\geq 0}\left[\sum_{\gamma\geq 0}\frac{\bar{\chi}_{d_0}(p^{k+2\gamma})\bar{\psi}_1(p^{k+2\gamma})}{\mathbb{N}p^{\gamma s_2+3\alpha w}}\frac{g(p^{k+2\gamma},p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}\right]$$

Since

$$g(p^{k+2\gamma},p^{3\alpha}) = \begin{cases} 1, & \text{if } \alpha=0\\ \phi(p^{3\alpha}), & \text{if } k+2\alpha\geq 3\alpha\\ -\mathbb{N}p^{3\alpha-1}, & \text{if } k+2\gamma=3\alpha-1, \alpha>0\\ 0, & \text{otherwise}\end{cases}$$

this reduces to

$$D(s_2,w;p^k) = \sum_{\substack{n',d'\\(n'd',p)=1}} \frac{G((n')^2,d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w}\left(\sum_{\gamma\geq 0}\frac{\bar{\chi}_{d_0}(p^{k+2\gamma})\bar{\psi}_1(p^{k+2\gamma})}{\mathbb{N}p^{\gamma s_2}}+\right.$$

$$\left.+\sum_{\alpha\geq 1}\left[\sum_{k+2\gamma\geq 3\alpha}\frac{\bar{\chi}_{d_0}(p^{k+2\gamma})\bar{\psi}_1(p^{k+2\gamma})}{\mathbb{N}p^{\gamma s_2+3\alpha w}}\frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}} - \frac{\mathbb{N}p^{3\alpha-1}}{\sqrt{\mathbb{N}p^{3\alpha}}}\frac{\bar{\chi}_{d_0}(p^{(3\alpha-1)})\bar{\psi}_1(p^{(3\alpha-1)})}{\mathbb{N}p^{(3\alpha-1-k)s_2/2+3\alpha w)}}\right]\right) \tag{6.12}$$

where the last term only occurs if $3\alpha-k-1/2$ is a positive integer. This is best expressed as a series of cases depending on the choice of α and k. That is, if $\alpha>0$,

$$\left[\sum_{k+2\gamma\geq 3\alpha}\frac{\bar{\chi}_{d_0}(p^{k+2\gamma})\bar{\psi}_1(p^{k+2\gamma})}{\mathbb{N}p^{\gamma s_2+3\alpha w}}\frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}} - \frac{\mathbb{N}p^{3\alpha-1}}{\sqrt{\mathbb{N}p^{3\alpha}}}\frac{\bar{\chi}_{d_0}(p^{(3\alpha-1)})\bar{\psi}_1(p^{(3\alpha-1)})}{\mathbb{N}p^{(3\alpha-1-k)s_2/2+3\alpha w)}}\right] =$$

$$= \begin{cases} \frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}\bar{\chi}_{d_0}(p^k)\bar{\psi}_1(p^k)L^{(p)}(s_2,\chi_{d_0}\psi_1), & \text{if } k\geq 3\alpha\\ \frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}\mathbb{N}p^{-(3\alpha-k)s_2/2}L^{(p)}(s_2,\chi_{d_0}\psi_1), & \text{if } 3\alpha-k \text{ even}\\ \frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}\bar{\chi}_{d_0}(p)\psi_1(p)\mathbb{N}p^{-(3\alpha+1-k)s_2/2}L^{(p)}(s_2,\chi_{d_0}\psi_1) - & \\ \qquad\frac{\mathbb{N}p^{3\alpha-1}}{\sqrt{\mathbb{N}p^{3\alpha}}}\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{-(3\alpha-1-k)s_2/2}, & \text{if } 3\alpha-k \text{ odd}\end{cases}$$

Note that in the last of the three cases, where $3\alpha-k$ is odd, we may extract the pth Euler factor of the L-series from both terms to obtain

$$L^{(p)}(s_2,\chi_{d_0}\psi_1)\Big[\mathbb{N}p^{3\alpha/2}\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{-(3\alpha+1-k)s_2/2} - \mathbb{N}p^{3\alpha/2-1}\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{-(3\alpha-1-k)s_2/2}\Big].$$

Putting this all back together again we have

$$D(s_2,w;p^k) = \sum_{\substack{n',d'\\(n'd',p)=1}} \frac{G((n')^2,d')\psi_1(n')\psi_2(d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w}L^{(p)}(s_2,\chi_{d_0}\psi_1)\left[\sum_{\alpha\geq 0}\frac{F(s_2;p,k,\alpha)}{\mathbb{N}p^{3\alpha w}}\right]$$

where

$$F(s_2;p,k,\alpha)=\begin{cases}\frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}\bar{\chi}_{d_0}(p^k)\bar{\psi}_1(p^k), & \text{if } k\geq 3\alpha\\ \frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}\mathbb{N}p^{-(3\alpha-k)s_2/2}, & \text{if } 3\alpha-k \text{ even}\\ \mathbb{N}p^{3\alpha/2}\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{-(3\alpha+1-k)s_2/2}- & \\ \qquad \mathbb{N}p^{3\alpha/2-1}\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{-(3\alpha-1-k)s_2/2}, & \text{if } 3\alpha-k \text{ odd}\end{cases}$$

If instead $p|d_1$, then we may write $d=p^{3\alpha+1}d'$ with $(p,d')=1$ and let $n=p^\gamma n'$ with $(p,n')=1$. Then

$$D_4(s_2,w;p^k)=\sum_{n,d}\frac{G(p^kn^2,d)\bar{\psi}_1(p^kn^2)\psi_2(d)}{\mathbb{N}n^{s_2}\mathbb{N}d^w}$$
$$=\sum_{\substack{n',d'\\ \alpha,\gamma\geq 0}}\frac{G(p^{k+2\gamma}(n')^2,p^{3\alpha+1}d')\bar{\psi}_1(p^{k+2\gamma}(n')^2)\psi_2(pd')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w\mathbb{N}p^{\gamma s_2+(3\alpha+1)w}}$$

But

$$G(p^{k+2\gamma}(n')^2,p^{3\alpha+1}d')=0\quad\text{unless } \gamma=(3\alpha-k)/2 \text{ with } \gamma \text{ integral}$$

If we can choose such a γ, then

$$D_4(s_2,w;p^k)=\sum_{\substack{n',d'\\ \alpha\geq 0}}\frac{G(p^{3\alpha}(n')^2,p^{3\alpha+1}d')\psi_1(n')\psi_2(pd')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w\mathbb{N}p^{(3\alpha-k)s_2/2+(3\alpha+1)w}}$$
$$=\sum_{n',d'}\frac{G((n')^2,pd')\psi_1(n')\psi_2(pd')}{(\mathbb{N}n')^{s_2}(\mathbb{N}p\mathbb{N}d')^w}\sum_{\substack{\alpha\\ 3\alpha\geq k}}\mathbb{N}p^{3\alpha/2-(3\alpha-k)s_2/2-3\alpha w}$$

Finally, if $p|d_2$, then we may write $d=p^{3\alpha+2}d'$ with $(p,d')=1$ and again let $n=p^\gamma n'$ with $(p,n')=1$. Then

$$D_4(s_2,w;p^k)=\sum_{n,d}\frac{G(p^kn^2,d)\psi_1(p^kn^2)\psi_2(d)}{\mathbb{N}n^{s_2}\mathbb{N}d^w}=$$
$$\sum_{\substack{n',d'\\ \alpha,\gamma\geq 0}}\frac{G(p^{k+2\gamma}(n')^2,p^{3\alpha+2}d')\bar{\psi}_1(p^{k+2\gamma}(n')^2)\psi_2(p^2d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w\mathbb{N}p^{\gamma s_2+(3\alpha+2)w}}$$

But

$$G(p^{k+2\gamma}(n')^2,p^{3\alpha+2}d')=0\quad\text{unless } \gamma=(3\alpha+1-k)/2 \text{ with } \gamma \text{ integral}$$

If we can choose such a γ, then

$$\begin{aligned}D_4(s_2,w;p^k) &= \sum_{\substack{n',d'\\ \alpha\geq 0}}\frac{G(p^{3\alpha+1}(n')^2,p^{3\alpha+2}d')\bar{\psi}_1(p(n')^2)\psi_2(p^2d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}d')^w\mathbb{N}p^{(3\alpha+1-k)s_2/2+(3\alpha+2)w}}\\ &= \sum_{n',d'}\frac{G((n')^2,pd')\overline{G((n')^2,p)}\bar{\psi}_1(p(n')^2)\psi_2(p^2d')}{(\mathbb{N}n')^{s_2}(\mathbb{N}p^2\mathbb{N}d')^w}\\ &\qquad\cdot\sum_{\substack{\alpha\\ 3\alpha\geq k}}\mathbb{N}p^{(3\alpha+1)/2-(3\alpha+1-k)s_2/2-3\alpha w}\end{aligned}$$

Incorporating our previous decomposition of the remaining series in each case (from the arguments used in taking limits of variables at infinity), we have shown:

Proposition 6.5. *Fix a positive integer k. The series $D_4(s_2, w; p^k)$ takes on the following forms according to the residue class of $ord_p(d)$ mod 3 and the parity of k.*

$$D_4(s_2, w; p^k) = \sum_{\substack{d \\ d=d_0d_3^3}} \frac{G(1, d_1)\overline{G(1.d_2)}L(s_2, \chi_{d_0}\psi_1)\bar{\psi}_1(d_2)\psi_2(d_1d_2)}{(\mathbb{N}d_1d_2)^w} \cdot$$

$$\cdot \prod_{\substack{q \text{ prime} \\ q^{\alpha_q}||d_3 \\ q \neq p}} \left[\begin{array}{c} \textit{same product terms as} \\ \textit{those coming from } D_4(s_2, w; 1) \end{array} \right] \cdot$$

$$\cdot \begin{cases} \sum_{\alpha \geq 0} \frac{F(s_2; p, k, \alpha)}{\mathbb{N}p^{3\alpha w}} & \text{if } ord_p(d) \equiv 0\ (3) \\ \sum_{\substack{\alpha \\ 3\alpha \geq k}} \mathbb{N}p^{3\alpha/2 - (3\alpha - k)s_2/2 - 3\alpha w} & \text{if } ord_p(d) \equiv 1\ (3),\ 3\alpha - k \text{ even} \\ \sum_{\substack{\alpha \\ 3\alpha+1 \geq k}} \mathbb{N}p^{(3\alpha+1)/2 - (3\alpha+1-k)s_2/2 - (3\alpha+1)w} & \text{if } ord_p(d) \equiv 2\ (3),\ 3\alpha - k \text{ odd} \end{cases}$$

where

$$F(s_2; p, k, \alpha) = \begin{cases} \frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}} \bar{\chi}_{d_0}(p^k)\bar{\psi}_1(p^k), & \text{if } k \geq 3\alpha \\ \frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}} \mathbb{N}p^{-(3\alpha-k)s_2/2}, & \text{if } 3\alpha - k \text{ even} \\ \mathbb{N}p^{3\alpha/2}\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{-(3\alpha+1-k)s_2/2} & \\ \qquad - \mathbb{N}p^{3\alpha/2-1}\chi_{d_0}(p)\psi_1(p)\mathbb{N}p^{-(3\alpha-1-k)s_2/2}, & \text{if } 3\alpha - k \text{ odd} \end{cases}$$

Recalling the key equality of Theorem 6.3, we want to ultimately compare series involving $D_6(s_2, w; p^{2r})$ for various powers of r with series involving $D_4(1 - s_2, w + s_2 - 1/2; p^r)$ for various r. To this end, we perform the transformation of variables $(s_2, w) \mapsto (1 - s_2, w + s_2 - 1/2)$ on the form of D_4 listed above.

Proposition 6.6. *For a fixed positive integer k, the series $D_4(1 - s_2, w + s_2 - 1/2; p^k)$ takes on the following forms according to the residue class of $ord_p(d)$ mod 3 and the parity of k. First, if k is even, we have*

$$D_4(1 - s_2, w + s_2 - 1/2; p^k) = \sum_{\substack{d \\ d=d_0d_3^3}} \frac{\overline{G_6(1, d_1)}G_6(1.d_2)L(s_2, \bar{\chi}_{d_0}\bar{\psi}_1)}{(\mathbb{N}d_1d_2)^w}$$

$$\prod_{\substack{q \text{ prime} \\ q^{\alpha_q}||d_3 \\ q \neq p}} \left[\begin{array}{c} \textit{same product terms as} \\ D_4(1 - s_2, w + s_2 - 1/2; 1) \end{array} \right] \zeta^{(p)}(6w + 3s_2 - 3) \cdot$$

$$\cdot \begin{cases} \sum_{\alpha \geq 0} \frac{\tilde{F}(1 - s_2; p, k, \alpha)}{\mathbb{N}p^{3\alpha(w+s_2-1/2)}} & \text{if } ord_p(d) \equiv 0\ (3) \\ \mathbb{N}p^{3j_1 + k/2 - (3j_1 + k/2)s_2 - 6j_1 w} & \text{if } ord_p(d) \equiv 1\ (3),\ j_1 = \lfloor \frac{k}{6} \rfloor \\ \mathbb{N}p^{3j_2 + k/2 - (3j_2 + k/2)s_2 - 6j_2 w} & \text{if } ord_p(d) \equiv 2\ (3),\ j_2 = \lfloor \frac{k-4}{6} \rfloor \end{cases} \tag{6.13}$$

where $\zeta_K^{(p)}(6w+3s_2-3)$ *denotes the* p^{th} *Euler factor of the indicated zeta function and*

$$\sum_{\alpha\geq 0}\frac{\tilde{F}(1-s_2;p,k,\alpha)}{\mathbb{N}p^{3\alpha(w+s_2-1/2)}}=\sum_{\substack{\alpha\\ 3\alpha<k}}\frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N}p^{3\alpha}}}\bar{\chi}_{d_0}(p^k)\bar{\psi}_1(p^k)\left[1-\mathbb{N}p^{3-3s_2-6w}\right]\mathbb{N}p^{3\alpha/2-3\alpha s_2-3\alpha w}+$$
$$\phi(p^{3j_1})\mathbb{N}p^{k/2-(3j_1+k/2)s_2-6j_1w}\left[1-\mathbb{N}p^{2-3s_2-6w}\right]+$$
$$\mathbb{N}p^{3j_2+k/2-(3j_2+k/2)s_2-6j_2w}\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\left[1-\bar{\chi}_{d_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{-s_2}\right] \quad (6.14)$$

with $j_1=\lfloor\frac{k}{6}\rfloor$ *and* $j_2=\lfloor\frac{k-4}{6}\rfloor$*. If instead, k is odd, we have*

$$D_4(1-s_2,w+s_2-1/2;p^k)=\sum_{\substack{d\\ d=d_0d_3^3}}\frac{\overline{G_6(1,d_1)}G_6(1.d_2)L(s_2,\bar{\chi}_{d_0}\bar{\psi}_1)\bar{\psi}_1(d_2)\psi_2(d_1d_2^2)}{(\mathbb{N}d_1d_2)^w}$$
$$\prod_{\substack{q\ prime\\ q^{\alpha_q}||d_3\\ q\neq p}}\begin{bmatrix}\textit{same product terms as}\\ D_4(1-s_2,w+s_2-1/2;1)\end{bmatrix}\zeta_K^{(p)}(6w+3s_2-3)\cdot$$
$$\cdot\begin{cases}\sum_{\alpha\geq 0}\frac{\tilde{F}(1-s_2;p,k,\alpha)}{\mathbb{N}p^{3\alpha(w+s_2-1/2)}} & \textit{if } ord_p(d)\equiv 0\ (3)\\ \mathbb{N}p^{3j_1+k/2-(3j_1+k/2)s_2-6j_1w} & \textit{if } ord_p(d)\equiv 1\ (3),\ j_1=\lfloor\frac{k-3}{6}\rfloor\\ \mathbb{N}p^{3j_2+k/2-(3j_2+k/2)s_2-6j_2w} & \textit{if } ord_p(d)\equiv 2\ (3),\ j_2=\lfloor\frac{k-1}{6}\rfloor\end{cases} \quad (6.15)$$

Proof: We will limit ourselves to the case where k is even, as the odd case follows by identical methods. From the previous proposition, we have that

$$D_4(s_2,w;p^k)=\sum_{\substack{d\\ d=d_0d_3^3}}\frac{G(1,d_1)\overline{G(1,d_2)}L(s_2,\chi_{d_0}\psi_1)\bar{\psi}_1(d_2)\psi_2(d_1d_2^2)}{(\mathbb{N}d_1d_2)^w}\cdot$$
$$\cdot\left(\prod_{\substack{q\text{ prime}\\ q^{\alpha_q}||d_3\\ q\neq p}}\begin{bmatrix}\text{same product terms as}\\ D_4(s_2,w;1)\end{bmatrix}\right)\cdot\begin{bmatrix}\text{terms corresp. to}\\ \text{disting. prime } p\end{bmatrix}$$

Performing the translation on the sum over d_0, we have

$$D_4(1-s_2,w+s_2-1/2;p^k)=\sum_{\substack{d\\ d=d_0d_3^3}}\frac{G(1,d_1)\overline{G(1,d_2)}L(1-s_2,\chi_{d_0}\psi_1)\bar{\psi}_1(d_2)\psi_2(d_1d_2^2)}{(\mathbb{N}d_1d_2)^{w+s_2-1/2}}$$
$$\to\sum_{\substack{d\\ d=d_0d_3^3}}\frac{G(1,d_1)^2\overline{G(1,d_2)^2}L(s_2,\bar{\chi}_{d_0}\bar{\psi}_1)\bar{\psi}_1(d_2)\psi_2(d_1d_2^2)(\mathbb{N}d_1d_2)^{s_2-1/2}}{(\mathbb{N}d_1d_2)^{w+s_2-1/2}}$$

by the functional equation for the cubic L-series. The above can be made exact by introducing the appropriate Gamma factors, but we omit them here to reduce notation. Canceling common factors and using the Hasse-Davenport relation (introduced in Section 2.2), this reduces to

$$\sum_{\substack{d \\ d=d_0 d_3^3}} \frac{\overline{G_6(1,d_1)} G_6(1,d_2) L(s_2, \bar{\chi}_{d_0} \bar{\psi}_1) \bar{\psi}_1(d_2) \psi_2(d_1 d_2^2)}{(\mathbb{N} d_1 d_2)^w}$$

Now on the level of correction factors, we simply transform those corresponding to primes $q \neq p$. For those terms corresponding to the prime p, it is evident from the previous proposition that each case depending on α and k contains a geometric sum from which we can factor the zeta Euler factor $\zeta_K^{(p)}(6w + 3s_2 - 3)$. Doing this in the case for k even leaves:

$$\begin{cases} \sum_{\alpha \geq 0} \frac{\tilde{F}(s_2; p, k, \alpha)}{\mathbb{N} p^{3\alpha w}} & \text{if } ord_p(d) \equiv 0 \ (3) \\ \mathbb{N} p^{6j_1/2 - (6j_1 - k)s_2/2 - 6j_1 w} & \text{if } ord_p(d) \equiv 1 \ (3),\ j_1 = \lfloor \frac{k}{6} \rfloor \\ \mathbb{N} p^{6j_2/2 - (6j_2 - k)s_2/2 - 6j_2 w} & \text{if } ord_p(d) \equiv 2 \ (3),\ j_2 = \lfloor \frac{k-4}{6} \rfloor \end{cases}$$

where

$$\begin{aligned} \sum_{\alpha \geq 0} \frac{\tilde{F}(s_2; p, k, \alpha)}{\mathbb{N} p^{3\alpha w}} &= \sum_{\substack{\alpha \\ 3\alpha < k}} \frac{\phi(p^{3\alpha})}{\sqrt{\mathbb{N} p^{3\alpha}}} \bar{\chi}_{d_0}(p^k) \bar{\psi}_1(p^k) \left[1 - \mathbb{N} p^{3 - 3s_2 - 6w}\right] \mathbb{N} p^{-3\alpha w} + \\ & \phi(p^{3j_1}) \mathbb{N} p^{-(6j_1 - k)s_2/2 - 6j_1 w} \left[1 - \mathbb{N} p^{2 - 3s_2 - 6w}\right] + \\ & \mathbb{N} p^{3j_2 - (6j_2 - k)s_2/2 - 6j_2 w} \bar{\chi}_{d_0}(p) \bar{\psi}_1(p) \left[1 - \bar{\chi}_{d_0}(p) \bar{\psi}_1(p) \mathbb{N} p^{-(1 - s_2)}\right] \end{aligned} \quad (6.16)$$

But the Euler factor has variables fixed by the translation $(s_2, w) \mapsto (1 - s_2, w + s_2 - 1/2)$. So simply substituting the appropriate variable changes in the above case analysis gives the result. □

6.4.4 Proving the Existence Theorem for $Z_6(s_1, s_2, w)$

At last, we have all the ingredients necessary to prove our theorem. We restate it in its equivalent form here for clarity.

Theorem 6.7. *There exist Dirichlet polynomials $T_l(s_2, w; p^{2k})$ satisfying the transformation property $T_l(s_2, w; p^{2k}) = T_l(s_2 + 2w - 1, 1 - w; p^{2k}) \mathbb{N} p^{l - 2lw}$ such that the coefficient of $\mathbb{N} p^{-ks_1}$ in Z_6, $Z_6(s_2, w; p^k)$, takes form*

$$\begin{aligned} Z_6(s_2, w; p^k) &= \sum_{l=0}^{k} T_l(s_2, w; p^{2k}) D_6(s_2, w; p^{2(k-l)}) \\ &= \sum_{\substack{l=0 \\ n,d}}^{k} \frac{S_l(1 - s_2, w + s_2 - 1/2; p^k) G(p^{k-l} n^2, d) \bar{\psi}_1(p^{k-l} n^2) \psi_2(d)}{\mathbb{N} n^{1 - s_2} \mathbb{N} d^{w + s_2 - 1/2}} \end{aligned}$$

Proof: From the previous analysis of both $D_6(s_2, w; p^r)$ and $D_4(1 - s_2, w + s_2 - 1/2; p^r)$ in the preceding sections, we see that the two sides of this equation are identical up to a finite number of terms corresponding to the distinguished prime p. Indeed, comparing the results from Proposition 6.4 and Proposition 6.6, we find that the sums over cube-free integers

d_0 match. Further, comparing the contributions over primes $q \neq p$ in Proposition 6.4 and Proposition 6.6, they correspond to the correction terms $D_6(s_2, w; 1)$ and $D_4(1 - s_2, w + s_2 - 1/2; 1)$ respectively. But translating from $Z_1(s_1, s_2, w)$, we find that

$$D_6(s_2, w; 1) = \frac{P_p(1 - s_1, 1 - s_2, d)}{(\mathbb{N}d_2 d_3^3)^{w+s_1+s_2-1}} = D_4(1 - s_2, w + s_2 - 1/2).$$

These contributions are equal according to the Rankin-Selberg argument of Section 3.4 (remembering that the correction coefficients are symmetric in s_1 and s_2, so the argument applies here as well). Moreover, note that the Euler factor for the zeta function $\zeta(6w+3s_2-3)$ in Proposition 6.6 can similarly be removed from D_6 according to the cases listed in the proposition. This Euler factor is fixed by the transformation $(s_2, w) \mapsto (1-s_2, w+s_2-1/2)$! Now comparing the list of finite terms that remain, the number of independent variables on the left-hand side corresponding to choices of coefficients of the T_l exactly match the number of equations given by the right-hand side according to choices of S_l, and the range of prime powers p^{-s_2} and p^{-w} are the same for D_4 and D_6, so a solution exists. We note that it may not be unique according to degenerate conditions among the equations, but we only need to guarantee its existence here. □

Corollary 6.8. *Let $Z_6(s_2, w; p^k)$, the coefficient of $\mathbb{N}p^{-ks_1}$ in $Z_6(s_1, s_2, w)$, be defined according to the previous theorem. Then*

$$Z_6(s_1, s_2, w) = Z_4(1 - s_1, 1 - s_2, w + s_1 + s_2 - 1)$$

Proof. First note that we may write

$$Z_6(s_1, s_2, w) = \sum_{M \equiv 1\ (3)} \frac{Z_6(s_2, w; M)}{\mathbb{N}M^{s_1}}$$

where

$$Z_6(s_2, w; M) = \sum_{l_1=0}^{k_1} \cdots \sum_{l_r=0}^{k_r} T_{l_1}(s_2, w; p_1^{2k_1}) \cdots T_{l_r}(s_2, w; p_r^{2k_r}) D_6(s_2, w; p_1^{2k_1-2l_1} \cdots p_r^{2k_r-2l_r})$$

for any $M = p_1^{k_1} \cdots p_r^{k_r}$. We simply take all possible products of Dirichlet polynomials $T_l(s_2, w; p^{2k})$ occurring in the coefficients of the primes p dividing N. This follows by an identical argument to Proposition 6.2. Then decomposing the Gauss sum series $D_6(s_2, w; p_1^{2k_1-2l_1} \cdots p_r^{2k_r-2l_r})$, we have exactly the desired contribution for each prime divisor p^{2k-2l}. For those primes q not dividing M, then $D_6(s_2, w; p_1^{2k_1-2l_1} \cdots p_r^{2k_r-2l_r})$ behaves just like $D_6(s_2, w; 1)$ and we see that our product terms associated to these primes q in the above expression are identical to those from the method of taking variables to infinity. We have a similar formulation for $Z_4(1 - s_1, 1 - s_2, w + s_1 + s_2 - 1)$ by combining terms according to the case method of Proposition 6.6. Then decomposing according to primes dividing d, it is clear that the result is a collection of factors which agree by comparing a finite number of distinguished prime powers according to the previous theorem. □

6.4.5 The Additional Functional Equation for $Z_6(s_1, s_2, w)$

In Section 6.2.1, we reformulated results of Patterson which led to an additional functional equation for $Z_4(s_1, s_2, w)$ as a series built from Fourier coefficients of metaplectic Eisenstein series. Here, we again have a series $Z_6(s_1, s_2, w)$ built out of such Fourier coefficients. This Dirichlet series is built out of Fourier coefficients of metaplectic Eisenstein series now on the six-fold cover of $GL(2)$. For this, we can use the work of Kazhdan and Patterson in [17], which generalizes the previously cited [23], to obtain a functional equation roughly of the form

$$Z_6(s_1, s_2, w; \psi_1, \psi_2) \to \sum_{\psi \in \Psi} Z_6(s_1 + 2w - 1, s_2 + 2w - 1, 1 - w; \psi_1 \psi, \bar{\psi}_2 \psi).$$

Define

$$D_6(w, \nu n) \stackrel{\text{def}}{=} \zeta_K(6w - 2) \sum_{\substack{d \equiv 1\ (3) \\ (d,6)=1}} \frac{G(\nu n, d)}{\mathbb{N}d^w}$$

where $\nu = \nu(\psi)$ as in our Definition 2.6. Further define

$$D_6^*(w, \nu n) = (2\pi)^{-5w} \Gamma(w - 1/3)\Gamma(w - 1/6)\Gamma(w)\Gamma(w + 1/6)\Gamma(w + 1/3) D_6(w, \nu m).$$

Then according to [17], Corollary II.2.4, translated into our notation by [11] in Section 2 of their paper, we have a functional equation for $D_6^*(w, \nu n^2)$ as $w \mapsto 1 - w$ as follows.

$$D_6^*(w, \nu n) = (1 - 3^{3-6w})(1 - 4^{3-6w})(1 - 3^{6w-3})^{-1}(1 - 4^{6w-3})^{-1}(\mathbb{N}n)^{1-2w} \sum_{\psi \in \Psi} \phi(w, \psi, \nu)\psi(m) D_6^*(1 - w, \nu(\psi)\nu m^2)$$

for constants $\phi(w, \psi, \nu) \ll 1$ for w with $\Re(w)$ bounded.

Then we may write our two-variable series $D_6(s_2, w; p^{2r}, \psi_1, \psi_2)$ in terms of these series studied by Kazhdan and Patterson.

$$D_6^*(s_2, w; p^{2r}, \psi_1, \psi_2) = (2\pi)^{-2s_2 - 2w + 1} \Gamma(s_2)\Gamma(s_2 + 2w - 1) \sum_{\substack{n \equiv 1\ (3) \\ (n,6)=1}} \frac{D_6^*(w, \nu(\psi_2) n^2 p^{2r}) \psi_1(n^2 p^{2r})}{\mathbb{N}n^{s_2}}$$

so that we may write

$$Z_6^*(s_2, w; p^{2k}, \psi_1, \psi_2) = \sum_{l=0}^{k} T_l(s_2, w; p^{2k} D_6^*(s_2, w; p^{2(k-l)})$$

Now applying the identical methods used in Section 6.2.2 and Section 6.2.3 to our series Z_6 composed of combinations of Dirichlet polynomials and Dirichlet series of the same essential form as D_6 above, we have the following result.

Proposition 6.9. *Let*

$$Z_6^*(s_1, s_2, w; \psi_1, \psi_2) = (2\pi)^{-2s_1 - 2w + 1} \Gamma(s_1)\Gamma(s_1 + 2w - 1) \sum_{n \equiv 1\ (3)} \frac{Z_6^*(s_2, w; m^2, \psi_1, \psi_2)}{\mathbb{N}m^{s_1}}$$

Then $Z_6^*(s_1, s_2, w; \psi_1, \psi_2)$ *satisfies the functional equation*

$$Z_6^*(s_1, s_2, w; \psi_1, \psi_2) = \sum_{\psi \in \Psi} \phi(w, \psi, \nu) Z_6^*(s_1 + 2w - 1, s_2 + 2w - 1, 1 - w; \psi_1 \psi, \bar{\psi}_2 \psi)$$

for constants $\phi(w, \psi, \nu) \ll 1$ *for* w *with* $\Re(w)$ *bounded.*

6.5 Functional Equations Related to $w \mapsto 1 - w$

In the previous sections, we have been considering functional equations for $Z_1(s_1, s_2, w)$ when expressed in the form

$$Z_1(s_1, s_2, w) = \sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\ (3)}} \frac{L(s_1, \chi_{d_0} \psi_1) L(s_2, \chi_{d_0} \psi_1) \psi_2(d) P(s_1, s_2, d)}{\mathbb{N} d^w},$$

We now consider the form of $Z_1(s_1, s_2, w)$ upon interchanging the order of summation:

$$\sum_{\substack{m,n \in \mathcal{O}_K \\ m,n \equiv 1\ (3)}} \frac{L(w, \chi_{\underline{mn}_0} \psi_2) \psi_1(mn) Q(w, m, n)}{\mathbb{N} m^{s_1} \mathbb{N} n^{s_2}}$$

Here, as noted previously, the natural functional equation comes from transforming $w \mapsto 1 - w$ in the L-series in the numerator. Using the functional equation for the L-series, we have

$$Z_1(s_1, s_2, w) = \sum_{m,n \equiv 1\ (3)} \frac{L(1 - w, \overline{\chi}_{\underline{mn}_0} \bar{\psi}_2) G(1, \underline{mn}_1) \overline{G(1, \underline{mn}_2)} \bar{\psi}_2(\underline{mn}_2) \psi_1(mn) Q(w, m, n)}{\mathbb{N} m^{s_1} \mathbb{N} n^{s_2} \mathbb{N} \underline{mn}_1^{1/2 - w} \mathbb{N} \underline{mn}_2^{1/2 - w}} \tag{6.17}$$

so if we define $Z_3(s_1, s_2, w)$ according to the relation

$$Z_3(s_1, s_2, w) = Z_1(s_1 + w - 1/2, s_2 + w - 1/2, 1 - w)$$

then from (6.17)

$$Z_3(s_1, s_2, w) = \sum_{m,n \equiv 1\ (3)} \frac{L(w, \overline{\chi}_{\underline{mn}_0} \bar{\psi}_2) G(1, \underline{mn}_1) \overline{G(1, \underline{mn}_2)} \bar{\psi}_2(\underline{mn}_2) \psi_1(mn) Q(1 - w; m, n)}{\mathbb{N} m^{s_1} \mathbb{N} n^{s_2} (\mathbb{N} \underline{mn}_2 \mathbb{N} \underline{mn}_3^3)^{w - 1/2}}$$

where m, n range over all integers. If we instead use the square-free heuristic (where all integers are both square-free and pairwise relatively prime, and we neglect the characters ψ_i) in an attempt to detect additional functional equations, we may write

$$Z_3(s_1, s_2, w) = \sum_{m,n} \frac{L(w, \overline{\chi}_{mn}) G(1, mn)}{\mathbb{N} m^{s_1} \mathbb{N} n^{s_2}} = \sum_{m,n} \frac{\overline{\chi}_{mn}(d) G(1, mn)}{\mathbb{N} d^w \mathbb{N} m^{s_1} \mathbb{N} n^{s_2}}.$$

Writing $M = mn$ and taking a sum over divisors of M, this becomes

$$\sum_{m,n} \frac{\overline{\chi}_M(d) G(1, M) \sum_{n \mid M} \mathbb{N} n^{s_1 - s_2}}{\mathbb{N} M^{s_1} \mathbb{N} d^w} = \sum_{m,n} \frac{\overline{\chi}_M(d) \tau(M) \sigma_{s_1 - s_2}(M)}{\mathbb{N} M^{s_1} \mathbb{N} d^w}$$

where $\sigma_{s_1-s_2}(M)$ is the usual divisor sum with indicated variable. In more symmetric form, we may write this as

$$Z_3(s_1,s_2,w)=\sum_{m,n}\frac{\overline{\chi}_M(d)\tau(M)\sigma_{s_1-s_2}(M)\mathbb{N}M^{(s_1-s_2)/2}}{\mathbb{N}M^{(s_1+s_2)/2}\mathbb{N}d^w}.$$

This shows that, for square-free integers, the numerator is realized as a Rankin-Selberg convolution of a twisted cubic theta function and a (non-metaplectic) Eisenstein series. This series has an additional functional equation into itself; it reads

$$Z_3(s_1,s_2,w)=Z_3(1-s_1,1-s_2,w+2s_1+2s_2-2).$$

We would like to show this functional equation for the full object $Z_3(s_1,s_2,w)$ summed over all integers (rather than just the square-free integers). Unfortunately, an exact functional equation would require significant additional information about the correction coefficients of $Q(w;m,n)$. We need to guarantee that such coefficients are compatible with a definition of Z_3 in terms of Dirichlet series similar to the square-free object above which possess the desired functional equation. This can be done by methods similar to those used in exhibiting a Z_6 functional equation from the definition of Z_4 done in the previous section. To execute this, however, we would have to follow the definition of Z_4 back to Z_1, then through the interchange of summation, and over to Z_3 using the transformation $w\mapsto 1-w$. This is a difficult bit of combinatorics. Fortunately, we only need an (inexact) asymmetrical functional equation for the purpose of analytic continuation (as we will make clear in the next section).

6.5.1 An Asymmetric Functional Equation

Recall that in the previous section, our careful transformation of $Z_1(s_1,s_2,w)$ under

$$(s_1,s_2,w)\mapsto(s_1+w-1/2,s_2+w-1/2,1-w)$$

resulted in a series Z_3 defined by

$$Z_3(s_1,s_2,w)=\sum_{\substack{m,n\in\mathcal{O}_K\\ m,n\equiv 1\ (3)}}\frac{L(w,\overline{\chi}_{\underline{mn}_0}\bar{\psi}_2)G(1,\underline{mn}_1)\overline{G(1,\underline{mn}_2)}\bar{\psi}_2(\underline{mn}_2)\psi_1(mn)Q(1-w;m,n)}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}(\mathbb{N}\underline{mn}_2\mathbb{N}\underline{mn}_3^3)^{w-1/2}}$$

where m,n range over all integers. Note further that the correction factor $Q(w;m,n)$ took form

$$Q(w;m,n)=\prod_{p^{\beta_3}||\underline{mn}_3}\sum_{l=0}^{\beta_3}b(p^l,m,n)\mathbb{N}p^{-lw}.$$

Hence rewriting this as a sum over divisors R of $\underline{mn}_3$, we have

$$\begin{aligned}\frac{Q(1-w;m,n)}{(\mathbb{N}\underline{mn}_2\mathbb{N}\underline{mn}_3^3)^{w-1/2}} &= \sum_{R|\underline{mn}_3^3\underline{mn}_2}b(R,m,n)\mathbb{N}R^{-1+w}(\mathbb{N}\underline{mn}_2\mathbb{N}\underline{mn}_3^3)^{1/2-w}\\ &= \sum_{R|\underline{mn}_3^3\underline{mn}_2}\frac{b(R,m,n)\mathbb{N}R^{-1/2}}{(\mathbb{N}\underline{mn}_2\mathbb{N}\underline{mn}_3^3/\mathbb{N}R)^{w-1/2}}\end{aligned}$$

Reinserting this into the form of $Z_3(s_1, s_2, w)$, we now have

$$Z_3(s_1,s_2,w) = \sum_{\substack{m,n,R \\ R|\underline{mn}_3^3\underline{mn}_2}} \frac{L(w,\overline{\chi}_{\underline{mn}_0}\bar{\psi}_2)G(1,\underline{mn}_1)\overline{G(1,\underline{mn}_2)}\bar{\psi}_2(\underline{mn}_2)\psi_1(mn)b(R;m,n)\mathbb{N}R^{-1/2}}{\mathbb{N}m^{s_1}\mathbb{N}n^{s_2}(\mathbb{N}\underline{mn}_2\mathbb{N}\underline{mn}_3^3/\mathbb{N}R)^{w-1/2}}$$

If we further simplify with an eye toward the square-free heuristic, we can write $mn = M$ so that $\underline{mn}_1\underline{mn}_2^2\underline{mn}_3^3 = M_1M_2^2M_3^3$. Z_3 can now be written

$$Z_3(s_1,s_2,w) = \sum_{\substack{M,R,d \\ R|M_3^3M_2 \\ mn=M}} \frac{\bar{\chi}_{M_0}(d)\bar{\psi}_2(d)G(1,M_1)\overline{G(1,M_2)}\bar{\psi}_2(\underline{mn}_2)\psi_1(mn)}{\mathbb{N}d^w\mathbb{N}M^{s_1}(\mathbb{N}M_2\mathbb{N}M_3^3/\mathbb{N}R)^{w-1/2}} \sum_{n|M} b(R;M/n,n)\mathbb{N}R^{-1/2}\mathbb{N}n^{s_1-s_2}$$

Anticipating a functional equation in s_1 and s_2, we want to reorder this series as an outer Dirichlet series in w and an inner series in s_1 and s_2 with the desired form. Then set $D = dM_2M_3^3/R$ and sum over all integers D and all M_1 from $M = M_1M_2^2M_3^3$. Separate the inner sum into a sum over $(M,D) = 1$ and a sum over $M|D^\infty$. If $(M_2M_3, D) = 1$, then $R = M_2M_3^3$. But we know from previous investigation that $b(M_2M_3^3, m, n) = 0$ if M_2 is non-trivial (cf. Chapter 1). So $M_2 = 1$ in this case and $R = M_3^3$. Moreover, $b(M_3^3, m, n) = \mathbb{N}M_3^2$ for any choice of m and n, so that $b(M_3^3, m, n)\mathbb{N}R^{-1/2} = \mathbb{N}M_3^{1/2}$. This leaves $Z_3(s_1, s_2, w) =$

$$\sum_{\substack{D \\ D=dM_2M_3^3/R}} \frac{\bar{\psi}_2(d)}{\mathbb{N}D^w} \sum_{\substack{M|D^\infty \\ R|M_3^3M_2 \\ mn=M}} \left[\frac{\bar{\chi}_{M_1}(d)G(1,M_1)\chi_{M_2}(d)\overline{G(1,M_2)}\bar{\psi}_2(M_2)\psi_1(M)}{\mathbb{N}M^{s_1}} \sum_{n|M} b(R;M/n,n)\mathbb{N}R^{-1/2}\mathbb{N}n^{s_1-s_2}\right] \sum_{\substack{(M,D)=1 \\ mn=M}} \frac{\bar{\chi}_{D_0}(M_1)\psi_1(M)G(1,M_1)M_3^{1/2}\sum_{n|M=M_1M_3^3} n^{s_1-s_2}}{(M_1M_3^3)^{s_1}} \tag{6.18}$$

since $(d, M_1) = 1$ so that $\bar{\chi}_{M_1}(d) = \bar{\chi}_d(M_1) = \bar{\chi}_D(M_1) = \bar{\chi}_{D_0}(M_1)$. For the inner sum, using the notation $\sigma_i(M) = \sum_{n|m}\mathbb{N}n^i$, noting that $G(1, M_1)\mathbb{N}M_3^{1/2}$ is precisely the M^{th} Fourier coefficient $\tau_3(M)$ of the cubic theta function, and also symmetrizing the denominator with respect to s_1 and s_2, we have

$$\sum_{\substack{D \\ D=dM_2M_3^3/R}} \frac{\bar{\psi}_2(d)}{\mathbb{N}D^w} \sum_{\substack{M|D^\infty \\ R|M_3^3M_2 \\ mn=M}} \left[\frac{\bar{\chi}_{M_1}(d)G(1,M_1)\chi_{M_2}(d)\overline{G(1,M_2)}\bar{\psi}_2(M_2)\psi_1(M)}{\mathbb{N}M^{s_1}} \sum_{n|M} b(R;M/n,n)\mathbb{N}R^{-1/2}\mathbb{N}n^{s_1-s_2}\right] \sum_{\substack{(M,D)=1 \\ mn=M}} \frac{\overline{\chi}_{D_0}(M)\psi_1(M)\tau_3(M)\sigma_{s_1-s_2}(M)\mathbb{N}M^{\frac{s_2-s_1}{2}}}{\mathbb{N}M^{\frac{s_1+s_2}{2}}} \tag{6.19}$$

The inner sum can now be viewed as an (imprimitive) Rankin-Selberg convolution of a twisted cubic theta function and an ordinary (non-metaplectic) Eisenstein series. (Recall that the ordinary Eisenstein series defined on a congruence subgroup of $GL(2)$ has divisor sums as Fourier coefficients.) Precisely stated, the Eisenstein series is $E(z,(s_1-s_2+1)/2)$. The primitive convolution of this form has functional equation

$$L(\frac{s_1+s_2}{2}, E\times\Theta_3\otimes\bar{\chi}_{D_0}\psi_1)\to L(1-\frac{s_1+s_2}{2}, E\times\Theta_3\otimes\bar{\chi}_{D_0}\psi_1)\mathbb{N}D_0^{2-2s_1-2s_2}$$

It is now more evident that, in order to achieve a perfect functional equation, we would ultimately need to realize the entire Dirichlet series above in terms of the Rankin-Selberg convolution. As mentioned in the previous section, this requires a more detailed knowledge of the correction coefficients $b(R;M/n,n)$ coming from the polynomial $Q(w;m,n)$, which can be obtained with significant effort using an existence argument. Instead, we perform the functional equation on the inner sum (after adding in missing Euler factors) and estimate the size of the rest of the series to obtain convergence estimates. These will be sufficient for our applications.

Adding in the missing Euler factors in the convolution to the inner sum of (6.19), we obtain

$$\sum_{\substack{D\\ D=dM_2M_3^3/R}}\frac{\bar{\psi}_2(d)}{\mathbb{N}D^w}\sum_{\substack{M|D^\infty\\ R|M_3^3M_2\\ mn=M}}\left[\frac{\bar{\chi}_{M_1}(d)G(1,M_1)\chi_{M_2}(d)\overline{G(1,M_2)}\bar{\psi}_2(M_2)\psi_1(M)}{\mathbb{N}M^{s_1}}\right.$$
$$\left.\sum_{n|M}b(R;M/n,n)\mathbb{N}R^{-1/2}\mathbb{N}n^{s_1-s_2}\right]\sum_{\substack{M\\ mn=M}}\frac{\bar{\chi}_{D_0}(M)\psi_1(M)\tau_3(M)\sigma_{s_1-s_2}(M)\mathbb{N}M^{\frac{s_2-s_1}{2}}}{\mathbb{N}M^{\frac{s_1+s_2}{2}}}$$
$$\cdot\prod_{p|D_3}\left[\sum_{\beta\geq 0}\frac{\bar{\chi}_{D_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{\beta/2}\sigma_{s_1-s_2}(p^{3\beta})\mathbb{N}p^{3\beta\frac{s_2-s_1}{2}}}{\mathbb{N}p^{3\beta\frac{s_1+s_2}{2}}}\right]^{-1} \tag{6.20}$$

where the inner sum is now over all integers M, indicating that the convolution is indeed primitive. We leave the series in this form here and note its convergence properties in section 9, where we complete the final step of the continuation.

6.6 Domains of Convergence

In this section, we prepare for the analytic continuation via functional equations by determining domains of convergence for the Dirichlet series discussed previously, but now cleared of all poles. Then we can apply convexity results to appropriately holomorphic functions.

Anticipating the collection of functional equations we will employ to obtain the continuation, we set

$$\Gamma_1(s_1,s_2,w)\stackrel{\text{def}}{=}(2\pi)^{-8s_1-8s_2-8w+6}\Gamma(s_1)\Gamma(s_2)\Gamma(w+s_1+s_2-4/3)\Gamma(w+s_1+s_2-7/6)$$
$$\Gamma(w+s_1+s_2-1)\Gamma(w+s_1+s_2-5/6)\Gamma(w+s_1+s_2-2/3)$$
$$\Gamma(s_1+w-1/2)\Gamma(s_2+w-1/2)\Gamma(w)$$
$$\Lambda(3w+3s_1-2)\Lambda(3w+3s_2-2)\Lambda(3s_1+3s_2+6w-5)\Lambda(6w+6s_1+6s_2-8)$$

where

$$\Lambda(s) = (2\pi)^{-s}\Gamma(s)\zeta_K(s)$$

and further define

$$Z_1^*(s_1, s_2, w) \stackrel{\text{def}}{=} \Gamma_1(s_1, s_2, w)Z_1(s_1, s_2, w)$$

This accounts for all of the normalizing zeta factors used to cancel natural poles in defining our Dirichlet series Z_3, Z_4, Z_5 and Z_6 and the image of their arguments under the involutions we plan to use. We are now ready to proceed with the continuation.

First, our bounds on the size of the correction coefficients of $P(s_1, s_2, d)$ show that the series $\sum_{d_3} \frac{P(s_1,s_2,d)}{\mathbb{N}d_3^{3w}}$ converges for $\Re(w) > 2/3$ so the convergence of $Z_1(s_1, s_2, w)$ is limited by bounds on the L-series rather than the correction factor. Using the usual upper bounds for $L(s, \chi_{d_0}\psi_1)$ obtained by the functional equation $s \mapsto 1 - s$ and Phragmen-Lindelöf convexity in twisted aspect, we have that the function

$$s_1(1 - s_1)s_2(1 - s_2)Z_1 * (s_1, s_2, w)$$

converges absolutely (and uniformly on compact subsets) in the region given by

$$\begin{array}{rrl} \Re(s_1) \geq 1, & \Re(s_2) \geq 1, & \Re(w) > 1; \\ 0 \leq \Re(s_1) < 1, & 0 \leq \Re(s_2) < 1, & \Re(w) > 2 - \Re(s_1)/2 - \Re(s_2)/2; \\ \Re(s_1) < 0, & \Re(s_2) < 0, & \Re(w) > 2 - \Re(s_1) - \Re(s_2); \end{array}$$

This we obtain by applying both functional equations for the L-series in s_1 and s_2. Additional regions of convergence can be obtained by taking the union of the above domain with that obtained from convexity and functional equations in one variable, but this is not the most expeditious path to a complete continuation, so we omit them here.

Interchanging the order of summation for $Z_1(s_1, s_2, w)$, the analogous estimates on the size of the correction coefficients of $Q(w; m, n)$ show that the series $\sum_{\underline{mn}_3} \frac{Q(w;m,n)}{\mathbb{N}m_3^{3s_1}\mathbb{N}n_3^{3s_2}}$ converges for $\Re(s_1) > 2/3$ and $\Re(s_2) > 2/3$. Hence the convergence of $Z_1(s_1, s_2, w)$ is limited by bounds on the L-series in w. Repeating the method above, we find that

$$w(1 - w)Z_1 * (s_1, s_2, w)$$

converges absolutely (and uniformly on compact subsets) in the region given by

$$\begin{array}{rrl} \Re(s_1) > 1, & \Re(s_2) > 1, & \Re(w) \geq 1; \\ \Re(s_1) + \Re(w)/2 > 3/2, & \Re(s_2) + \Re(w)/2 > 3/2, & 0 \leq \Re(w) < 1; \\ \Re(s_1) + \Re(w) > 3/2, & \Re(s_2) + \Re(w) > 3/2, & \Re(w) < 0; \end{array}$$

Combining this information, we have that

$$s_1(1 - s_1)s_2(1 - s_2)w(1 - w)\Gamma_1(s_1, s_2, w)Z_1(s_1, s_2, w)$$

is a holomorphic function absolutely convergent on the union of these two domains. (We obtain the analytic continuation to the union since these regions intersect in the domain given by $\Re(s_1), \Re(s_2), \Re(w) > 1$.)

Similarly, the Dirichlet series $Z_6(s_1, s_2, w)$ contains a 6th order Gauss sum which appears as a Fourier coefficient of an Eisenstein series on the six-fold cover of $GL(2)$. This series was investigated by Patterson [23] and subsequently by Kazhdan-Patterson [17]. Again, by estimates on the size of the correction coefficients, the domain of convergence is restricted by the bound on the series involving Gauss sums. According to Kazhdan-Patterson, these bounds imply that the series $(w - 2/3)(w - 4/3)Z_6(s_1, s_2, w)$ converges absolutely (and uniformly on compact subsets) in the region given by

$$\begin{array}{lll} \Re(s_1) > 1, & \Re(s_2) > 1, & \Re(w) \geq 1; \\ \Re(s_1) + \Re(w) > 2, & \Re(s_2) + \Re(w) > 2, & 0 \leq \Re(w) < 1; \\ \Re(s_1) + 2\Re(w) > 2, & \Re(s_2) + 2\Re(w) > 2, & \Re(w) < 0; \end{array}$$

Before interchanging the order of summation to obtain Gauss sums, we had a realization of $Z_6(s_1, s_2, w)$ as a Dirichlet series containing products of L-series. Then using the functional equation of these L-series again, together with Phragmen-Lindelöf convexity, shows that the function $s_1(s_1 - 1)s_2(s_2 - 1)Z_6(s_1, s_2, w)$ converges absolutely (and uniformly on compact subsets) in the region given by

$$\begin{array}{lll} \Re(s_1) \geq 1, & \Re(s_2) \geq 1, & \Re(w) > 1; \\ 0 \leq \Re(s_1) < 1, & 0 \leq \Re(s_2) < 1, & \Re(w) > 2 - \Re(s_1)/2 - \Re(s_2)/2; \\ \Re(s_1) < 0, & \Re(s_2) < 0, & \Re(w) > 2 - \Re(s_1) - \Re(s_2); \end{array}$$

Again, we can continue the holomorphic function

$$s_1(s_1 - 1)s_2(s_2 - 1)(w - 2/3)(w - 4/3)Z_6(s_1, s_2, w)$$

to the union of these two domains.

6.7 Analytic Continuation to a Half-Plane

Our ultimate goal is a continuation of the function to all of complex three-space. Because of the difficulty of visualizing these domains in three-space (and even as tube domains drawn in terms of their real parts in real three-space), we will perform our arguments in the plane spanned by the lines $s_1 - s_2 = 0, w = 0$ and $s_1 = s_2 = 0$. In this section, we show that the convexity estimates of the previous section, upon mapping them between Z_1 and Z_6 via functional equation, yield a half-plane contained in this plane. Because the functional equations and domains of convergence are symmetric in s_1 and s_2 for Z_1 and Z_6, this poses no real difficulty. Now thinking of the analytic function on the half-plane as a function of two variables, $s_1 = s_2 = s$ and w, any extension of this region has a convex hull equal to the entire space. By a convexity theorem of several complex variables (see Hormander), this guarantees an analytic continuation of our function to the whole plane. Once this is achieved, any additional region of continuation into either half-space cut by the plane will similarly give a convex hull in three-space which is the entire half-space. Showing this for each half-space gives a continuation to the entire plane.

Here we take a first step toward this by continuing the function $Z_1(s, w)$, defined by $Z_1(s_1, s_2, w)$ setting $s_1 = s_2$, to a half plane. Then by the convexity estimates in the previous section, we have that

$$s^2(1 - s)^2 w(1 - w)Z_1(s, w)$$

converges absolutely in the region given by

$$\begin{aligned} \Re(s)+\Re(w) &> 3/2, & \Re(w) &< 0; \\ \Re(s)+\Re(w)/2 &> 3/2, & 0 \le \Re(w) &< 1; \\ \Re(s) &\ge 1, & \Re(w) &> 1; \\ 0 \le \Re(s) &< 1, & \Re(w) &> 2-\Re(s); \\ \Re(s_1) &< 0, & \Re(w) &> 2-2\Re(s); \end{aligned}$$

As indicated in the figure below, the convex hull of this region is given according to the following boundary lines:

$$\{(s,w) \mid \Re(w) > \max(2-2\Re(s), 2-4/3\Re(s), 3/2-\Re(s))\}$$

Similarly define $Z_6(s,w)$ by setting $s_1 = s_2$ in $Z_6(s_1,s_2,w)$. From our earlier convexity estimates on $Z_6(s_1,s_2,w)$, we have that

$$s^2(s-1)^2(w-2/3)(w-4/3)Z_6(s,w)$$

converges absolutely in the region given by

$$\begin{aligned} \Re(s)+2\Re(w) &> 2, & \Re(w) &< 0; \\ \Re(s)+\Re(w) &> 2, & 0 \le \Re(w) &< 1; \\ \Re(s) &\ge 1, & \Re(w) &> 1; \\ 0 \le \Re(s) &< 1, & \Re(w) &> 2-\Re(s); \\ \Re(s) &< 0, & \Re(w) &> 2-2\Re(s); \end{aligned}$$

Since all of our functional equations are linear combinations, we can determine the result of applying $A : Z_1 \to Z_6$ by determining the image under A of any two collinear points on the piece-wise linear boundary. Here $A : (s,w) \mapsto (1-s, w+2s-1)$. Then

$$\begin{aligned} (-1,4) &\to (2,1), & (0,2) &\to (1,1), \\ (3/2,0) &\to (-1/2,2), & (5/2,-1) &\to (-3/2,3) \end{aligned}$$

Now we take the resulting points and the boundary defined by the lines between them and map them under $B : Z_6 \to Z_6$ where B is given by $B : (s,w) \mapsto (s+2w-1, 1-w)$. This gives

$$\begin{aligned} (2,1) &\to (3,0), & (1,1) &\to (2,0), \\ (-1/2,2) &\to (5/2,-1), & (-3/2,3) &\to (7/2,-2) \end{aligned}$$

Because the function Z_1 converges on the above region, the function $Z_6(s,w)$ converges on the union of the original region from convexity and the two transformed regions under the functional equation. The convex hull of this region is the set of points describing a half-plane:

$$\{(s,w) \mid \Re(w) > 3/2-\Re(s)\}$$

Because this line is fixed by the transformation A, we have the identical half-plane of absolute convergence for the function $Z(s,w)$.

6.8 Completing the Continuation

Recall that after manipulating $Z_1(s_1+w-1/2, s_2+w-1/2, 1-w) \stackrel{\text{def}}{=} Z_3(s_1,s_2,w)$ we ended up with the equation (6.19). We recopy this below but now under the assumption that $s_1 = s_2 = s$ (that is, we restrict to the plane $s_1 = s_2$ according to our strategy for continuing Z_1).

$$\sum_{\substack{D \\ D=dM_2M_3^3/R}} \frac{\bar{\psi}_2(d)}{\mathbb{N}D^w} \sum_{\substack{M|D^\infty \\ R|M_3^3M_2 \\ mn=M}} \left[\frac{\bar{\chi}_{M_1}(d)G(1,M_1)\chi_{M_2}(d)\overline{G(1,M_2)}\bar{\psi}_2(M_2)\psi_1(M)}{\mathbb{N}M^{s_1}} \right.$$
$$\left. \sum_{n|M} b(R;M/n,n)\mathbb{N}R^{-1/2}\mathbb{N}n^{s_1-s_2} \right] L(s, E\times\Theta_3\otimes\bar{\chi}_{D_0}\psi_1)$$
$$\cdot \prod_{p|D_3}\left[\sum_{\beta\geq 0} \frac{\bar{\chi}_{D_0}(p)\bar{\psi}_1(p)\mathbb{N}p^{\beta/2}\sigma_{s_1-s_2}(p^{3\beta})\mathbb{N}p^{3\beta\frac{s_2-s_1}{2}}}{\mathbb{N}p^{3\beta\frac{s_1+s_2}{2}}}\right]^{-1} \tag{6.21}$$

We want to find the domain of convergence for this series upon taking absolute values. Since the divisor function $\sigma_0(N)$ has growth bounded by $\mathbb{N}N^\epsilon$ for any $\epsilon > 0$, then the geometric sums in the product converge for $\Re(s) > 1/3$. Taking absolute values on the middle sum reduces this to

$$\sum_{\substack{M|D^\infty \\ R|M_3^3M_2 \\ mn=M}} \sum_{n|M} \frac{b(R;M/n,n)\mathbb{N}R^{-1/2}}{\mathbb{N}M^s}$$

Using our convergence assumption on the coefficients $b(R;M/n,n)$ first stated at the end of Chapter 4 and proven via the existence argument in Chapter 5, we have that the coefficients $b(R;M/n,n)\mathbb{N}R^{-1/2} \ll \mathbb{N}M^2(\mathbb{N}M^3)^{-1/2} = \mathbb{N}M^{1/2}$, so this sum will converge for values of s such that $\Re(s) > 1/2$.

Finally, the L-series $L(s, E\times\Theta_3\otimes\bar{\chi}_{D_0}\psi_1)$ converges absolutely for $\Re(s) > 7/6$ according to a trivial estimate on the size of the Fourier coefficient of the cubic theta function. Recall this L-series satisfied the functional equation

$$L(s, E\times\Theta_3\otimes\bar{\chi}_{D_0}\psi_1) \to L(1-s, E\times\Theta_3\otimes\bar{\chi}_{D_0}\psi_1)\mathbb{N}D_0^{2-4s}$$

Then according to the Phragmen-Lindelöf convexity bound, we have a continuation in s such that the growth in $\mathbb{N}D_0$ in the vertical strip $-1/6 < \Re(s) < 7/6$ is bounded by $\mathbb{N}D_0^{7/3-2s}$. This implies that the entire multiple Dirichlet series $Z_3(s_1,s_2,w)$ converges absolutely in the region given by

$$\Re(s) > 7/6, \Re(w) > 1; \quad 1/2 < \Re(s) \leq 7/6, \Re(w) + 2\Re(s) > 10/3.$$

From the functional equations between Z_1 and Z_6, we were able to obtain a region of convergence in the plane $s_1 = s_2 = s$ which extended to the half-plane $\Re(w)+\Re(s) > 3/2$. Now using the functional equation $Z_1(s+w-1/2, 1-w) = Z_3(s,w)$ according to the definition of Z_3, convergence for Z_1 in the half-plane $\Re(w)+\Re(s) > 3/2$ implies convergence for Z_3 in the half-plane $\Re(s) > 1$. This overlaps the region given above, so now taking the convex hull of their union, we have an analytic continuation for $Z_3(s_1, s_2, w)$ up to the half-plane $\Re(s) > 1/2$. Applying the functional equation one more time from $Z_3(s, w)$ to $Z_1(s, w)$ gives a continuation of $Z_1(s, w)$ up to the half-plane $\Re(w) + \Re(s) > 1$.

Now using the fact that $Z_1(s_1, s_2, w)$, as a function of three-variables, converges absolutely for $\Re(s_1) \geq 1, \Re(s_2) \geq 1, \Re(w) > 1$, the two-variable estimates immediately give a continuation to the convex hull of the region bounded by the half-spaces $\Re(w) + \Re(s_1) > 1, \Re(s_2) > 1$ and $\Re(w) + \Re(s_2) > 1, \Re(s_1) > 1$.

6.9 The Determination of Polar Planes

Because Z_1, using either order of summation, contains L-series in the numerator with arguments s_1 and s_2 or w, then it has poles whenever any of these arguments take the value 0 or 1. Now we need to apply all of the functional equations of Z_1 into itself used to obtain the continuation. If we reflect these polar planes at $s_i = 0$, $s_i = 1$, $w = 0$ and $w = 1$ according to these transformations, then we will determine all the poles associated to the poles coming from L-series in the numerator of $Z_1(s_1, s_2, w)$. Recall that we presented the diagram earlier:

$$\begin{array}{rcll} Z_1(s_1, s_2, w) & \xleftrightarrow{(1-s_1,1-s_2,w+s_1+s_2-1)} & Z_6(s_1, s_2, w) \circlearrowleft & (s_1+2w-1,s_2+2w-1,1-w) \\ \text{interchange} \Big\| & & & \\ Z_1(s_1, s_2, w) & \xleftrightarrow{(s_1+w-1/2,s_2+w-1/2,1-w)} & Z_3(s_1, s_2, w) \circlearrowright & (1-s_1,1-s_2,w+2s_1+2s_2-2) \end{array}$$

But Z_6 contains a sixth order Gauss sum in the numerator, so its cube-free part is essentially the Dirichlet series associated to an Eisenstein series on the 6-fold cover of $GL(2)$. That is, the numerator roughly took the form $\overline{G(m^2n^2, d)}$. Thus according to the Selberg theory, as a sum over integers d, it has a functional equation as $w \to 1 - w$ and poles at $w = 1/2 + 1/6$ (cf. [13]). Similarly, Z_3 contains a cubic Gauss sum in the numerator and its cube-free part is essentially the Mellin transform of a Rankin-Selberg convolution of a twisted cubic theta function and a non-metaplectic Eisenstein series $E(z, (s_1 - s_2 + 1)/2)$. As mentioned previously, this object has a $GL(4)$ functional equation as $s_i \to 1 - s_i$ so that $w \to w + 2s_1 + 2s_2 - 2$ and poles at $2s_i - 1/2 = 1/2 + 1/3$ according to the usual Fourier analysis which transforms the argument of the Eisenstein series. In total, our functional equations used to obtain the continuation are given by the transformations:

$$\begin{aligned} A &: Z_1(s_1, s_2, w) \to Z_6(1 - s_1, 1 - s_2, w + s_1 + s_2 - 1), \\ B &: Z_6(s_1, s_2, w) \to Z_6(s_1 + 2w - 1, s_2 + 2w - 1, 1 - w), \\ C &: Z_1(s_1, s_2, w) \to Z_3(s_1 + w - 1/2, s_2 + w - 1/2, 1 - w), \\ D &: Z_3(s_1, s_2, w) \to Z_3(1 - s_1, 1 - s_2, w + 2s_1 + 2s_2 - 2). \end{aligned}$$

Each of these transformations is an involution. This implies that the total collection of functional equations of Z_1 into itself is described by the set of transformations $\{ABA, ABACDC, ABACDCABA, \ldots\}$ and $\{CDC, CDCABA, CDCABACDC, \ldots\}$. This produces the following list of functional equations from Z_1 into Z_1.

$$\begin{aligned} (s_1, s_2, w) &\longrightarrow (s_1 + 2s_2 + 2w - 2, 2s_1 + s_2 + 2w - 2, -2s_1 - 2s_2 - 3w + 4) \\ (s_1, s_2, w) &\longrightarrow (1 - s_1, 1 - s_2, 1 - w) \\ (s_1, s_2, w) &\longrightarrow (3 - s_1 - 2s_2 - 2w, 3 - 2s_1 - s_2 - 2w, -3 + 2s_1 + 2s_2 + 3w) \end{aligned}$$

Applying this set of functional equations to the polar planes $s_i = 0, 1$ and $w = 0, 1$ (the poles associated to Z_1 in its original form with L-series in the numerator) produces the first twelve poles in our list in theorem 1. That is, we obtain the polar planes

$$\begin{array}{llll} s_1 = 1, & s_1 = 0, & s_1 + 2s_2 + 2w - 3 = 0, & s_1 + 2s_2 + 2w - 2 = 0, \\ s_2 = 1, & s_2 = 0, & 2s_1 + s_2 + 2w - 3 = 0, & 2s_1 + s_2 + 2w - 2 = 0, \\ w = 1, & w = 0, & 2s_1 + 2s_2 + 3w - 3 = 0, & 2s_1 + 2s_2 + 3w - 4 = 0, \end{array}$$

We can similarly generate all of the functional equations of Z_6. They are given by the sets of transformations $\{B, BACDCA, \ldots\}$ and $\{ACDCA, ACDCAB, \ldots\}$. This yields transformations which all take $w \to 1 - w$. Z_6 now has both the original polar plane $w = 2/3$ and $1 - w = 2/3$. These polar planes translate to polar planes of Z_1 via the involution A taking $w \to w + s_1 + s_2 - 1$, producing the pair of planes:

$$w + s_1 + s_2 - 5/3 = 0, \quad w + s_1 + s_2 - 4/3 = 0.$$

Lastly, Z_3 functional equations into itself are given by the sets $\{C, CDABAD, \ldots\}$ and $\{DABAD, DABADC, \ldots\}$ and all take $s_i \to 1 - s_i$. Then the polar planes are given by $s_i = 2/3$ and $1 - s_i = 2/3$ and translated to Z_1 by the involution C taking $s_i \to 1 - s_i$. This produces the final four planes in our original list in Theorem 1. That is, we obtain the polar planes:

$$\begin{array}{ll} w + s_1 - 7/6 = 0, & w + s_1 - 5/6 = 0, \\ w + s_2 - 7/6 = 0, & w + s_2 - 5/6 = 0. \end{array}$$

Because we did not explicitly require the use of the functional equations involving Z_4 and Z_5, the natural polar planes coming from these objects must, in fact, not appear as poles of the object $Z_1(s_1, s_2, w)$. Letting $\Lambda(s) = (2\pi)^{-s}\Gamma(s)\zeta_K(s)$, we obtain the following theorem:

Theorem 6.10. *Let $K = \mathbb{Q}(\sqrt{-3})$ with ring of integers $\mathcal{O}_K$. Given an integer $d \in \mathcal{O}_K$, write $d = d_1 d_2^2 d_3^3$ with d_1 and d_2 cube-free. Let $\chi_{d_0} = \chi_{d_1}\bar{\chi}_{d_2}$ denote the product of cubic residue characters with conductor $d_1 d_2$. Let ψ_1 and ψ_2 be primitive cubic characters of a fixed conductor $N|9$. Define the function*

$$Z_1(s_1, s_2, w; \psi_1, \psi_2) = \sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1\,(3)}} \frac{L(s_1, \chi_{d_0}\psi_1) L(s_2, \chi_{d_0}\psi_1) \psi_2(d) P(s_1, s_2; d, \psi_1)}{\mathbb{N}d^w}$$

where $P(s_1, s_2; d, \psi_1)$ is a certain finite, Eulerian Dirichlet polynomial in two variables s_1 and s_2 depending only on the indicated quantities. Then defining

$$Z_1^*(s_1, s_2, w; \psi_1, \psi_2) \stackrel{def}{=} \Gamma_1(s_1, s_2, w) Z_1(s_1, s_2, w; \psi_1, \psi_2)$$

where

$$\begin{aligned} \Gamma_1(s_1, s_2, w) \stackrel{def}{=} (2\pi)^{-8s_1 - 8s_2 - 8w + 6} \Gamma(s_1)\Gamma(s_2)\Gamma(w + s_1 + s_2 - 4/3)\Gamma(w + s_1 + s_2 - 7/6) \\ \Gamma(w + s_1 + s_2 - 1)\Gamma(w + s_1 + s_2 - 5/6)\Gamma(w + s_1 + s_2 - 2/3) \\ \Gamma(s_1 + w - 1/2)\Gamma(s_2 + w - 1/2)\Gamma(w) \\ \Lambda(3w + 3s_1 - 2)\Lambda(3w + 3s_2 - 2)\Lambda(3s_1 + 3s_2 + 6w - 5)\Lambda(6w + 6s_1 + 6s_2 - 8), \end{aligned}$$

$Z_1^*(s_1, s_2, w; \psi_1, \psi_2)$ has a meromorphic continuation to a region of $\mathbb{C}^3$ containing the point $(1/2, 1/2, 1/2)$.

Moreover, the function $Z_1(s_1, s_2, w; 1, 1)$ is analytic in this region except for the following 18 polar planes:

$$\begin{array}{rrrr} s_1 = 1, & s_1 = 0, & s_1 + 2s_2 + 2w - 3 = 0, & s_1 + 2s_2 + 2w - 2 = 0, \\ s_2 = 1, & s_2 = 0, & 2s_1 + s_2 + 2w - 3 = 0, & 2s_1 + s_2 + 2w - 2 = 0, \\ w = 1, & w = 0, & 2s_1 + 2s_2 + 3w - 3 = 0, & 2s_1 + 2s_2 + 3w - 4 = 0, \end{array}$$

$$\begin{array}{rr} w + s_1 + s_2 - 5/3 = 0, & w + s_1 + s_2 - 4/3 = 0, \\ w + s_1 - 7/6 = 0, & w + s_1 - 5/6 = 0, \\ w + s_2 - 7/6 = 0, & w + s_2 - 5/6 = 0. \end{array}$$

Chapter 7

Mean-Value Estimates for L-series

To finish, we offer one application of the resulting continuation. By specializing to the line $(1/2, 1/2, w)$, we can count poles of $Z(1/2, 1/2, w)$ according to values of w. Then inserting this function of one variable into an integral transform, we get mean-value estimates using contour integration.

Proposition 7.1. *Let $\sigma > 0$ be a positive real number. Then*

$$F(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(w)x^w dw = e^{-1/x}$$

where $\Gamma(w)$ denotes the usual Gamma function.

Proof: The result follows from a simple exercise in contour integration, showing that the Gamma function and $e^{-1/x}$ are inverse Mellin transforms of each other. Given any value for x, then moving the line of integration to the left to the line $\Re(w) = -R$ and taking the limit, the integral along the line $\Re(w) = -R$ goes to 0 according to the damping of the Gamma function and we pick up poles at each of the negative integers from the Gamma function and the residue there is given by

$$x^{-k}\mathrm{Res}_{z=-k}\Gamma(z) = \frac{(-1)^k x^{-k}}{k!}.$$

Then summing over all such negative integers k, we obtain $e^{-1/x}$ and the result follows. $\square$

It follows that

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Z_1(1/2, 1/2, w)\Gamma(w)x^w dw = \sum_{d\equiv 1\ (3)} L(1/2, \chi_{d_0})^2 P(1/2, 1/2, d)\, e^{-\mathbb{N}d/x}$$

since

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Z_1(1/2, 1/2, w)\Gamma(w)x^w dw =$$

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \sum_{d\equiv 1\ (3)} L(1/2, \chi_{d_0})^2 P(1/2, 1/2, d)\Gamma(w)\left(\frac{x}{\mathbb{N}d}\right)^w dw$$

and $|Z_1(1/2, 1/2, w)| \ll |t|^k$ for some k. (The latter assertion can be seen by a Phragmen-Lindelöf convexity argument similar to the one used in the previous chapter together with arguments given in Proposition 4.11 of [7].)

Now moving the line of integration to $\Re(w) = 1/2 + \epsilon$, we need to account for any poles with $1/2 < \Re(w) < 2$. Recall that we determined that $Z_1(s_1, s_2, w)$ is analytic for these points $(1/2, 1/2, w)$ except for points occurring on the polar planes. They satisfied the equations:

$$\begin{array}{rrrr} s_1 = 1, & s_1 = 0, & s_1 + 2s_2 + 2w - 3 = 0, & s_1 + 2s_2 + 2w - 2 = 0, \\ s_2 = 1, & s_2 = 0, & 2s_1 + s_2 + 2w - 3 = 0, & 2s_1 + s_2 + 2w - 2 = 0, \\ w = 1, & w = 0, & 2s_1 + 2s_2 + 3w - 3 = 0, & 2s_1 + 2s_2 + 3w - 4 = 0, \end{array}$$

$$\begin{array}{rr} w + s_1 + s_2 - 5/3 = 0, & w + s_1 + s_2 - 4/3 = 0, \\ w + s_1 - 7/6 = 0, & w + s_1 - 5/6 = 0, \\ w + s_2 - 7/6 = 0, & w + s_2 - 5/6 = 0. \end{array}$$

We now want to find the planes containing points with $s_1 = s_2 = 1/2$. The planes $s_1 = 0, s_1 = 1, s_2 = 0$, and $s_2 = 1$ are the only planes which do not include such a point $(1/2, 1/2, w)$. This leaves 14 remaining planes and one can check that they have w values

$$\{1, 3/4 \text{ (2 times)}, 2/3 \text{ (4 times)}, 1/3 \text{ (4 times)}, 1/4 \text{ (2 times)}, 0\}$$

Values of w which occur more than once indicate the pole may be non-simple at $(1/2, 1/2, w)$.

Estimating the integral on the line $\Re(w) = 1/2 + \epsilon$, we find that

$$\frac{1}{2\pi i} \int_{1/2+\epsilon-i\infty}^{1/2+\epsilon+i\infty} Z_1(1/2, 1/2, w)\Gamma(w)x^w dw \ll x^{1/2+\epsilon},$$

since $Z_1(1/2, 1/2, w)$ has polynomial growth in t along this line which is damped by the Gamma function so that our growth estimates depend solely on the size of x.

The poles contribute $c_1 x + c_2 x^{3/4} \log x + c_3 x^{3/4} + c_4 x^{2/3} F(\log x)$ where c_1, c_2, c_3, and c_4 are explicitly computable constants according to the residue of the associated poles and F is an explicit monic polynomial of degree 3 with real coefficients depending on the precise form of the correction factor. Hence we have shown the following:

Theorem 7.2. *According to the contour integration methods above,*

$$\sum_{d \equiv 1 \ (3)} L(1/2, \chi_{d_0})^2 P(1/2, 1/2, d) e^{-\mathbb{N}d/X} = c_1 X + c_2 X^{3/4} + c_3 X^{2/3} F(\log X) + O(X^{1/2+\epsilon})$$

where c_1, c_2, and c_3 are explicit constants with c_1, c_2 non-zero and $F(X)$ is a polynomial with $deg(F) \leq 3$.

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