18.781 In-Class Discussion, September 5, 2007

```
parisize = 4000000, primelimit = 500000
(13:42) gp > P(n)=c=1;for(k=2,n/2,if(isprime(2*k-1), c++));c
(13:44) gp > P(5)
%1 = 2
(13:44) gp > P(50)
%2 = 15
(13:44) gp > P(500)
%3 = 95
(13:45) gp > P(5000)
%4 = 669
(13:45) gp > P(5*10^4)
%5 = 5133
(13:45) gp > P(5*10^5)
%6 = 41538
(13:45) gp > P(5*10^6)
%7 = 348513
(13:47) gp > ##
  ***   last result computed in 1mn, 41,340 ms.
(13:47) gp > P(5*10^7)
%8 = 3001134
(14:06) gp > ##
  ***   last result computed in 18mn, 4,480 ms.
```

In the calculation above, note that PARI knows a table of primes up to 500,000 ("primelimit") so calculations with such primes are ultrafast. But for larger n, we are forced to define our own function, like $P(n)$.

We can slightly modify the program, defining a function $L(n)$ to count the number of primes of the form $5k + 2$ which are less than n.

```
 GP/PARI CALCULATOR Version 2.1.6 (released)
                i686 running linux (ix86 kernel) 32-bit version
                (readline v4.3 enabled, extended help available)

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comes WITHOUT ANY WARRANTY WHATSOEVER.

Type ? for help, \q to quit.
Type ?12 for how to get moral (and possibly technical) support.

   realprecision = 28 significant digits
   seriesprecision = 16 significant terms
   format = g0.28

parisize = 4000000, primelimit = 500000
? L(n)=c=0;for(k=0,(n-1)/5,if(isprime(5*k+2),c++));c
? L(5)
%1 = 1
? L(50)
%2 = 5
? L(500)
%3 = 25
? L(5000)
%4 = 170
? L(5*10^4)
%5 = 1289
? L(5*10^5)
%6 = 10404
? L(5*10^6)
%7 = 87179
? ##
  ***   last result computed in 4,864 ms.
? L(5*10^7)
%8 = 750395
? ##
  ***   last result computed in 50,500 ms.
```

This suggests that the number of primes of form $5k + 2$ for integer k may, in fact, be infinite.

But comparing our data from the first two PARI sessions, we can make a much more precise conjecture.

The following table suggests that, as one might guess, 1/4 of all primes have form $5k + 2$.

n	P(n)	L(n)	P(n) / L(n)	Li(n)
50	15	5	3	18.4687
500	95	25	3.8	101.794
5000	669	170	3.93529	684.281
5*10^4	5133	1289	3.98215	5166.55
5*10^5	41538	10404	3.99250	41606.3
5*10^6	348513	87179	3.99767	348638
5*10^7	3001134	750395	3.99940	3001560

I've also included the value of the integral

$$Li(x) = \int_2^x \frac{dt}{\log\ t}$$

used in the Prime Number Theorem.