KEY IDENTITY : $\frac{1}{q-1} \sum_{\substack{\omega \\ \omega^{q-1}=1}} \omega^{-\nu(a)} \log(L_\omega(s)) = \sum_{\substack{p \\ p^m \equiv a \pmod q}} \sum_{m=1}^{\infty} m^{-1} p^{-ms}$ for $\mathrm{Re}(s) > 1$.

(start with Euler product $L_\omega(s) = \prod_p (1 - \omega^{\nu(p)} p^{-s})^{-1}$

take logs, use fact that $\sum_\omega \omega^{-\nu(a)+\nu(n)} = \begin{cases} q-1 & \text{if } n \equiv a \ (q) \\ 0 & \text{else} \end{cases}$)

(STEP II) Note $L_1(s) = (1 - q^{-s})\,\zeta(s)$

For $\omega \neq 1$, $L_\omega(s)$ conv. for $s > 0$ (as Dirichlet series $\sum_{\substack{n \not\equiv 0 \\ (\mathrm{mod}\ q)}} \frac{\omega^{\nu(n)}}{n^s}$)

(unif. conv. for $s \geq \delta > 0$, so $L_\omega(s)$ continuous on this region.

so to show $\log(L_\omega(s))$ bounded below, suffices to show $L_\omega(1) \neq 0$.)
(not "$= -\infty$")

(STEP III) cases: if ω : complex, pf by contradiction. (Know $\prod_\omega L_\omega(s) \geq 1$)

if $L_\omega(1) = 0$, $L_{\bar\omega}(1) = 0$, growth rates contradict product identity

(STEP IV) : $\omega = -1$. (Legendre symbol)

Use the property of Gauss sums: $g(n,q) = \left(\frac{n}{q}\right) g(1,q)$

$$\Rightarrow L(1) = \frac{1}{g(1,q)} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \sum_{n=1}^{\infty} \frac{1}{n} e^{2\pi i mn/q}$$

$$= \frac{-1}{g(1,q)} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \left[\log\left(2 \sin \frac{\pi m}{q}\right) + i\left(\frac{\pi m}{q} - \frac{\pi}{2}\right)\right]$$

complex analytic version of log power series.

(STEP V) Use Gauss' eval. of Gauss sum to break into cases $q \equiv 1, 3 \ (4)$.

$$g(1,q) = \begin{cases} q^{1/2} & q \equiv 1 \ (4) \\ i q^{1/2} & q \equiv 3 \ (4) \end{cases}$$

$q \equiv 3 \ (4)$: $\displaystyle L(1) = -\frac{\pi}{q^{3/2}} \sum_{m=1}^{q-1} m \cdot \left(\frac{m}{q}\right)$ showed $\displaystyle \sum_{m=1}^{q-1} m \cdot \left(\frac{m}{q}\right)$ odd.

(no simple pf. that it's negative)

$q \equiv 1 \ (4)$: $\displaystyle -\frac{1}{q^{1/2}} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \cdot \log\left(2 \sin \frac{\pi m}{q}\right)$

Breaking into res./non-res. : $\displaystyle \log \underbrace{\left[\frac{\prod_{N:\text{Non-R}} \sin(\pi N/q)}{\prod_{R:\text{Res.}} \sin(\pi R/q)} \right]}_{Q} \cdot q^{-1/2}$

Must show $Q \neq 1$.

use another result of Gauss on roots of unity. :

$$\prod_{R:\text{Res}} \left(x - e^{2\pi i R/q}\right) = \tfrac{1}{2}\left[Y(x) - q^{1/2} Z(x) \right]$$

$$\prod_{N:\text{Non-res}} \left(x - e^{2\pi i N/q}\right) = \tfrac{1}{2}\left[Y(x) + q^{1/2} Z(x) \right]$$

then $\displaystyle Q = \frac{Y(1) + q^{1/2} Z(1)}{Y(1) - q^{1/2} Z(1)}$ with $Z(1) \neq 0$. ✓.

Word about the evaluation of Gauss sum:

Proved before that $g(1,q)^2 = (-1)^{q-1/2} \cdot q = \left(\frac{-1}{q}\right) \cdot q$.

Elementary pfs. of evaluation : Ch. 6.4. Ireland/Rosen has pf. using rts. of unity.

Better pf: use Poisson summation formula.

$$g(1,q) = \sum_{R:\text{Res.}} e^{2\pi i R/q} - \sum_{N:\text{Non-res.}} e^{2\pi i N/q} = 1 + 2\cdot \sum_{R:\text{Res.}} e^{2\pi i R/q}$$

$$\left(\text{since } \sum_{m=1}^{q-1} e^{2\pi i m/q} = -1\right).$$

i.e. $$g(1,q) = \sum_{x=0}^{q-1} e^{2\pi i x^2/q}$$

(since x^2 hits QR's twice as $x \in [1, q-1]$.)

What Dirichlet proves (w/ general modulus N instead of q):

$$S(N) = \sum_{x=0}^{N-1} e^{2\pi i x^2/N}. \qquad S(N) = \left(\frac{1+i^{-N}}{1+i^{-1}}\right)\cdot N^{1/2}$$

$$= N^{1/2}\cdot \begin{cases} (1+i) & N \equiv 0 \ (4) \\ 1 & N \equiv 1 \ (4) \\ 0 & N \equiv 2 \ (4) \\ i & N \equiv 3 \ (4) \end{cases}$$

Poisson summation formula: for sufficiently nice f,

$$\sum_{n=A}^{B}{}' f(n) = \sum_{m=-\infty}^{\infty} \int_A^B f(x)\, e^{2\pi i m x}\, dx$$

looks like Fourier coeff.

$'$: replace $f(A), f(B)$ by $\frac{1}{2} f(A), \frac{1}{2} f(B)$.

idea: let $A=0$, $B=1$. Define $g(x) = f(x)$ for $0 \le x < 1$

but $g(x)$ periodic, w/ period 1.

(so has discont. at integers)

Then $g(x)$ has a Fourier series

$$\frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos 2\pi\nu x + b_\nu \sin 2\pi \nu x)$$

with $\frac{1}{2} a_\nu = \int_0^1 f(x) \cos 2\pi\nu x \, dx$, $\frac{1}{2} b_\nu = \int_0^1 f(x) \sin 2\pi\nu x\, dx$

Basic thm. of Fourier analysis: $g(x)$ = above series for all x with $g(x)$ continuous, $= \dfrac{g_-(a) + g_+(a)}{2}$ at discontinuity $x = a$.

Taking $x = 0$ in Fourier series:

$$\frac{1}{2}\left[f(0) + f(1) \right] = \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} a_\nu = \sum_{\nu=-\infty}^{\infty} \int_0^1 f(x) \cos 2\pi \nu x \, dx$$

(i.e. case $A=0$, $B=1$ in Poisson summation formula)

to prove in general, replace $f(x)$ by $f(x+n)$

for $n = A, A+1, \dots, B-1$ and add results.

To use Poisson summation for Gauss sum evaluation, pick

$f_1(x) = \cos 2\pi x^2/N$, $f_2(x) = \sin 2\pi x^2/N$. Do Poisson summation. combine results.

$$S(N) = \sum_{x=0}^{N-1} e^{2\pi i x^2/N} = \sum_{x=0}^{N-1} \cos 2\pi x^2/N + i \sin 2\pi x^2/N$$

$$= \sum_{\nu=-\infty}^{\infty} \int_0^N e^{2\pi i \nu x} \cdot e^{2\pi i x^2/N} \, dx$$

Poisson summation

$$= N \cdot \sum_{-\infty}^{\infty} \int_0^1 e^{2\pi i N (x^2 - \nu x)} \, dx$$

change of vars: $x + \frac{1}{2}\nu = y$

$$= N \sum_{-\infty}^{\infty} \underbrace{e^{-\frac{1}{2}\pi i N \nu^2}} \int_{1/2\nu}^{1+1/2\nu} e^{2\pi i N y^2} \, dy.$$

analyze according to parity of ν.