18.781, Fall 2007 Problem Set 9
Due: WEDNESDAY, November 14

This set of problems concludes our work on reciprocity laws and algebraic number theory.

1. Show that
$$p^{-1}\sum_{t=0}^{p-1}\zeta_p^{t(x-y)} = \delta(x,y)$$
where
$$\delta(x,y) = \begin{cases} 1 & x \equiv y \ (p) \\ 0 & x \not\equiv y \ (p) \end{cases}$$

2. Let χ be a character on $\mathbb{Z}/p\mathbb{Z}$. (That is, a multiplicative homomorphism from $(\mathbb{Z}/p\mathbb{Z})^\times$ to the non-zero complex numbers.) Define, as in class,
$$g(a,\chi) = \sum_{t \ (\mathrm{mod}\ p)} \chi(t)\zeta_p^{at}.$$
Prove that if $a \neq 0$ and $\chi \neq 1_p$, the trivial character mod p, that
$$g(a,\chi) = \chi(a^{-1})g(1,\chi)$$

3. The Jacobi sum (used in the cubic case for our proof of reciprocity) is defined by
$$J(\chi,\lambda) = \sum_{\substack{a,b\ (p) \\ a+b=1}} \chi(a)\lambda(b)$$
where χ and λ are characters of $\mathbb{Z}/p\mathbb{Z}$. Prove the following:
   (a) $J(1_p, 1_p) = p$ (again 1_p is the trivial character which is always 1 mod p. By convention, we take $1_p(0) = 1$ for this problem, though that is not always the standard convention.)
   (b) $J(1_p, \chi) = 0$
   (c) $J(\chi, \chi^{-1}) = -\chi(-1)$
   (d) If $\chi \cdot \lambda \neq 1_p$, then
$$J(\chi,\lambda) = \frac{g(\chi)g(\lambda)}{g(\chi\lambda)}$$

4. Let $p \equiv 1\ (3)$, and set $p = \pi\bar{\pi}$, with π a primary prime in R. Show that

$$x^3 \equiv a\ (p)$$

is solvable in the integers if and only if

$$\left(\frac{a}{\pi}\right) = 1.$$

5. The congruence

$$x^3 \equiv 2 + 3\omega\ (11)$$

is difficult to solve in R since there are 121 residue classes mod 11. By cubic reciprocity

$$\left(\frac{2+3\omega}{11}\right) = \left(\frac{11}{2+3\omega}\right).$$

Use this fact, together with your knowledge of the congruence

$$x^3 \equiv 11\ (7)$$

to prove that the initial congruence mod 11 in R is not solvable.

6. Prove $g(1,p) = \sum_{t \pmod p} \zeta_p^{t^2}$ by evaluating the sum

$$\sum_{t \pmod p} [1 + \left(\frac{t}{p}\right)]\zeta_p^t$$

(Recall that ζ_p denotes a pth root of unity $e^{2\pi i/p}$ and the Gauss sum

$$g(a,p) = \sum_{t \pmod p} \left(\frac{t}{p}\right) \zeta_p^{at}$$

7. Let f be a function from $\mathbb{Z} \to \mathbb{C}$, the complex numbers. Suppose that for p prime, $f(n+p) = f(n)$ for all integers n. Define

$$\hat{f}(a) = p^{-1} \sum_{t \pmod p} f(t)\zeta_p^{-at}.$$

Prove that

$$f(t) = \sum_{a \pmod p} \hat{f}(a)\zeta_p^{at}$$

(Note the similarity with Fourier analysis for periodic functions on $\mathbb{R}$.)