Instructions and Sample Questions for Midterm I
18.103, Spring 2007

The purpose of this document is give you some idea of the format and character of the coming midterm exam (held in class, Wednesday, March 21).

As mentioned in class, due to time constraints, there will only be 4-5 questions on the exam. TWO of these will be to write down proofs of some of the major results needed in the development of measure and integration theory so far. A list of ten of the most important theorems (from which these 2 questions will be somewhat randomly drawn) is as follows:

1. The Weak and Strong Laws of Large Numbers
2. Outer measure is a measure on $\mathcal{M}_F$, the completion of a ring with respect to the distance function on sets.
3. Given a set A expressible as the countable union of sets in $\mathcal{M}_F$, the outer measure of a set A, $\mu^*(A)$, is finite if and only if $A \in \mathcal{M}_F$
4. The Borel-Cantelli Lemmas
5. Simple functions approximate measurable functions (A-G Thm. 6, Sect. 2.2)
6. Countable additivity of Lebesgue integrals of (non-neg.) measurable functions
7. The Monotone Convergence Theorem
8. Fatou's Lemma (assuming the Monotone Convergence Theorem)
9. Lebesgue Dominated Convergence Theorem (assuming Fatou's Lemma)
10. Lebesgue integration generalizes Riemann integration for bounded functions. (A-G Thm. 1, Sect. 2.4)

Then the additional 2 or 3 questions will be something like the following examples:

1. Recall that an event is "plausible" if its occurrence can be described by a measurable set in a given measure theoretic model. Given a Bernoulli sequence B, let S_B be the infinite series

$$S_B = \sum_{n=1}^{\infty} \frac{(-1)^{a_n}}{2^n}$$

where a_n is the nth digit in the binary expansion of B. Show that for any $\epsilon > 0$, the event $|S_B| < \epsilon$ is plausible using the Lebesgue measure on $[0, 1]$. Then compute its probability.

2. (a) Let f be a bounded function on E, $\mu(E) < \infty$. Prove that f is measurable if and only if

$$\inf_{f \leq \psi} \int_E \psi \, d\mu = \sup_{f \geq \phi} \int_E \phi \, d\mu$$

for all simple functions ϕ and ψ.

(b) Use this fact to give a very short proof that if f is a bounded, Riemann-integrable function on $[a, b]$ then f is Lebesgue integrable, with the values of the Riemann integral and Lebesgue integral of f on $[a, b]$ equal.

For additional practice problems, a good start would be the unassigned questions from Adams and Guillemin. Many other standard textbooks on measure theory and integration cover very similar material to our book and provide an additional source of good problems. For that matter, going back over assigned homework questions is never a bad idea either.