18.103 MIDTERM II

Wednesday, May 2, 2007

Name: ______________________________

Numeric Student ID: ____________________

Instructor's Name: _____________________

I agree to abide by the terms of the honor code:

Signature: ___________________________

Instructions: Print your name, student ID number and instructor's name in the space provided. During the test you may not use notes, books or calculators. Read each question carefully and **show all your work**; full credit cannot be obtained without sufficient justification for your answer unless explicitly stated otherwise. Underline your final answer to each question. There are 5 questions. You have 55 minutes to do all the problems.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5 (Bonus)		5
Total		40

1. State and carefully prove the "Strong Law of Large Numbers" expressed in terms of random variables and expectation values.

   Solution:

   This is theorem 1 of Section 2.7 in your book. See page 111 for a solution. Many people forgot to assert that the random variables f_i are bounded, in the assumptions of the theorem. This is not, in general, necessary, but the proof in the book (which everyone used) requires it to conclude that the integrals, e.g.

$$\int f_i^2 \, d\mu$$

   are indeed finite.

2. Define $\mathcal{L}^p(X, \mu)$ and prove that for $1 \leq p < \infty$, $\mathcal{L}^p$ is a Banach space.

   Solution:

   This is proved on p. 187 of your book, in the appendix "On L^p Matters," with the assumption that the space X is σ-finite. In class, we did a more general proof without this assumption, so you can refer to class notes as well. Either proof was completely acceptable on the test.

3. Given a constant $c > 0$, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dy}{c^2 + y^2}$$

using the Fourier transform of the function

$$f_c(x) = \begin{cases} e^{-cx}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Solution:

The Fourier transform of $f_c(x)$ is

$$\begin{aligned} \hat{f}_c(x) &= \int_{-\infty}^{\infty} f_c(x)\, e^{-ixy}\, dx \\ &= \int_0^{\infty} e^{-(c+iy)x}\, dx \\ &= \frac{1}{c+iy} \end{aligned}$$

So in particular,

$$|\hat{f}_c(x)|^2 = \frac{1}{c^2 + y^2}$$

Now, using Plancherel's theorem, which states that

$$||\hat{f}||_2^2 = 2\pi ||f||_2^2$$

for functions f in the Schwartz class, we have

$$\int_{-\infty}^{\infty} \frac{dy}{c^2 + y^2} = 2\pi \int_0^{\infty} |e^{-cx}|^2\, dx = \frac{\pi}{c}.$$

Many people forgot to justify why this could be used for f_c, which is not strictly in the Schwartz class, as there is a discontinuity at the origin. But owing to time constraints, this was not counted against you. How would you show this?

4. Let $X = Y = [0, 1]$, the unit interval, and $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, the Borel sets in the unit interval. Consider the measure spaces $(X, \mathcal{M}, \mu)$ with $\mu = \mu_L$, Lebesgue measure, and $(Y, \mathcal{N}, \nu)$ with ν the counting measure (i.e., the number of elements in the set). Set

$$D = \{(x, y) \in X \times Y \mid x = y\}$$

with 1_D the characteristic function of D.

Which, if any, of the following integrals are equal?

$$\int_0^1 \int_0^1 1_D \, d\mu \, d\nu, \quad \int_0^1 \int_0^1 1_D \, d\nu \, d\mu, \quad \int_{X \times Y} 1_D \, d(\mu \times \nu).$$

Here, $\mu \times \nu$ is the product measure, defined as usual by extending the definition on "rectangles" via outer measure. Be sure to carefully explain your answer.

Solution:

First, note that Fubini's theorem does not apply because the counting measure ν is not σ-finite on the unit interval. So no two of these integrals are necessarily equal, and that's exactly what we can show by direct computation. Recall that

$$\int_0^1 1_D \, d\mu = \mu(E_y) = 0$$

since the slice at fixed y, E_y, is a single point, which has Lebesgue measure 0. This immediately implies

$$\int_0^1 \int_0^1 1_D \, d\mu \, d\nu = \int_0^1 0 \, d\nu = 0.$$

Similarly,

$$\int_0^1 1_D \, d\nu = \nu(E_x) = 1$$

since the slice at fixed x, E_x, also contains a single point, which has counting measure 1. Hence,

$$\int_0^1 \int_0^1 1_D \, d\nu \, d\mu = \int_0^1 1 \, d\mu = \mu_L([0, 1]) = 1.$$

Finally, to compute the value for the product measure, recall that $\mu \times \nu$ is an extension of the measure defined by

$$\mu \times \nu(A \times B) = \mu(A)\nu(B)$$

on sets $A \times B$, with $A \in \mathcal{M}, B \in \mathcal{N}$, so-called "rectangles." So

$$\int_{X \times Y} 1_D \, d(\mu \times \nu) = \mu \times \nu(D)$$

which can be computed by finding a sequence of rectangles converging in measure to D. There are many ways to do this. My first thought was to imagine a sequence where the nth term contains 2^n squares of width $1/2^n$ covering the diagonal. The measure of each of these squares, with respect to $\mu \times \nu$, is $\mu(I)\nu(I) = \infty \cdot 1/2^n = \infty$, so the product measure of the diagonal, which is the limit of the measures of this converging sequence of squares, is also ∞.

5. (Bonus) Define the convolution operator $*$ for functions $f, g \in \mathcal{L}^1(\mathbb{R})$ by

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\, dy$$

Prove $f * g$ is in $\mathcal{L}^1(\mathbb{R})$ and that

$$||f * g||_1 \leq ||f||_1 ||g||_1$$

Solution:

This exercise is handled on page 193 of the book. Many people tried to use Fubini's theorem, which is the right idea, but first you need to prove that the integrand to which you are applying Fubini's theorem is indeed integrable.